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On Recursive Hyperbolic Fibonacci Quaternions

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Abstract
Many quaternions with the coefficients selected from special integer sequences such as Fibonacci and Lucas sequences have been investigated by a great number of researchers. This article presents new classes of quaternions whose components are composed of symmetrical hyperbolic Fibonacci functions. In addition, the Binet's formulas, certain generating matrices, generating functions, Cassini's and d'Ocagne's identities for these quaternions are given.

Keywords: Binet's formula, Cassini's identity, generating Matrix, hyperbolic Fibonacci functions, quaternion
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1. Introduction

Quaternions are defined as a 4-tuple of real numbers and represented by a linear combination of the elements of the standard orthonormal basis such as

$$q = q_0 + iq_1 + jq_2 + kq_3$$

with the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \tag{1.1}$$

where q_0, q_1, q_2 and q_3 are any real numbers, q_0 is called the scalar part of q and $iq_1 + jq_2 + kq_3$ is the vector part. Note that its scalar and vector parts are abbreviated as $Sc(q)$ and $Vec(q)$, respectively. The conjugate of q is

$$q^* = q_0 - iq_1 - jq_2 - kq_3$$

and its norm is

$$N(q) = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{1.2}$$

Eq. (1.1) indicates that the real quaternions form a non-commutative division algebra, even the skew field in the set of quaternions. Due to the loss of commutativity, it is very difficult to study them.

Quaternions are useful tools in many science areas such as mathematics, physics and computer sciences. The monographs in [1] and [2] present well-known systematic investigations on the subject. In recent decades, several researchers investigate different types of quaternions. In [3], Horadam gave the Fibonacci quaternions in the form

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

where F_n is the usual Fibonacci number defined as

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geqslant 0$$

by the initial terms $F_0 = 0$ and $F_1 = 1$. In addition, the Lucas numbers are defined by the same recurrence relation but with the initial conditions $L_0 = 2$ and $L_1 = 1$. Certain important results on the Fibonacci quaternions are presented in the references [3]-[8] and many other related references which are not given here.

In the current literature, many interesting generalizations of the Fibonacci and Lucas numbers and their various types can be found. However, one of the most interesting approaches to the topics is given by Stakhov and Tkachenko in [9]. By Binet's formulas of the Fibonacci and Lucas numbers, they introduced a new concept called hyperbolic Fibonacci and hyperbolic Lucas functions. The monograph [10] offers a detailed review. Furthermore, Stakhov and Rozin improve this approach to symmetrical hyperbolic Fibonacci and symmetrical hyperbolic Lucas functions. According to the authors in [11], the symmetrical hyperbolic functions are defined as follows:

$$\text{Symmetrical Fibonacci sine functions: } sFs(x) = \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}, \tag{1.3}$$

$$\text{Symmetrical Fibonacci cosine functions: } cFs(x) = \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}}, \tag{1.4}$$

$$\text{Symmetrical Lucas sine functions: } sLs(x) = \alpha^x - \alpha^{-x} \tag{1.5}$$

and

$$\text{Symmetrical Lucas cosine functions: } cLs(x) = \alpha^x + \alpha^{-x}, \tag{1.6}$$

where x is any real number and α is the golden ratio. It should be noted that $sFs(2k) = F_{2k}$, $cLs(2k) = L_{2k}$, $cFs(2k+1) = F_{2k+1}$, and $sLs(2k+1) = L_{2k+1}$ can be written for $k \in Z$, respectively. Since the hyperbolic functions play a great role in modern sciences such as mathematics and physics, these special functions are very important. Note that throughout the paper, we will omit the letter "s" in right-hand sides of the representations "$sFs(x)$" and "$cFs(x)$" for combinatorial simplicity, e.g. $sF(x)$ and $cF(x)$, respectively.

Based on the above developments, Daşdemir introduced the symmetrical hyperbolic Lucas sine and cosine quaternions as

$$sPs(x) = sLs(x) + isLs(x+1) + jsLs(x+2) + ksLs(x+3)$$

and

$$cPs(x) = cLs(x) + icLs(x+1) + jcLs(x+2) + kcLs(x+3),$$

respectively [12]. Here, x is any real number, and $sLs(x)$ and $cLs(x)$ were respectively defined in (1.5) and (1.6).

In this paper, we define new classes of quaternions. The coefficients of these quaternions are chosen from the symmetrical hyperbolic Fibonacci functions. Note that our definitions give the quaternions regarded as a combinations of real valued-functions, not integer valued-functions. Further, we give hyperbolic properties and some identities including the Binet's formulas, the generating functions, the Cassini's and d'Ocagne's identities for these quaternions.

2. Main Results

Consider the symmetrical hyperbolic Fibonacci functions given in (1.3) and (1.4). Hence, we give the following definition.

Definition 2.1. *The symmetrical hyperbolic Fibonacci sine and cosine quaternion functions are defined by the relations*

$$S(x) = sF(x) + isF(x+1) + jsF(x+2) + ksF(x+3) \tag{2.1}$$

and

$$C(x) = cF(x) + icF(x+1) + jcF(x+2) + kcF(x+3), \tag{2.2}$$

respectively.

For simplicity, we shall call these quaternions in (2.1) and (2.2) s-Fibonacci quaternions and c-Fibonacci quaternions, respectively. It should be noted that x is regarded as any real number throughout the paper.

We can present the following fundamental properties of the quaternions defined above.

Theorem 2.2. *Let x be any real number. Hence, we have the following relations:*

$$S(x+2) = C(x+1) + S(x) \tag{2.3}$$

and

$$C(x+2) = S(x+1) + C(x), \tag{2.4}$$

respectively.

Proof. The proof is completed by employing the Binet's formulas given in (1.3) and (1.4). □

Remark 2.3. *According to Theorem 2.2, it is possible to exchange symmetrically $S(x)$ with $C(x)$ and $C(x)$ with $S(x)$ for all linear relations of the s-Fibonacci and c-Fibonacci quaternions.*

Note that the recurrence relations given in (2.3) and (2.4) are inhomogeneous. If we apply the corresponding relation to the first term of its right-hand-side, we obtain new structure of each equation, in the homogeneous form, as

$$S(x+2) = 3S(x) - S(x-2) \tag{2.5}$$

and

$$C(x+2) = 3C(x) - C(x-2). \tag{2.6}$$

We can represent the recurrence relations in (2.3)-(2.6) by generating matrix technique. To do this, we introduce the matrices

$$\mathbf{Mc}(x) = \begin{bmatrix} C(x+1) & S(x) \\ S(x) & C(x-1) \end{bmatrix},$$

$$\mathbf{Ms}(x) = \begin{bmatrix} S(x+1) & C(x) \\ C(x) & S(x-1) \end{bmatrix},$$

$$\mathbf{Ns}(x) = \begin{bmatrix} S(x+1) & S(x) \\ S(x-1) & S(x-2) \end{bmatrix}$$

and

$$\mathbf{Nc}(x) = \begin{bmatrix} C(x+1) & C(x) \\ C(x-1) & C(x-2) \end{bmatrix}.$$

Hence, we can write

$$\mathbf{Mc}(x) = \mathbf{P}.\mathbf{Ms}(x-1),\ \mathbf{Ms}(x) = \mathbf{P}.\mathbf{Mc}(x-1), \tag{2.7}$$

$$\mathbf{Ns}(x) = \mathbf{R}.\mathbf{Ns}(x-2) \text{ and } \mathbf{Nc}(x) = \mathbf{R}.\mathbf{Nc}(x-2), \tag{2.8}$$

where $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$. Hence, we can give the following theorem.

Theorem 2.4. *Let x and n be a real positive integer, respectively. Then,*

$$\mathbf{Mc}(x) = \mathbf{P}^m . \begin{cases} \mathbf{Ms}(\mu), & \textit{if } m \textit{ is odd} \\ \mathbf{Mc}(\mu), & \textit{if } m \textit{ is even} \end{cases},$$

$$\mathbf{Ms}(x) = \mathbf{P}^m . \begin{cases} \mathbf{Mc}(\mu), & \textit{if } m \textit{ is odd} \\ \mathbf{Ms}(\mu), & \textit{if } m \textit{ is even} \end{cases},$$

$$\mathbf{Ns}(x) = \mathbf{R}^m . \mathbf{Ns}(\mu),$$

and

$$\mathbf{Nc}(x) = \mathbf{R}^m . \mathbf{Nc}(\mu).$$

Proof. Extending the right side of the matrix equations in (2.7) and (2.8) to the right, the desired results are obtained by a property of matrix multiplication. □

Remark 2.5. *It follows from Theorem 2.4 that all the determinantal identities obtained for s-Fibonacci quaternions are equal to negative sign of those for c-Fibonacci quaternions. Hence, it is sufficient that we only give the next results for the s-Fibonacci quaternions. But keep in mind that negative signs of right-hand side for these equations are valid for c-Fibonacci quaternions.*

The Binet's formulas for the *s*-Fibonacci and *c*-Fibonacci quaternions are given in the following theorem.

Theorem 2.6 (Binet's formula)**.** *The Binet's formulas of s-Fibonacci and c-Fibonacci quaternions are given as follows:*

$$S(x) = \frac{\boldsymbol{A}\alpha^{x} - \boldsymbol{B}\alpha^{-x}}{\sqrt{5}} \text{ and } C(x) = \frac{\boldsymbol{A}\alpha^{x} + \boldsymbol{B}\alpha^{-x}}{\sqrt{5}}, \tag{2.9}$$

where $\boldsymbol{A} = 1 + i\alpha + j\alpha^2 + k\alpha^3$, $\boldsymbol{B} = 1 - i\beta + j\beta^2 - k\beta^3$ *and* $\beta = -\alpha^{-1}$.

Proof. From the definition of *s*-Fibonacci quaternions and the Binet's formula of $sF(x)$, we can write

$$\begin{aligned} S(x) &= sF(x) + isF(x+1) + jsF(x+2) + ksF(x+3) \\ &= \tfrac{\alpha^{x}-\alpha^{-x}}{\sqrt{5}} + i\tfrac{\alpha^{x+1}-\alpha^{-(x+1)}}{\sqrt{5}} + j\tfrac{\alpha^{x+2}-\alpha^{-(x+2)}}{\sqrt{5}} + k\tfrac{\alpha^{x+3}-\alpha^{-(x+3)}}{\sqrt{5}} \\ &= \tfrac{\alpha^{x}\left(1+i\alpha+j\alpha^2+k\alpha^3\right)-\alpha^{-x}\left(1+i\alpha^{-1}+j\alpha^{-2}+k\alpha^{-3}\right)}{\sqrt{5}}. \end{aligned}$$

Substituting $\beta = -\alpha^{-1}$ into the last equation, the first equation is attained. By employing the same procedure, the other is obtained. So, the proof is completed. □

From the Binet's formulas in (2.9), we conclude that the *c*-Fibonacci quaternions are an even function, but nothing can be said for the *s*-Fibonacci quaternions. In addition, considering the Binet's formulas given in (2.9), we can write

$$C(x) = S(x) + \frac{2\mathbf{B}}{\sqrt{5}}\beta^{x}. \tag{2.10}$$

This result indicates that a study of the one involves familiarity with the other one. Note that all the results obtained in this paper are transformed to the other by employing Eq. (2.10).

From the Binet's formulas given in (2.9), we conclude that the *s*-Fibonacci and the *c*-Fibonacci quaternions have the same form except for the sign of α^{-x}. Hence, we can enter an auxiliary function that possesses 1 and -1 for consecutive integer values of x into the Binet's formulas to guarantee continuous condition. To do this, the function $\cos(\pi x)$ may be the best choice. Consequently, the following definition arises naturally.

Definition 2.7. *The quasi-sine Fibonacci quaternion is defined as*

$$\mathscr{Q}(x) = \frac{\boldsymbol{A}\alpha^{x} - \cos(\pi x)\boldsymbol{B}\alpha^{-x}}{\sqrt{5}} \tag{2.11}$$

Here, we can say that the definition in (2.11) satisfies the same recurrence relation in (2.3). For even or odd integer values of x, the quasi-sine Fibonacci quaternion reduces to the *s*-Fibonacci and the *c*-Fibonacci quaternions, respectively.

Theorem 2.8. *For any real number x, we have*

$$N(S(x)) = \sqrt{3cF(2x+3) - \frac{8}{5}} \tag{2.12}$$

and

$$N(C(x)) = \sqrt{3cF(2x+3) + \frac{8}{5}}. \tag{2.13}$$

Proof. Considering definition in (1.2), we can write

$$\begin{aligned} [N(S(x))]^2 &= [sF(x)]^2 + [sF(x+1)]^2 + [sF(x+2)]^2 + [sF(x+3)]^2 \\ &= \left[\frac{\alpha^{x} - \alpha^{-x}}{\sqrt{5}}\right]^2 + \left[\frac{\alpha^{(x+1)} - \alpha^{-(x+1)}}{\sqrt{5}}\right]^2 + \left[\frac{\alpha^{(x+2)} - \alpha^{-(x+2)}}{\sqrt{5}}\right]^2 + \left[\frac{\alpha^{(x+3)} - \alpha^{-(x+3)}}{\sqrt{5}}\right]^2 \end{aligned}$$

Here, expanding the square terms and using property $\alpha^2+1=\sqrt{5}\alpha$ yields to

$$[N(S(x))]^2 = \frac{\alpha^{2x}\left(1+\alpha^2+\alpha^4+\alpha^6\right)+\alpha^{-2x}\left(1+\alpha^{-2}+\alpha^{-4}+\alpha^{-6}\right)-8}{5}$$
$$= \frac{3\sqrt{5}\left(\alpha^{(2x+3)}+\alpha^{-(2x+3)}\right)-8}{5}.$$

The last equation gives Eq. (2.12). Repeating the same procedure, (2.13) can be demonstrated. □

Next theorem only presents some linear elementary properties for the s-Fibonacci quaternions to reduce the size of the paper. Note that according to Remark 2.3, a property known for the one can also be written for the other due to the exchange rule for all the linear properties.

Theorem 2.9. *Let x be any real numbers. Then, we have*

$$sPs(x) = S(x+1)+S(x-1), \tag{2.14}$$
$$2S(x+1) = C(x)+sPs(x),$$
$$5S(x) = sPs(x+1)+sPs(x-1)$$

and

$$cPs(x)+5S(x) = 2sPs(x+1).$$

Proof. We only prove Eq. (2.14) since the others can be showed similarly. From the Binet's formula in (2.9), we can write

$$\frac{\mathbf{A}\alpha^{x+1}-\mathbf{B}\alpha^{-(x+1)}}{\sqrt{5}}+\frac{\mathbf{A}\alpha^{x-1}-\mathbf{B}\alpha^{-(x-1)}}{\sqrt{5}}=\frac{\left(\alpha^2+1\right)\left(\mathbf{A}\alpha^{x-1}-\mathbf{B}\alpha^{-x-1}\right)}{\sqrt{5}}$$

and using $\alpha^2+1=\sqrt{5}\alpha$, the proof is completed. □

Theorem 2.10 (Hyperbolic version of the Pythagorean Theorem)**.** *The following property holds for any real number x:*

$$C(x)^2-S(x)^2 = -\frac{4}{\sqrt{5}}[C(0)]^*.$$

Proof. To show the validity of theorem, we use an interesting property of quaternions: Let p be any quaternion of the components p_0, p_1, p_2, and p_3. Then, $p^2=2p_0p-[N(p)]^2$. Hence, we can write

$$C(x)^2-S(x)^2 = 2cF(x)C(x)-[N(C(x))]^2-2sF(x)S(x)-[N(S(x))]^2$$
$$= 2cF(x)C(x)-2sF(x)S(x)-\frac{16}{5}.$$

It can be showed that $cF(x)cF(x+y)-sF(x)sF(x+y)=\frac{2}{\sqrt{5}}cF(y)$ for all real numbers x any y by using the Binet's formulas in (1.3) and (1.4). Considering this identity, we obtain

$$C(x)^2-S(x)^2 = 2cF(x)C(x)-2sF(x)S(x)-\frac{16}{5}=\frac{4}{\sqrt{5}}C(0)-\frac{16}{5}$$

Since $\frac{2}{\sqrt{5}}cF(0)=\frac{4}{5}$, the proof is completed. □

Theorem 2.11 (Moivre-type formula)**.** *Let x be any real number and n be any* $(positive\,or\,negative)$ *integers. Then,*

$$[C(x)+S(x)]^n = \left(\frac{2}{\sqrt{5}}\boldsymbol{A}\right)^{n-1}(C(nx)+S(nx)) \tag{2.15}$$

and

$$[C(x)-S(x)]^n = \left(\frac{2}{\sqrt{5}}\boldsymbol{B}\right)^{n-1}(C(nx)-S(nx)). \tag{2.16}$$

Proof. Using the Binet's formulas in (2.9), we can write

$$\begin{aligned}\left[\frac{\mathbf{A}\alpha^{x}+\mathbf{B}\alpha^{-(x)}}{\sqrt{5}}+\frac{\mathbf{A}\alpha^{x}-\mathbf{B}\alpha^{-x}}{\sqrt{5}}\right]^{n} &= \left[\frac{2\mathbf{A}}{\sqrt{5}}\right]^{n}\alpha^{nx}\\ &= \left[\frac{2\mathbf{A}}{\sqrt{5}}\right]^{n-1}\frac{2\mathbf{A}\alpha^{nx}+\mathbf{B}\alpha^{-nx}-\mathbf{B}\alpha^{-nx}}{\sqrt{5}}\\ &= \left[\frac{2\mathbf{A}}{\sqrt{5}}\right]^{n-1}\frac{\mathbf{A}\alpha^{nx}+\mathbf{B}\alpha^{-nx}+\mathbf{A}\alpha^{nx}-\mathbf{B}\alpha^{-nx}}{\sqrt{5}},\end{aligned}$$

which is Eq. (2.15). Similarly, Eq. (2.16) can be proved. □

Now we give two important theorems that will be reduced to some special cases.

Theorem 2.12 (Vajda identity)**.** *Let x, y, z, and t be any real numbers. Then, we have*

$$S(x+z)S(y-t)-S(x)S(y+z-t)=\frac{2}{\sqrt{5}}[C(u)-C(v)]^{*}+\frac{1}{\sqrt{5}}(sF(u)-sF(v))(2i+5k), \tag{2.17}$$

where $u=x-y+z+t$ and $v=x-y-z+t$.

Proof. By employing the Binet's formula in (1.3) for each part after applying the multiplication rule in (1.1) to the left-hand side of Eq. (2.17), the desired result is obtained directly. □

From Vajda identity, we also have the following special identities:

- For $y-t=y$ and $z=1$, we recover the d'Ocagne's identity:

$$S(x+1)S(y)-S(x)S(y+1)=\frac{1}{\sqrt{5}}\left\{2[S(x-y)]^{*}+cF(x-y)(2i+5k)\right\}.$$

- For $x=y$ and $r=s$, we find the Catalan's identity:

$$S(x+z)S(x-z)-S(x)^{2}=\frac{1}{\sqrt{5}}\left\{2[C(2z)]^{*}+sF(2z)(2i+5k)+2\gamma\right\},$$

  where $\gamma=-\frac{2}{\sqrt{5}}+i+\frac{3}{\sqrt{5}}j+2k$.

- For $x=y$ and $r=s=1$, we find the Cassini's identity:

$$S(x+1)S(x-1)-S(x)^{2}=\frac{1}{5}\left(2-8j-\sqrt{5}k\right).$$

Theorem 2.13 (Mixed-Vajda identity)**.** *Let x, y, z, and t be any real numbers. Then, the mixed-Vajda identity of first kind is*

$$S(x+z)S(y-t)-C(x)C(y+z-t)=\frac{2}{\sqrt{5}}[C(u)+C(v)]^{*}+\frac{1}{\sqrt{5}}(sF(u)+sF(v))(2i+5k)$$

and the mixed-Vajda identity of second kind is

$$C(x+z)S(y-t)-C(x)S(y+z-t)=\frac{2}{\sqrt{5}}[S(u)-S(v)]^{*}+\frac{1}{\sqrt{5}}(cF(u)-cF(v))(2i+5k),$$

where $u=x-y+z+t$ and $v=x-y-z+t$.

Proof. The proof can be completed by repeating the same procedure in Theorem 2.12. □

Particular cases of the mixed-Vajda identity of first kind are as follows:

- For $y-t=y$ and $z=1$, we obtain the d'Ocagne's identity of first kind:

$$S(x+1)S(y)-C(x)C(y+1)=\frac{1}{\sqrt{5}}\left\{\begin{array}{l}2[C(x-y+1)+C(x-y-1)]^{*}\\+(sF(x-y+1)+sF(x-y-1))(2i+5k)\end{array}\right\}.$$

- For $x = y$ and $r = s$, we find the Catalan's identity of first kind:

$$S(x+z)S(x-z) - C(x)^2 = \frac{1}{\sqrt{5}}\left\{2[C(2z)]^* + sF(2z)(2i+5k) - 2\gamma\right\}.$$

where $\phi = i - \sqrt{5}j + k$.

- For $x = y$ and $r = s = 1$, we find the Cassini's identity of first kind:

$$S(x+1)S(x-1) - C(x)^2 = \frac{1}{\sqrt{5}}\left(2\sqrt{5} - 4i - 4\sqrt{5}j - 9k\right).$$

Similarly, we have the following particular cases of the mixed-Vajda identity of second kind are as follows:

- For $y - t = y$ and $z = 1$, we obtain the d'Ocagne's identity of second kind:

$$C(x+1)S(y) - C(x)S(y+1) = \frac{1}{\sqrt{5}}\left\{2[C(x-y)]^* + sF(x-y)(2i+5k)\right\}.$$

- For $x = y$ and $r = s$, we find the Catalan's identity of second kind:

$$C(x+z)S(x-z) - C(x)S(x) = \frac{1}{\sqrt{5}}\left\{2[S(2z)]^* + cF(2z)(2i+5k) - \frac{2}{\sqrt{5}}\phi\right\}.$$

where $\phi = i - \sqrt{5}j + k$.

- For $x = y$ and $r = s = 1$, we find the Cassini's identity of second kind:

$$C(x+1)S(x-1) - C(x)S(x) = \frac{1}{5}\left(2\sqrt{5} - 4i - 4\sqrt{5}j - 9k\right).$$

Theorem 2.14 (General bilinear formula)**.** *Let x, y, z, t, and w be any integers satisfying $x + y = z + t$. Then, we have*

$$sF(x)C(y) - cF(z)S(t) = sF(x-w)C(y-w) - cF(z-w)S(t-w)$$

and

$$cF(x)S(y) - sF(z)C(t) = cF(x-w)S(y-w) - sF(z-w)C(t-w).$$

Proof. The proof is seen easily by applying the similar technique used in Theorem 2.12. □

We define the following functions:

$$G_s(x,t) = \sum_{n=0}^{\infty} S(x+n)t^n,\ G_c(x,t) = \sum_{n=0}^{\infty} C(x+n)t^n \tag{2.18}$$

and

$$g_s(x,t) = \sum_{n=0}^{\infty} S(x+n)\frac{t^n}{n!} \text{ and } g_c(x,t) = \sum_{n=0}^{\infty} C(x+n)\frac{t^n}{n!}.$$

Note that generating functions are powerful tools for solving linear homogeneous recurrence relations with constant coefficients. Let us introduce generating functions of the s-Fibonacci and c-Fibonacci quaternions. Hence we can write the following theorem.

Theorem 2.15. *The generating functions for the s-Fibonacci and c-Fibonacci quaternions are as follows:*

$$G_s(x,t) = \frac{S(x) + S(x+1)t - S(x-2)t^2 - S(x-1)t^3}{(1+x-x^2)(1-x-x^2)} \tag{2.19}$$

and

$$G_c(x,t) = \frac{C(x) + C(x+1)t - C(x-2)t^2 - C(x-1)t^3}{(1+x-x^2)(1-x-x^2)}.$$

Proof. We only show the validity of Eq. (2.19) since other can be proven in a similar way. First, we take $t^4 G_s(x,t)$ and $-3t^2 G_s(x,t)$ into account. Hence, from Eqs. (2.5), (2.18) and the last equations, we have

$$\left(1-3x^2+x^4\right) G_s(x,t) = S(x)+S(x+1)t-S(x-2)t^2-S(x-1)t^3,$$

which is desired result. □

We give the exponential generating function for $S(x)$ and $C(x)$ in the following theorem.

Theorem 2.16. *The exponential generating functions for the s-Fibonacci and c-Fibonacci quaternions are as follows:*

$$g_s(x,t) = \frac{\boldsymbol{A}\alpha^x e^{\alpha t} - \boldsymbol{B}\alpha^{-x} e^{-\beta t}}{\sqrt{5}}$$

and

$$g_c(x,t) = \frac{\boldsymbol{A}\alpha^x e^{\alpha t} + \boldsymbol{B}\alpha^{-x} e^{-\beta t}}{\sqrt{5}},$$

where t is any real number and e is the famous Euler's constant.

Proof. Considering MacLaurin expansion for the exponential function, we can write

$$\begin{aligned} g_s(t) = \sum_{n=0}^{\infty} S(x+n)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\mathbf{A}\alpha^{x+n} - \mathbf{B}\alpha^{-(x+n)}}{\sqrt{5}} \frac{t^n}{n!} &= \frac{1}{\sqrt{5}}\left(\mathbf{A}\alpha^x \sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!} - \mathbf{B}\alpha^{-x}\sum_{n=0}^{\infty}\frac{\left(\alpha^{-1}t\right)^n}{n!}\right) \\ &= \frac{1}{\sqrt{5}}(\mathbf{A}\alpha^x e^{\alpha t} - \mathbf{B}\alpha^{-x} e^{\alpha^{-1}t}), \end{aligned}$$

which is the first equation. Using the same procedure, the second equation can be obtained. □

We give the Honsberger formula for s-Fibonacci and c-Fibonacci quaternions in the following theorem. Note that there are many applications in physic and statistics.

Theorem 2.17 (Honsberger formula)**.** *Let x and y be any real numbers. Then,*

$$S(x+y) = sF(x+1)C(y) + cF(x)S(y-1)$$

and

$$C(x+y) = sF(x+1)S(y) + cF(x)C(y-1).$$

Proof. Using the Binet's formulas in (1.3), (1.4) and (2.9), we can write

$$\begin{aligned} sFs(x+1)C(y)+cFs(x)S(y-1) &= \frac{\alpha^{x+1}-\alpha^{-(x+1)}}{\sqrt{5}}\frac{\mathbf{A}\alpha^y+\mathbf{B}\alpha^{-y}}{\sqrt{5}} + \frac{\alpha^x+\alpha^{-x}}{\sqrt{5}}\frac{\mathbf{A}\alpha^{y-1}-\mathbf{B}\alpha^{-(y-1)}}{\sqrt{5}} \\ &= \frac{1}{5}\left\{\mathbf{A}\alpha^{x+y+1} - \mathbf{B}\alpha^{-(x+y+1)} + \mathbf{A}\alpha^{x+y-1} - \mathbf{B}\alpha^{-(x+y-1)}\right\} \\ &= \frac{1}{5}\left\{\mathbf{A}\alpha^{x+y-1}\left(\alpha^2+1\right) - \mathbf{B}\alpha^{-(x+y-1)}\left(\alpha^{-2}+1\right)\right\}. \end{aligned}$$

It can be proved easily that $\alpha^2+1=\sqrt{5}\alpha$ and $\alpha^{-2}+1=\sqrt{5}\alpha^{-1}$ are satisfied. Substituting these properties, the result follows. The second equation can be proved similarly. □

Let z be any real number. Substituting $(x,y)=(x-z,y+z)$ into the Honsberger formulas we can give a more general version of Theorem 2.17 in the following.

Corollary 2.18. *For any real numbers x, y and z, we have*

$$S(x+y) = sF(x-z+1)C(y+z) + cF(x-z)S(y+z-1)$$

and

$$C(x+y) = sF(x-z+1)S(y+z) + cF(x-z)C(y+z-1).$$

We now give summation formula in the following theorem.

Theorem 2.19. *Let x be any real number and n be any positive integer. Then, we have*

$$\sum_{k=0}^{n-1} S(x+k) = C(x+n+1) + C(x+n) - C(x) - C(x-1)$$

and

$$\sum_{k=0}^{n-1} C(x+k) = S(x+n+1) + S(x+n) - S(x) - S(x-1).$$

Proof. Summing all the equations after writing Eq. (2.5) for $x, x+1, \ldots, x+n$, with some mathematical arrangements, we obtain

$$\sum_{k=0}^{n} S(x+k) = -S(x) - S(x+1) + S(x+n+1) + S(x+n+2) + S(x-2) + S(x-1) - S(x+n-1) - S(x+n).$$

Applying the recurrence relation in (2.3) to the last equation, the result follows. □

3. Conclusion

In this paper, we defined the hyperbolic Fibonacci and the quasi-sine Fibonacci quaternion and try to develop some matrix equations to these definitions. Also, we investigated some identities including Binet's formulas, the generating functions, the Pythagorean-type and Moivre-type formulas. In particular, we presented Vajda-type identities that can be reduced to some important well-known forms, including Catalan's or Cassini's identities.

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Author's contributions

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