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Some Inequalities for Statistical Submanifolds in Metallic-like Statistical Manifolds

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Abstract. In this study, we introduce metallic-like statistical manifolds and provide some examples, which are a generalized version of metallic manifolds. We also obtain the first Chen inequality and a Chen inequality for the $\delta(2,2)$ invariant for statistical submanifolds in metallic-like statistical manifolds.

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1. Introduction

A member of the metallic means family, the golden mean has been used in many ancient cultures as the basis of proportion to compose music, design sculptures and paintings, or build temples and palaces. It is assumed that some members of metallic means family, particularly the golden mean and silver mean, are intrinsically related to the theoretical explanation of behavior in quantum physics. Some relatives of the golden mean have been used by physicists in their recent research trying to analyze the behavior of nonlinear dynamic systems in the transition from periodicity to semi-periodicity [29, 30].

Information geometry is a method of exploring the world of information by means of modern geometry. Concept of information have so far been studied using mostly algebraic, logical, analytical and probabilistic methods. Because geometry examines the mutual relations between elements such as distance and curvature, it should provide powerful tools to the information science. Information geometry has appeared from studies of invariant geometric structures involved in statistical inference. It presents a Riemannian metric together with dually coupled affine connections in a manifold of probability distributions. These are important structures not only in statistical inference but also in the broader fields of information science such as machine learning, signal processing, optimization, and even neuroscience [1].

Amari introduced the concept of statistical manifolds via information geometry (see [1]). Statistical manifolds are endowed with dual connections an analogue to conjugate connections in affine geometry (see [24]). The fact that dual connections are not metric, it is not easy to give a good definition of sectional curvature using the canonical Riemannian geometry stuff. So in [25], B. Opozda introduced a method to define sectional curvature of statistical manifolds. In the study of differential geometric properties of a submanifold, relationships between the intrinsic and the extrinsic invariants are very important questions to ask, and a giant number of such relations are discovered in the past decades. For example, let M be a surface in Euclidean 3-space, we know the Euler inequality: $K \leq |H|^2$, where H is the mean

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curvature (extrinsic property) and K is the Gaussian curvature (intrinsic property). The equality holds at points where M is congruent to an open piece of a plane or a sphere (umbilical points). B.-Y Chen [10] obtained the same inequality for submanifolds of real space forms. Then in [11], B.-Y Chen obtained the Chen-Ricci inequality, which is a sharp relation between the squared mean curvature and the Ricci curvature of a Riemannian submanifold of a real space form. For more on Chen inequalities, we refer the reader to see [18, 22, 23, 26].

Recently, statistical manifolds are being studied very actively. In [32], Takano defined a new type statistical manifolds which including almost complex and almost contact structure. In 2015, A.D. Vîlcu and G.E. Vîlcu [34] investigated on statistical manifolds with quaternionic settings and suggested a few obvious problems. While searching for answers to one of these obvious problems, M. Aquib [3] found some of the curvature properties of submanifolds and a couple of inequalities for totally real statistical submanifolds of quaternionic Kaehler-like statistical space forms. A recent time, B.-Y Chen et al. [12] obtained a Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. Following the same paper, H. Aytimur et al. [7] derived the same inequalities for statistical submanifolds of Kaehler-like statistical manifolds. Very recently, in 2020, C. W. Lee and J. W. Lee obtained inequalities on Sasakian statistical manifolds in terms of casorati curvatures. For more on statistical submanifolds, we refer the reader to see [2–7, 9, 12, 21, 28, 32]. So far no study has been done on statistical concepts for metallic structures.

Motivated by these papers, we introduced metallic-like statistical manifolds which generalized version of golden-like statistical manifolds. The structure of the paper is as follows. In the second section, we revisit the definitions and other basic notions. In the third section, we define metallic-like statistical manifolds and construct certain examples of such manifolds. In the fourth section, we derive our main inequalities. We also discuss the equality case.

2. Preliminaries

A tensor field ϑ of type $(1,1)$ is called a polynomial structure if it satisfies the following equation on m- dimensional Riemannian manifold (M, g) with $b_1, \ldots, b_n$ real numbers

$$Q(X) := X^n + b_n X^{n-1} + \ldots + b_2 X + b_1 I = 0,$$

where I is the identity transformation (for instance see [13, 15]). We notice that:

- if $Q(\vartheta) = \vartheta^2 + I$, then ϑ is an almost complex structure.
- if $Q(\vartheta) = \vartheta^2 - I$, then ϑ is an almost product structure.
- if $Q(\vartheta) = \vartheta^2 - p\vartheta - qI$, then ϑ is an metallic structure,

where p and q are two integers [14, 16, 19]. The Riemannian metric g is called $\vartheta-$ compatible if

$$g(\vartheta X, Y) = g(X, \vartheta Y), \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$. Let g be $\vartheta-$ compatible and ϑ be metallic structure on Riemannian manifold M. Then (M, g) is called a metallic Riemannian manifold. Using the equation (2.1), we have

$$g(\vartheta X, \vartheta Y) = g(\vartheta^2 X, Y) = p\, g(X, \vartheta Y) + q\, g(X, Y).$$

It is to note that by putting $p = q = 1$ in above equations, a metallic structure reduces to a Golden structure [31].
Members of the metallic family are classified as follows:

- the golden structure $\vartheta = \frac{1+\sqrt{5}}{2}$ for $p = q = 1$, determined by the ratio of two consecutive classical Fibonacci numbers;
- the silver structure $\kappa_{2,1} = 1 + \sqrt{2}$ if $p = 2$ and $q = 1$, determined by the ratio of two consecutive Pell numbers;
- the bronze structure $\kappa_{3,1} = \frac{3+\sqrt{13}}{2}$ with $p = 3$ and $q = 1$;
- the subtle structure $\kappa_{4,1} = 2 + \sqrt{5}$ if $p = 4$ and $q = 1$;
- the copper structure $\kappa_{1,2} = 2$ with $p = 1$ and $q = 2$;
- the nickel structure $\kappa_{1,3} = \frac{1+\sqrt{13}}{2}$ if $p = 1$ and $q = 3$ and so on [16].

Definition 2.1. (i) Let ϑ be a metallic structure on M and let ∇ be a linear connection on M. Then ∇ is called a ϑ-connection if $\nabla\vartheta = 0$, i.e. ϑ is covariantly constant with respect to ∇.

(ii) The metallic Riemannian manifold (M, g, ϑ) is called a locally metallic Riemannian manifold if the Levi-Civita connection ∇ of g is a ϑ-connection (see [8]).

Proposition 2.2. *Let M_p and M_q be real space forms with constant sectional curvatures c_1 and c_2, respectively. Similar to semi-Riemannian product space form (see [33]), We can easily obtain the Riemannian curvature tensor R of a metallic product space form $(M = M_p(c_p) \times M_q(c_q), g, \vartheta)$ as the following:*

$$\begin{aligned} R(X,Y)Z = \frac{1}{4}(c_1+c_2)\Big[&g(Y,Z)X - g(X,Z)Y + \frac{4}{(2\kappa_{p,q}-p)^2}\{g(\vartheta Y,Z)\vartheta X - g(\vartheta X,Z)\vartheta Y\} \\ &+ \frac{p^2}{(2\kappa_{p,q}-p)^2}\{g(Y,Z)X - g(X,Z)Y\} + \frac{2p}{(2\kappa_{p,q}-p)^2}\{g(\vartheta X,Z)Y + g(X,Z)\vartheta Y \\ &- g(\vartheta Y,Z)X - g(Y,Z)\vartheta X\}\Big] \pm \frac{1}{2}(c_1-c_2)\Big[\frac{1}{(2\kappa_{p,q}-p)}\{g(Y,Z)\vartheta X - g(X,Z)\vartheta Y\} \\ &+ \frac{1}{(2\kappa_{p,q}-p)}\{g(\vartheta Y,Z)X - g(\vartheta X,Z)Y\} + \frac{p}{2\kappa_{p,q}-p}\{g(X,Z)Y - g(Y,Z)X\}\Big]. \end{aligned}$$

One can easily obtain the curvature tensor R^* for dual connection just by replacing ϑ by ϑ^*.

Let M^n be statistical submanifold of (N^m, g, ϑ). The Gauss and Weingarten formulae are

$$\nabla_X Y = \nabla_X Y + h(X,Y), \quad \nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$
$$\nabla^*_X Y = \nabla^*_X Y + h^*(X,Y), \quad \nabla^*_X \xi = -A^*_\xi X + \nabla^{*\perp}_X \xi,$$

for any $X, Y \in TM$ and $\xi \in T^\perp M$, respectively. Furthermore, we have the following equations

$$\begin{aligned} &Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \\ &g(h(X,Y),\xi) = g(A^*_\xi X, Y), \quad g(h^*(X,Y),\xi) = g(A_\xi X, Y), \\ &Xg(\xi,\eta) = g(\nabla^\perp_X \xi, \eta) + g(\xi, \nabla^{*\perp}_X \eta), \end{aligned}$$

for any $\eta \in T^\perp M$. The mean curvature vector fields for orthonormal tangent frame $\{e_1, e_2, \ldots, e_n\}$ and normal frame $\{e_{n+1}, \ldots, e_m\}$, respectively, are defined as

$$H = \frac{1}{n}\sum_{i=1}^n h(e_i,e_i) = \frac{1}{n}\sum_{\gamma=n+1}^m \left(\sum_{i=1}^n h^\gamma_{ii}\right)\xi_\gamma, \ h^\gamma_{ij} = g(h(e_i,e_j), e_\gamma)$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^n h^*(e_i,e_i) = \frac{1}{n}\sum_{\gamma=n+1}^m \left(\sum_{i=1}^n h^{*\gamma}_{ii}\right)\xi_\gamma, \ h^{*\gamma}_{ij} = g(h^*(e_i,e_j), e_\gamma),$$

for $1 \le i, j \le n$ and $n+1 \le \gamma \le m$. Moreover, we have $2h^0 = h + h^*$ and $2H^0 = H + H^*$, where the second fundamental form h^0 and the mean curvature H^0 are calculated with respect to Levi-Civita connection ∇^0 on M.

The squared mean curvatures are defined as

$$\|H\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^m \left(\sum_{i=1}^n h^\gamma_{ii}\right)^2, \quad \|H^*\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^m \left(\sum_{i=1}^n h^{*\gamma}_{ii}\right)^2.$$

If we suppose that $\mathcal{W}$ is a d-dimensional subspace of TM, $d \ge 2$, and $\{e_1, e_2, \ldots, e_d\}$ is an orthonormal basis of $\mathcal{W}$. Then the scalar curvature of the d-plane section is given as

$$\tau(\mathcal{W}) = \sum_{1 \le < u < v \le d} K(e_u \wedge e_v).$$

A point $x \in M$ is called as quasi-umbilical point, if at x there exist $m-n$ mutually orthogonal unit normal vectors e_i, $i \in \{n+1, \ldots, m\}$ in a way the shape operators with respect to all vectors e_i have an eigenvalue with multiplicity $n-1$ and for each e_i the distinguished eigen vector is the same.

Now, we state the following fundamental results on statistical manifolds.

Proposition 2.3. *[6] Let M be statistical submanifold of (N, g, ϑ). Let R and R^* be the Riemannian curvature tensors on N for ∇ and ∇^*, respectively. Then we have the following.*

$$g(R^*(X,Y)Z, W) = g(R(X,Y)Z, W) + g(h(X,Z), h^*(Y,W)) - g(h^*(X,W), h(Y,Z)),$$
$$g(R(X,Y)Z, W) = g(R^*(X,Y)Z, W) + g(h^*(X,Z), h(Y,W)) - g(h(X,W), h^*(Y,Z)),$$

$$g(R^{\perp}(X,Y)\xi,\eta) = g(R(X,Y)\xi,\eta) + g([A^*_{\xi}, A_{\eta}]X,Y),$$
$$g(R^{*\perp}(X,Y)\xi,\eta) = g(R^*(X,Y)\xi,\eta) + g([A_{\xi}, A^*_{\eta}]X,Y),$$

where $[A_{\xi}, A^*_{\eta}] = A_{\xi}A^*_{\eta} - A^*_{\eta}A_{\xi}$ *and* $[A^*_{\xi}, A_{\eta}] = A^*_{\xi}A_{\eta} - A_{\eta}A^*_{\xi}$, *for* $X,Y,Z,W \in TM$ *and* $\xi,\eta \in T^{\perp}M$.

Now, we state two important lemma's which we use to prove the main results in the upcoming sections.

Lemma 2.4. *[12] Let $n \geq 3$ be an integer and $b_1, b_2, \dots, a_n$ are n real numbers. Then, we have*

$$\sum_{1\leq i<j\leq n}^{n} b_i b_j - b_1 b_2 \leq \frac{n-2}{2(n-1)}\Big(\sum_{i=1}^{n} b_i\Big)^2.$$

Moreover, the equality holds if and only if $b_1 + b_2 = b_3 = \ldots = b_n$.

Lemma 2.5. *[22] Let $n \geq 4$ be an integer and $b_1, b_2, \dots, b_n$ are n real numbers. Then, we have*

$$\sum_{1\leq i<j\leq n}^{n} b_i b_j - b_1 b_2 - b_3 b_4 \leq \frac{n-3}{2(n-2)}\Big(\sum_{i=1}^{n} b_i\Big)^2.$$

Moreover, the equality holds if and only if $b_1 + b_2 = b_3 + b_4 = b_5 = \cdots = b_n$.

3. Metallic-like Statistical Manifolds

Takano introduced generalized almost complex and almost contact statistical manifolds, calling them Kahler-like statistical manifold and Sasaki-like statistical [32]. Inspired by this study, We will describe metallic-like statistical manifolds, which are a generalized version of metallic manifolds.

Definition 3.1. Let (M, g, ϑ) be a locally metallic semi-Riemannian manifold endowed with a tensor field ϑ^* of type (1,1) satisfying

$$g(\vartheta X, Y) = g(X, \vartheta^* Y), \tag{3.1}$$

for vector fields X and Y. In view of (3.1), we easily derive

$$(\vartheta^*)^2 X = p\vartheta^* X + qX, \tag{3.2}$$
$$g(\vartheta X, \vartheta^* Y) = p\, g(\vartheta X, Y) + q\, g(X,Y). \tag{3.3}$$

Then (M, g, ϑ) is called metallic-like statistical manifold.

According to (3.2) and (3.3), the tensor fields $\vartheta + \vartheta^*$ and $\vartheta - \vartheta^*$ are symmetric and skew symmetric with respect to g, respectively. The equations (3.1), (3.2) and (3.3) imply the following proposition.

Proposition 3.2. *(M, g, ϑ) is a metallic-like statistical manifold if and only if so is (M, g, ϑ^*).*

We remark that if we choose $\vartheta = \vartheta^*$ in a metallic-like statistical manifold, then we have a metallic semi-Riemannian manifold.

We first present some examples for metallic Riemannian manifolds.

Example 3.3. Consider the Euclidean 6−space $\mathbb{R}^6$ with standard coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$. Let ϑ be an $(1,1)$ tensor field on $\mathbb{R}^6$ defined by

$$\vartheta(x_1, x_2, x_3, x_4, x_5, x_6) = (\kappa x_1, (p-\kappa)x_2, \kappa x_3, (p-\kappa)x_4, \kappa x_5, (p-\kappa)x_6),$$

for any vector field $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6$, where $\kappa = \frac{p+\sqrt{p^2+4q}}{2}$ and $p - \kappa = \frac{p-\sqrt{p^2+4q}}{2}$ are the metallic numbers. Then we obtain $\vartheta^2 = p\vartheta + qI$. Moreover, we can easily see that standard metric $\langle\,,\rangle$ on $\mathbb{R}^6$ is ϑ compatible. Hence, $(\mathbb{R}^6, \langle\,,\rangle, \vartheta)$ is a metallic Riemannian manifold.

Example 3.4. (Clifford algebras). Let $C^{\gamma}(n)$ be the real Clifford algebra of the positive definite form $\sum_{k=1}^{m}\left(\mu^k\right)^2$ of $\mathbb{R}^m$ [17]. According to the Clifford product, the standard base of $\mathbb{R}^m$ satisfies the following relations:

$$E_k^2 = 1 \quad , \quad k = l$$
$$E_k E_l + E_l E_k = 0 \quad , \quad k \neq l.$$

Thus, using $\vartheta_i = \frac{1}{2}\left(p + \sqrt{p^2+4q}E_i\right)$ and above equation, we derive a new representation of the Clifford algebra:

$$\begin{cases} \vartheta_k, \quad \text{metallic structure} \quad , \quad k = l \\ \vartheta_k\vartheta_l + \vartheta_l\vartheta_k = p\left(\vartheta_k + \vartheta_l\right) - \frac{p^2}{2} \quad , \quad k \neq l, \end{cases}$$

where E_1 and E_2, orthonormal basis vectors of $\mathbb{R}^2_2$, are as follows [27]:

$$1 = I_2 \quad , \quad E_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad E_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and thus we obtain

$$\text{(i) } \vartheta_1 = \frac{1}{2}\left(p + \sqrt{p^2+4q}E_1\right) = \begin{pmatrix} \frac{p+\sqrt{p^2+4q}}{2} & 0 \\ 0 & \frac{p-\sqrt{p^2+4q}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{p,q} & 0 \\ 0 & p - \sigma_{p,q} \end{pmatrix}$$

$$\text{(ii) } \vartheta_2 = \frac{1}{2}\left(p + \sqrt{p^2+4q}E_2\right) = \frac{1}{2}\begin{pmatrix} p & \sqrt{p^2+4q} \\ \sqrt{p^2+4q} & p \end{pmatrix}.$$

Next, we construct some examples for metallic-like statistical manifolds.

Example 3.5. Consider the semi-Euclidean space $\mathbb{R}^3_1$ with standard coordinates (x_1, x_2, x_3) and the semi-Riemannian metric g with the signature $(-, +, +)$. Let ϑ be an $(1, 1)$ tensor field on $\mathbb{R}^3_1$ defined by

$$\vartheta(x_1, x_2, x_3) = \frac{1}{2}(px_1 + (2\kappa - p)x_2, px_2 + (2\kappa - p)x_1, (p - \kappa)x_3),$$

for any vector field $(x_1, x_2, x_3) \in \mathbb{R}^3_1$, where $\kappa = \frac{p+\sqrt{p^2+4q}}{2}$ are the members of the metallic means family. Then we obtain $\vartheta^2 = p\vartheta + qI$. Also we can easily see that structure is compatible with the metric. This implies that ϑ is a metallic structure on $\mathbb{R}^3_1$.

Now, we define an $(1, 1)$ tensor field ϑ^* on $\mathbb{R}^3_1$ by

$$\vartheta^*(x_1, x_2, x_3) = \frac{1}{2}(px_1 + (p - 2\kappa)x_2, px_2 + (p - 2\kappa)x_1, (p - \kappa)x_3).$$

Thus, we have $\vartheta^{*2} = p\vartheta^* + qI$. Moreover, we have the equation (3.1). Hence, $(\mathbb{R}^3_1, g, \vartheta)$ is a metallic-like statistical manifold.

Now, we give a generalized example of the above example.

Example 3.6. Let $\mathbb{R}^{2n+m}_n$ be a $(2n + m)$- dimensional affine space with the coordinate system

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m).$$

If we define a semi-Riemannian metric g and the tensor field ϑ as follows:

$$g = \begin{pmatrix} -\kappa\delta_{ij} & 0 & 0 \\ 0 & \kappa\delta_{ij} & 0 \\ 0 & 0 & (p-\kappa)\delta_{ij} \end{pmatrix}, \quad \vartheta = \frac{1}{2}\begin{pmatrix} p\delta_{ij} & (2\kappa - p)\delta_{ij} & 0 \\ (2\kappa - p)\delta_{ij} & p\delta_{ij} & 0 \\ 0 & 0 & \kappa\delta_{ij} \end{pmatrix},$$

where $\kappa = \frac{p+\sqrt{p^2+4q}}{2}$ are the members of the metallic means family. Then ϑ is metallic structure on $\mathbb{R}^{2n+m}_n$. Morever, if the conjugate tensor field ϑ^* is defined as

$$\vartheta^* = \frac{1}{2}\begin{pmatrix} p\delta_{ij} & (p - 2\kappa)\delta_{ij} & 0 \\ (p - 2\kappa)\delta_{ij} & p\delta_{ij} & 0 \\ 0 & 0 & \kappa\delta_{ij} \end{pmatrix}.$$

Then, we can easily see that $(\mathbb{R}^{2n+m}_n, g, \vartheta)$ and $(\mathbb{R}^{2n+m}_n, g, \vartheta^*)$ are metallic-like statistical manifolds. Also, this verifies the Proposition 3.2.

4. Main Inequalities

Let $\{e_1, e_2, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ be the orthonormal frames of TM and $T^{\perp}M$, respectively. Given a point $p \in M$ and a plane section $\pi \subset T_pM$, we fix $\Psi(\pi) = g(\vartheta X, X)g(\vartheta Y, Y)$. Also, we set $\Theta(\pi) = g^2(\vartheta X, Y)$, where X, Y are any orthonormal vectors spanning π (for instance see [20]).

For any tangent vector field $X \in \Gamma(TM)$, we can write $\vartheta X = \mathcal{T}X + \mathcal{F}X$, where $\mathcal{T}X$ and $\mathcal{F}X$ are the tangential and normal components of ϑX, respectively. If $\mathcal{T} = 0$, the submanifold is said to be an anti-invariant submanifold and if $\mathcal{F} = 0$, the submanifold is said to be an invariant submanifold. The squared norm of $\mathcal{T}$ at $p \in M$ is defined as

$$\|\mathcal{T}\|^2 = \sum_{i,j=1}^{n} g^2(\vartheta e_i, e_j).$$

The scalar curvature corresponding to the sectional K-curvature is

$$\tau = \frac{1}{2} \sum_{1 \le i < j \le n} \Big[g(R(e_i, e_j)e_j, e_i) + g(R^*(e_i, e_j)e_j, e_i) - 2g(R^0(e_i, e_j)e_j, e_i) \Big].$$

Thus we have

$$\begin{aligned}
R(e_i, e_j, e_j, e_i) = {} & \frac{1}{4}(c_1 + c_2)\Big[g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i) \\
& + \frac{4}{(2\kappa_{p,q} - p)^2}\{g(\vartheta e_j, e_j)g(\vartheta e_i, e_i) - g(\vartheta e_i, e_j)g(\vartheta e_j, e_i)\} \\
& + \frac{p^2}{(2\kappa_{p,q} - p)^2}\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} + \frac{2p}{(2\kappa_{p,q} - p)^2}\{g(\vartheta e_i, e_j)g(e_j, e_i) \\
& + g(e_i, e_j)g(\vartheta e_j, e_i) - g(\vartheta e_j, e_j)g(e_i, e_i) - g(e_j, e_j)g(\vartheta e_i, e_i)\}\Big] \\
& \pm \frac{1}{2}(c_1 - c_2)\Big[\frac{1}{(2\kappa_{p,q} - p)}\{g(e_j, e_j)g(\vartheta e_i, e_i) - g(e_i, e_j)g(\vartheta e_j, e_i)\} \\
& + \frac{1}{(2\kappa_{p,q} - p)}\{g(\vartheta e_j, e_j)g(e_i, e_i) - g(\vartheta e_i, e_j)g(e_j, e_i)\} + \frac{p}{2\kappa_{p,q} - p}\{g(e_i, e_j)g(e_j, e_i) \\
& - g(e_j, e_j)g(e_i, e_i)\}\Big] + g(h^*(e_i, e_i), h(e_j, e_j)) - g(h(e_j, e_i), h^*(e_j, e_i)).
\end{aligned}$$

The curvature tensor with respect to dual connection, i.e., $R^*(e_i, e_j, e_j, e_i)$ can be obtained from the above equation just by replacing ϑ by ϑ^*. Using (3.1) and Gauss equation, after doing some straightforward computations, we deduce

$$\begin{aligned}
\tau = {} & \frac{1}{4}\frac{(c_1 + c_2)(n^2 - n)}{p^2 + 4q}\Big\{ 1 + p^2 + \frac{4}{n^2 - n}\{\mathrm{tr}^2(\vartheta) - \|\mathcal{T}\|^2 - \frac{4p}{n}\mathrm{tr}(\vartheta)\} \\
& \pm 2\sqrt{p^2 + 4q}(2\,\mathrm{tr}(\vartheta) - np)\Big\} - \tau_0 + \frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{1 \le i < j \le n}\Big[h_{ii}^{*\gamma}h_{jj}^{*\gamma} + h_{ii}^{\gamma}h_{jj}^{*\gamma} - 2h_{ij}^{*\gamma}h_{ij}^{\gamma} \Big],
\end{aligned}$$

which can be written as

$$\begin{aligned}
\tau = {} & \frac{1}{4}\frac{(c_1 + c_2)(n^2 - n)}{p^2 + 4q}\Big\{ 1 + p^2 + \frac{4}{n^2 - n}\{\mathrm{tr}^2(\vartheta) - \|\mathcal{T}\|^2 - \frac{4p}{n}\mathrm{tr}(\vartheta)\} \\
& \pm 2\sqrt{p^2 + 4q}(2\,\mathrm{tr}(\vartheta) - np)\Big\} - \tau_0 + 2\sum_{\gamma=n+1}^{m}\sum_{1 \le i < j \le n}[h_{ii}^{0\gamma}h_{jj}^{0\gamma} - (h_{ij}^{0\gamma})^2] \\
& - \frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{1 \le i < j \le n}[\{h_{ii}^{\gamma}h_{jj}^{\gamma} + (h_{ij}^{\gamma})^2\} + \{h_{ii}^{*\gamma}h_{jj}^{*\gamma} - (h_{ij}^{*\gamma})^2\}].
\end{aligned}$$

By using Gauss equation for the Levi-Civita connection, we have

$$\tau = \frac{1}{4}\frac{(c_1+c_2)(n^2-n)}{p^2+4q}\Big\{1+p^2+\frac{4}{n^2-n}\{\mathrm{tr}^2(\vartheta)-\|\mathcal{T}\|^2-\frac{4p}{n}\mathrm{tr}(\vartheta)\}\pm 2\sqrt{p^2+4q}(2\,\mathrm{tr}(\vartheta) \\ -np)\Big\}-2\hat{\tau}_0-\frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{1\le i<j\le n}[\{h^{\gamma}_{ii}h^{*\gamma}_{jj}-(h^{\gamma}_{ij})^2\}+\{h^{*\gamma}_{ii}h^{*\gamma}_{jj}-(h^{*\gamma}_{ij})^2\}], \tag{4.1}$$

where $\hat{\tau}_0$ is the scalar curvature according to main statistical manifold. Now, the sectional K-curvature $K(\pi)$ of the plane section π is

$$K(\pi) \quad = \quad \frac{1}{2}\Big[g(R(e_1,e_2)e_2),e_1)+g(R^*(e_1,e_2)e_2),e_1)-2g(R^0(e_1,e_2)e_2),e_1)\Big].$$

Thus we have

$$\begin{aligned}R(e_1,e_2,e_2,e_1)=&\frac{1}{4}(c_1+c_2)\Big[g(e_2,e_2)g(e_1,e_1)-g(e_1,e_2)g(e_2,e_1)\\&+\frac{4}{(2\kappa_{p,q}-p)^2}\{g(\vartheta e_2,e_2)g(\vartheta e_1,e_1)-g(\vartheta e_1,e_2)g(\vartheta e_2,e_1)\}\\&+\frac{p^2}{(2\kappa_{p,q}-p)^2}\{g(e_2,e_2)g(e_1,e_1)-g(e_1,e_2)g(e_2,e_1)\}+\frac{2p}{(2\kappa_{p,q}-p)^2}\{g(\vartheta e_1,e_2)g(e_2,e_1)\\&+g(e_1,e_2)g(\vartheta e_2,e_1)-g(\vartheta e_2,e_2)g(e_1,e_1)-g(e_2,e_2)g(\vartheta e_1,e_1)\}\Big]\\&\pm\frac{1}{2}(c_1-c_2)\Big[\frac{1}{(2\kappa_{p,q}-p)}\{g(e_2,e_2)g(\vartheta e_1,e_1)-g(e_1,e_2)g(\vartheta e_2,e_1)\}\\&+\frac{1}{(2\kappa_{p,q}-p)}\{g(\vartheta e_2,e_2)g(e_1,e_1)-g(\vartheta e_1,e_2)g(e_2,e_1)\}+\frac{p}{2\kappa_{p,q}-p}\{g(e_1,e_2)g(e_2,e_1)\\&-g(e_2,e_2)g(e_1,e_1)\}\Big]+g(h^*(e_1,e_1),h(e_2,e_2))-g(h(e_2,e_1),h^*(e_2,e_1)).\end{aligned}$$

$R^*(e_1,e_2,e_2,e_1)$ can be obtained from the above equation just by replacing ϑ by ϑ^*.

Using (3.1) and Gauss equation, after doing some straightforward computations, we deduce

$$\begin{aligned}K(\pi)=&\frac{1}{4}(c_1+c_2)+\frac{c_1+c_2}{4(p^2+4q)}[4\Psi(\pi)-4\Theta^2(\pi)+p^2-2p\,\mathrm{tr}(\vartheta_{|_\pi})]\\&\pm\frac{1}{2}\frac{c_1-c_2}{\sqrt{p^2+4q}}[\mathrm{tr}(\vartheta_{|_\pi})-p]-K_0(\pi)+\frac{1}{2}\sum_{\gamma=n+1}^{m}\Big[h^{\gamma}_{11}h^{*\gamma}_{22}+h^{*\gamma}_{11}h^{\gamma}_{22}-2h^{*\gamma}_{12}h^{\gamma}_{12}\Big].\end{aligned}$$

Using $h+h^*=2h^0$, we get

$$\begin{aligned}K(\pi)=&\frac{1}{4}(c_1+c_2)+\frac{c_1+c_2}{4(p^2+4q)}[4\Psi(\pi)-4\Theta^2(\pi)+p^2-2p\,\mathrm{tr}(\vartheta_{|_\pi})]\\&\pm\frac{1}{2}\frac{c_1-c_2}{\sqrt{p^2+4q}}[\mathrm{tr}(\vartheta_{|_\pi})-p]-K_0(\pi)+2\sum_{\gamma=n+1}^{m}\Big[h^{0\gamma}_{11}h^{0\gamma}_{22}-(h^{0\gamma}_{12})^2\Big]\\&-\frac{1}{2}\sum_{\gamma=n+1}^{m}\Big\{\Big[h^{\gamma}_{11}h^{\gamma}_{22}-(h^{\gamma}_{12})^2\Big]+\Big[h^{*\gamma}_{11}h^{*\gamma}_{22}-(h^{*\gamma}_{12})^2\Big]\Big\}.\end{aligned}$$

In view of Gauss equation with respect to Levi-Civita connection, we have

$$\begin{aligned} K(\pi) = K_0(\pi) + \frac{1}{4}(c_1 + c_2) + \frac{c_1 + c_2}{4(p^2 + 4q)}[4\Psi(\pi) - 4\Theta^2(\pi) + p^2 - 2p\,\mathrm{tr}(\vartheta_{|_\pi})] \\ \pm \frac{1}{2}\frac{c_1 - c_2}{\sqrt{p^2 + 4q}}[\mathrm{tr}(\vartheta_{|_\pi}) - p] - 2\hat{K}_0(\pi) - \frac{1}{2}\sum_{\gamma=n+1}^{m}\left[h^{\gamma}_{11}h^{\gamma}_{22} - (h^{\gamma}_{12})^2\right] \\ - \frac{1}{2}\sum_{\gamma=n+1}^{m}\left[h^{*\gamma}_{11}h^{*\gamma}_{22} - (h^{*\gamma}_{12})^2\right], \end{aligned} \tag{4.2}$$

where $\hat{K}_0$ is the sectional curvature according to main statistical manifold. From (4.1) and (4.2), we have

$$\begin{aligned} (\tau - K(\pi)) - (\tau_0 - k_0(\pi)) = \frac{1}{4}\frac{(c_1 + c_2)(n^2 - n)}{p^2 + 4q}\Big\{1 + p^2 + \frac{4}{n^2 - n}\{\mathrm{tr}^2(\vartheta) - \|\mathcal{T}\|^2 - \frac{4p}{n}\mathrm{tr}(\vartheta)\} \\ \pm 2\sqrt{p^2 + 4q}(2\,\mathrm{tr}(\vartheta) - np)\Big\} - \frac{1}{4}(c_1 + c_2) - \frac{c_1 + c_2}{4(p^2 + 4q)}[4\Psi(\pi) - 4\Theta^2(\pi) + p^2 - 2p\,\mathrm{tr}(\vartheta_{|_\pi})] \\ \pm \frac{1}{2}\frac{c_2 - c_1}{\sqrt{p^2 + 4q}}[\mathrm{tr}(\vartheta_{|_\pi}) - p] - \frac{1}{2}\sum_{\gamma=n+1}^{m}\left[h^{\gamma}_{ii}h^{\gamma}_{jj} - (h^{\gamma}_{ij})^2\right] - \frac{1}{2}\sum_{\gamma=n+1}^{m}\left[h^{*\gamma}_{ii}h^{*\gamma}_{jj} - (h^{*\gamma}_{ij})^2\right] \\ + \frac{1}{2}\sum_{\gamma=n+1}^{m}\sum_{\alpha=1}^{3}\Big\{\left[h^{\gamma}_{11}h^{\gamma}_{22} - (h^{\gamma}_{12})^2\right] + \left[h^{*\gamma}_{11}h^{*\gamma}_{22} - (h^{*\gamma}_{12})^2\right]\Big\} + 2\hat{K}_0(\pi) - 2\hat{\tau}_0. \end{aligned}$$

Using lemma 2.4, we can get the above equation in simplied form as

$$\begin{aligned} (\tau - K(\pi)) - (\tau_0 - k_0(\pi)) \geq \frac{1}{4}\frac{(c_1 + c_2)(n^2 - n)}{p^2 + 4q}\Big\{1 + p^2 + \frac{4}{n^2 - n}\{\mathrm{tr}^2(\vartheta) - \|\mathcal{T}\|^2 - \frac{4p}{n}\mathrm{tr}(\vartheta)\} \\ \pm 2\sqrt{p^2 + 4q}(2\,\mathrm{tr}(\vartheta) - np)\Big\} - \frac{1}{4}(c_1 + c_2) - \frac{c_1 + c_2}{4(p^2 + 4q)}[4\Psi(\pi) - 4\Theta^2(\pi) + p^2 - 2p\,\mathrm{tr}(\vartheta_{|_\pi})] \\ \pm \frac{1}{2}\frac{c_2 - c_1}{\sqrt{p^2 + 4q}}[\mathrm{tr}(\vartheta_{|_\pi}) - p] - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2 + \|H^*\|^2] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0. \end{aligned}$$

Summarizing, we can state the following:

Theorem 4.1. *Let N be a metallic-like statistical manifold of dimension m and M be its statistical submanifold of dimension n. Then, we have the following*

$$\begin{aligned} (\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \geq \frac{1}{4}\frac{(c_1 + c_2)(n^2 - n)}{p^2 + 4q}\Big\{1 + p^2 + \frac{4}{n^2 - n}\{\mathrm{tr}^2(\vartheta) - \|\mathcal{T}\|^2 - \frac{4p}{n}\mathrm{tr}(\vartheta)\} \\ \pm 2\sqrt{p^2 + 4q}(2\,\mathrm{tr}(\vartheta) - np)\Big\} - \frac{1}{4}(c_1 + c_2) - \frac{c_1 + c_2}{4(p^2 + 4q)}[4\Psi(\pi) - 4\Theta^2(\pi) + p^2 - 2p\,\mathrm{tr}(\vartheta_{|_\pi})] \\ \pm \frac{1}{2}\frac{c_2 - c_1}{\sqrt{p^2 + 4q}}[\mathrm{tr}(\vartheta_{|_\pi}) - p] - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2 + \|H^*\|^2] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0. \end{aligned}$$

Moreover, the equalities holds for any $\gamma \in \{n+1, \dots, m\}$ if and only if

$$h^{\gamma}_{11} + h^{\gamma}_{22} = h^{\gamma}_{33} = \cdots = h^{\gamma}_{nn},$$

$$h^{*\gamma}_{11} + h^{*\gamma}_{22} = h^{*\gamma}_{33} = \cdots = h^{*\gamma}_{nn},$$

$$h^{\gamma}_{ij} = h^{*\gamma}_{ij} = 0, i \neq j,\ (i, j) \neq (1, 2), (2, 1),\ 1 \leq i < j \leq n.$$

Corollary 4.2. *Let M^n be a totally real statistical submanifold of a metallic-like statistical manifold N of dimension m. Then, we have the following*

$$(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \geq \frac{1}{4}\frac{(c_1+c_2)(n^2-n)}{p^2+4q}\left\{1+p^2 \mp 2\,np\sqrt{p^2+4q}\right\} - \frac{1}{4}(c_1+c_2) - \frac{c_1+c_2}{4(p^2+4q)}p^2 \pm \frac{1}{2}\frac{c_1-c_2}{\sqrt{p^2+4q}}p - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2+\|H^*\|^2] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0.$$

Moreover, the equalities holds for any $\gamma \in \{n+1,\ldots,m\}$ if and only if.

$$h^{\gamma}_{11} + h^{\gamma}_{22} = h^{\gamma}_{33} = \cdots = h^{\gamma}_{nn},$$

$$h^{*\gamma}_{11} + h^{*\gamma}_{22} = h^{*\gamma}_{33} = \cdots = h^{*\gamma}_{nn},$$

$$h^{\gamma}_{ij} = h^{*\gamma}_{ij} = 0, i \neq j,\ (i,j) \neq (1,2),(2,1),\ 1 \leq i < j \leq n.$$

Corollary 4.3. *Let M^n be a totally real statistical submanifold of of a metallic-like statistical manifold N of dimension m. If there exists a point $p \in M$ and $\pi \subset T_pM$ a plane such that*

$$\tau - K(\pi) < \tau_0 - K_0(\pi) + \frac{1}{4}\frac{(c_1+c_2)(n^2-n)}{p^2+4q}\left\{1+p^2 \mp 2\,np\sqrt{p^2+4q}\right\} - \frac{1}{4}(c_1+c_2) - \frac{c_1+c_2}{4(p^2+4q)}p^2 \pm \frac{1}{2}\frac{c_1-c_2}{\sqrt{p^2+4q}}p + 2(\hat{K}_0(\pi) - \hat{\tau}_0).$$

Then M is non-minimal, i.e., $H \neq 0$ or $H^* \neq 0$.

4.1. **Chen's $\delta(2,2)$ Inequality.** Let $p \in M$, $\pi_1, \pi_2 \subset T_pM$ be mutually orthogonal planes spanned by $\{e_1, e_2\}$ and $\{e_3, e_4\}$, respectively. Also, let $\{e_1, e_2, \cdots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$, respectively.

By doing simple calculations for $K(\pi_1)$ and $K(\pi_2)$ and using lemma 2.5, we can obtain the following inequality, which represents the Chen $\delta(2,2)$ inequality for statistical submanifold in a metallic-like statistical manifold.

Theorem 4.4. *Let M^n be a statistical submanifold in a metallic-like statistical manifold N of dimension n, then we have*

$$(\tau - K(\pi_1) - K(\pi_2) - (\tau_0 - K_0(\pi_1) - K_0(\pi_2)) \geq \frac{1}{4}\frac{(c_1+c_2)(n^2-n)}{p^2+4q}\Big\{1+p^2+\frac{4}{n^2-n}\{\mathrm{tr}^2(\vartheta) - \|\mathcal{T}\|^2 - \frac{4p}{n}\mathrm{tr}(\vartheta)\} \pm 2\sqrt{p^2+4q}(2\,\mathrm{tr}(\vartheta) - np)\Big\} - \frac{1}{2}(c_1+c_2) - \frac{c_1+c_2}{4(p^2+4q)}[4(\Psi(\pi_1)+\Psi(\pi_2)) - 4(\Theta^2(\pi_1)+\Theta^2(\pi_2)) + p^2 - 2p(\mathrm{tr}(\vartheta_{|\pi_1}) + \mathrm{tr}(\vartheta_{|\pi_2}))] \pm \frac{1}{2}\frac{c_2-c_1}{\sqrt{p^2+4q}}[\mathrm{tr}(\vartheta_{|\pi_1}) + \mathrm{tr}(\vartheta_{|\pi_2}) - 2p] - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2+\|H^*\|^2] - 2[\hat{\tau}_0 - \hat{K}_0(\pi_1) - \hat{K}_0(\pi_2)].$$

Moreover, the equalities holds for any $\gamma \in \{n+1,\ldots,m\}$ if and only if

$$h^{\gamma}_{11} + h^{\gamma}_{22} = h^{\gamma}_{33} + h^{\gamma}_{44} = h^{\gamma}_{55} \cdots = h^{\gamma}_{nn},$$

$$h^{*\gamma}_{11} + h^{*\gamma}_{22} = h^{*\gamma}_{33} + h^{*\gamma}_{44} = h^{*\gamma}_{55} \cdots = h^{*\gamma}_{nn},$$

$$h^{\gamma}_{ij} = h^{*\gamma}_{ij} = 0, i \neq j,\ (i,j) \neq (1,2),(2,1),(3,4),(4,3),\ 1 \leq i < j \leq n.$$

Corollary 4.5. *Let M^n be a totally real statistical submanifold in a metallic-like statistical manifold N of dimension n, then we have*

$$(\tau - K(\pi_1) - K(\pi_2) - (\tau_0 - K_0(\pi_1) - K_0(\pi_2)) \geq \frac{1}{4}\frac{(c_1+c_2)(n^2-n)}{p^2+4q}\Big\{1 + p^2 \mp 2\,np\sqrt{p^2+4q}\Big\}$$
$$- \frac{1}{2}(c_1+c_2) - \frac{c_1+c_2}{4(p^2+4q)}[p^2] \pm \frac{1}{2}\frac{c_1-c_2}{\sqrt{p^2+4q}}[2p] - \frac{n^2(n-2)}{4(n-1)}[\|H\|^2 + \|H^*\|^2]$$
$$- 2[\hat{\tau}_0 - \hat{K}_0(\pi_1) - \hat{K}_0(\pi_2)].$$

Moreover, the equalities holds for any $\gamma \in \{n+1,\ldots,m\}$ if and only if

$$h^{\gamma}_{11} + h^{\gamma}_{22} = h^{\gamma}_{33} + h^{\gamma}_{44} = h^{\gamma}_{55}\cdots = h^{\gamma}_{nn},$$
$$h^{*\gamma}_{11} + h^{*\gamma}_{22} = h^{*\gamma}_{33} + h^{*\gamma}_{44} = h^{*\gamma}_{55}\cdots = h^{*\gamma}_{nn},$$
$$h^{\gamma}_{ij} = h^{*\gamma}_{ij} = 0, i \neq j,\ (i,j) \neq (1,2),(2,1),(3,4),(4,3),\ 1 \leq i < j \leq n.$$

Corollary 4.6. *Let M^n be a totally real statistical submanifold of a metallic-like statistical manifold N of dimension m. If there exists a point $p \in M$ and $\pi_1, \pi_2 \subset T_pM$ mutually orthogonal planes such that*

$$\tau - K(\pi_1) - K(\pi_2) < \tau_0 - K_0(\pi_1) - K_0(\pi_2) + \frac{1}{4}\frac{(c_1+c_2)(n^2-n)}{p^2+4q}\Big\{1 + p^2 \mp 2\,np\sqrt{p^2+4q}\Big\}$$
$$- \frac{1}{2}(c_1+c_2) - \frac{c_1+c_2}{4(p^2+4q)}[p^2] \pm \frac{1}{2}\frac{c_1-c_2}{\sqrt{p^2+4q}}[2p] - 2[\hat{\tau}_0 - \hat{K}_0(\pi_1) - \hat{K}_0(\pi_2)].$$

Then M is non-minimal, i.e., $H \neq 0$ or $H^* \neq 0$.

Remark 4.7. Considering the family of metallic structures for all results obtained in the study, similar relationships can also be obtained for golden, silver, bronze, subtle, copper and nickel structures, as well.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

I have read and agreed to the published version of the manuscript.

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