Turk. J. Math. Comput. Sci.
15(1)(2023) 125–144
© MatDer
DOI : 10.47000/tjmcs.999069

Padovan, Perrin and Pell-Padovan Dual Quaternions

Zehra İşbilir[1], Nurten Gürses[2,*]

[1]Department of Mathematics, Faculty of Arts and Sciences, Düzce University, 81620, Düzce, Türkiye.
[2]Department of Mathematics, Faculty of Arts and Sciences, Yildiz Technical University, 34220, Istanbul, Türkiye.

Received: 22-09-2021 • Accepted: 27-02-2023

Abstract. In this present study, we intend to determine the Padovan, Perrin and Pell-Padovan dual quaternions with non-negative and negative subscripts. In line with this purpose, we construct some new properties such as; special determinant equalities, new recurrence relations, matrix formulas, Binet-like formulas, generating functions, exponential generating functions, summation formulas, and binomial properties for these special dual quaternions.

2010 AMS Classification: 11B37, 11K31, 11R52

Keywords: Padovan numbers, Perrin numbers, Pell-Padovan numbers, dual quaternions.

1. Introduction

Over the years, researchers have taken a strong interest in special numbers. Many studies have been done and are ongoing on the special numbers (or sequences), which are different orders such as second-order (Fibonacci, Jacobsthal, Pell, Horadam and etc.), third-order (Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Pell-Padovan and etc.), and the other higher-order recurrence sequences (Tetranacci, Pentanacci, and etc.).

The Padovan sequence (A000931 in [33] and see [4, 25, 26, 29–31, 39, 40]), the Perrin sequence (A001608 and A078712 in [33] and see [22, 27, 29–31]) and the Pell-Padovan sequence (A066983 in [33], see [1, 2, 7–9, 30–32, 37]) are the sequences of integers. They are the third-order linear recurrence sequences and defined recursively, as follows, respectively:

$$P_{n+3} = P_n + P_{n+1}, \quad P_0 = P_1 = P_2 = 1, \quad \forall n \in \mathbb{N}, \tag{1.1}$$
$$R_{n+3} = R_n + R_{n+1}, \quad R_0 = 3, R_1 = 0, R_2 = 2, \quad \forall n \in \mathbb{N}, \tag{1.2}$$
$$T_{n+3} = 2T_{n+1} + T_n, \quad T_0 = T_1 = T_2 = 1, \quad \forall n \in \mathbb{N}. \tag{1.3}$$

The initial values of the Padovan numbers can be given as $P_0 = 0, P_1 = 0, P_2 = 1$, $\forall n \in \mathbb{N}$ in some studies. If Padovan, Perrin and Pell-Padovan numbers are extended to negative subscripts, the following recurrence relations are given $\forall n \in \mathbb{Z}^+$:

$$P_{-n} = P_{-(n-3)} - P_{-(n-1)},$$
$$R_{-n} = R_{-(n-3)} - R_{-(n-1)},$$
$$T_{-n} = T_{-(n-3)} - 2T_{-(n-1)}.$$

*Corresponding Author
Email addresses: zehraisbilir@duzce.edu.tr (Z. İşbilir), nbayrak@yildiz.edu.tr (N. Gürses)

It should be noted that, the recurrence relations (1.1), (1.2), (1.3) are valid $\forall n \in \mathbb{Z}$. This situation can be checked from Table 1, which includes several values of the Padovan, Perrin and Pell-Padovan numbers.

Table 1. Some values of the Padovan, Perrin and Pell-Padovan numbers

n	...	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	...
P_n	...	−1	1	0	0	1	0	1	1	1	2	2	3	4	...
R_n	...	−2	4	−3	2	1	−1	3	0	2	3	2	5	5	...
T_n	...	25	−15	9	−5	3	−1	1	1	1	3	3	7	9	...

Matrix sequences of Padovan and Perrin numbers are examined in the studies [45–48]. Also, matrix representations of the Padovan numbers (see [20, 28, 34–36, 44]) and Perrin numbers (see [19, 20, 24, 34]) are studied.

On the other hand, the famous mathematician W. R. Hamilton investigated quaternions in 1843, [15]. Quaternion algebra is associative and non-commutative 4-dimensional Clifford algebra. The set of all real quaternions is represented by $\mathcal{H} = \{q = q_0 + iq_1 + jq_2 + kq_3; q_0, q_1, q_2, q_3 \in \mathbb{R}\}$, where i, j, k are quaternionic units which satisfy the following rules:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Dual quaternions are examined by Majernik [23] and determined by the set:

$$\mathcal{H}_{\mathcal{D}} = \{q = q_0 + iq_1 + jq_2 + kq_3;\ q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}\},$$

where i, j, k are quaternionic units which satisfy the following rules ([12, 23]):

$$i^2 = j^2 = k^2 = ijk = 0, \quad ij = ji = jk = kj = ki = ik = 0. \tag{1.4}$$

By means of dual quaternions, one can express the Galilean transformation in one quaternionic equation, [23]. Quaternions with special number components are examined in many research: Fibonacci quaternions [16], Pell quaternions and Pell-Lucas quaternions [6], Jacobsthal quaternions [41], the Horadam quaternions [14], generalized Tribonacci quaternions [5], Padovan, Perrin and Pell-Padovan quaternions [5, 10, 11, 13, 42], generalized dual Fibonacci quaternions [49], the dual third order Jacobsthal quaternions and dual third order Jacobsthal-Lucas quaternions [3].

In this paper, we investigate the Padovan, Perrin and Pell-Padovan dual quaternions with non-negative and negative subscripts. This study is organized into 4 sections; in the first two sections, we remark on some of the definitions and properties concerning the Padovan, Perrin and Pell-Padovan numbers and also the dual quaternions. In Section 3, we introduce the Padovan and Perrin dual quaternions, and then we construct new special properties for them. Moreover, we give special determinant equalities, recurrence relations, matrix formulations, Binet-like formulas, generating functions, exponential generating functions, summation formulas, and also binomial properties. As for Section 4, we examine the Pell-Padovan dual quaternions and the similar concepts relating to them. In Appendix A, we give some necessary information and notations regarding the Padovan, Perrin and Pell-Padovan numbers.

2. Preliminaries

The characteristic equation of the Padovan and Perrin numbers is $x^3 - x - 1 = 0$ with roots r_1, r_2, r_3, where

$$r_1 = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}},$$

$$r_2 = -\sqrt[3]{\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}\right),$$

$$r_3 = -\sqrt[3]{\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}\right),$$

and $r_1 + r_2 + r_3 = 0$, $r_1 r_2 r_3 = 1$, $r_1 r_2 + r_1 r_3 + r_2 r_3 = -1$. The ratio of two consecutive Padovan or Perrin numbers converges to the value $r_1 \approx 1.3247...$ which is named *plastic ratio*[1] [26]: $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \lim_{n\to\infty} \frac{R_{n+1}}{R_n} \approx 1.3247...$ According to these, $\forall n \in \mathbb{Z}$, Binet-like formulas for Padovan and Perrin numbers are as follows:

$$P_n = ar_1^n + br_2^n + cr_3^n, \qquad (2.1)$$
$$R_n = r_1^n + r_2^n + r_3^n,$$

where $a = \dfrac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}$, $b = \dfrac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}$, $c = \dfrac{(r_1-1)(r_2-1)}{(r_3-r_1)(r_3-r_2)}$, [45,47,48].

The characteristic equation of the Pell-Padovan numbers is $x^3 - 2x - 1 = 0$. The roots of this equation are $\widetilde{r}_1 = (1+\sqrt{5})/2$, $\widetilde{r}_2 = (1-\sqrt{5})/2$, $\widetilde{r}_3 = -1$, where $\widetilde{r}_1 + \widetilde{r}_2 + \widetilde{r}_3 = 0$, $\widetilde{r}_1\widetilde{r}_2 + \widetilde{r}_1\widetilde{r}_3 + \widetilde{r}_2\widetilde{r}_3 = -2$, $\widetilde{r}_1\widetilde{r}_2\widetilde{r}_3 = 1$. Thereafter, $\forall n \in \mathbb{Z}$, the Binet-like formula of the Pell-Padovan numbers is given by:

$$T_n = w_1\widetilde{r}_1^n + w_2\widetilde{r}_2^n - \widetilde{r}_3^n, \qquad (2.2)$$

where $w_1 = (\sqrt{5}-1)/\sqrt{5}$, $w_2 = (\sqrt{5}+1)/\sqrt{5}$, [37].

Additionally, the dual quaternion $q \in \mathcal{H}_{\mathcal{D}}$ is written as $q = S_q + \overrightarrow{V}_q$, where $S_q = q_0$ is called the scalar part and $\overrightarrow{V}_q = iq_1 + jq_2 + kq_3$ is called the vector part. Also for $q, p \in \mathcal{H}_{\mathcal{D}}$, the algebraic operations are given as:

- $q = p \Leftrightarrow q_0 = p_0,\ q_1 = p_1,\ q_2 = p_2,\ q_3 = p_3,$
- $q \pm p = q_0 \pm p_0 + i(q_1 \pm p_1) + j(q_2 \pm p_2) + k(q_3 \pm p_3),$
- $\lambda q = \lambda q_0 + i\lambda q_1 + j\lambda q_2 + k\lambda q_3,\ \lambda \in \mathbb{R},$
- $qp = q_0p_0 + i(q_0p_1 + q_1p_0) + j(q_0p_2 + q_2p_0) + k(q_0p_3 + q_3p_0),$

whereby the rules (1.4) in the multiplication of any two dual quaternions. Thus, this implies that $S_{q\pm p} = S_q \pm S_p$ and $\overrightarrow{V}_{q\pm p} = \overrightarrow{V}_q \pm \overrightarrow{V}_p$. The conjugate of q is $\overline{q} = q_0 - iq_1 - jq_2 - kq_3$. Then, $\overline{q} = S_q - \overrightarrow{V}_q$. Also, the norm q is: $N_q = q\overline{q} = \overline{q}q = q_0^2$.

3. Padovan and Perrin Dual Quaternions

In this section, we investigate the Padovan and Perrin dual quaternions with non-negative and negative subscripts. Also, we obtain various features and relations associated with them.

Definition 3.1. $\forall n \in \mathbb{N}$, the nth Padovan and the nth Perrin dual quaternions are defined as follows:

$$\widehat{P}_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}, \qquad (3.1)$$

$$\widehat{R}_n = R_n + iR_{n+1} + jR_{n+2} + kR_{n+3}, \qquad (3.2)$$

respectively. Besides, $\forall n \in \mathbb{Z}^+$, the $-n$th Padovan and the $-n$th Perrin dual quaternions with negative subscripts are given by:

$$\widehat{P}_{-n} = P_{-n} + iP_{-n+1} + jP_{-n+2} + kP_{-n+3},$$
$$\widehat{R}_{-n} = R_{-n} + iR_{-n+1} + jR_{-n+2} + kR_{-n+3},$$

respectively. Here, P_n and R_n are the nth Padovan and nth Perrin numbers, P_{-n} and R_{-n} are the $-n$th Padovan and $-n$th Perrin numbers with negative subscripts, and i, j, k are quaternionic units that satisfy the rules (1.4). The set of all Padovan and Perrin dual quaternions are represented with $\widehat{\mathcal{P}}_{\mathcal{D}}$ and $\widehat{\mathcal{R}}_{\mathcal{D}}$, respectively.

$\forall n \in \mathbb{Z}$, several values of the Padovan and Perrin dual quaternions can be seen in Table 2.

[1]The plastic ratio was originally worked on by Gérard Cordonnier in 1924.

TABLE 2. Some values of $\widehat{P}_n$ and $\widehat{R}_n$ for $n \in \mathbb{Z}$

n	$\widehat{P}_n$	$\widehat{R}_n$
$\vdots$	$\vdots$	$\vdots$
-3	$i + k$	$2 + i - j + k3$
-2	$1 + j + k$	$1 - i + j3$
-1	$i + j + k$	$-1 + i3 + k2$
0	$1 + i + j + k2$	$3 + j2 + k3$
1	$1 + i + j2 + k2$	$i2 + j3 + k2$
2	$1 + i2 + j2 + k3$	$2 + i3 + j2 + k5$
3	$2 + i2 + j3 + k4$	$3 + i2 + j5 + k5$
$\vdots$	$\vdots$	$\vdots$

Throughout this study,

- the examined properties of the Padovan dual quaternions are also valid for Perrin dual quaternions in some parts, so we omit them. However, we give Binet-like formulas, generating functions, exponential generating functions separately.
- some proofs are omitted in the theorems which include more than one property since they can be obtained like the others.

Let consider $\widehat{P}_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$, $\widehat{P}_m = P_m + iP_{m+1} + jP_{m+2} + kP_{m+3} \in \widehat{\mathcal{P}}_{\mathcal{D}}$, $\forall n \in \mathbb{Z}$. Then, P_n is called scalar and $iP_{n+1} + jP_{n+2} + kP_{n+3}$ is called vector parts of $\widehat{P}_n$. The conjugate of $\widehat{P}_n$ is given by:

$$\overline{\widehat{P}}_n = P_n - iP_{n+1} - jP_{n+2} - kP_{n+3}. \tag{3.3}$$

Addition (also subtraction) of $\widehat{P}_n$ and $\widehat{P}_m$ is $\widehat{P}_n \pm \widehat{P}_m = P_n \pm P_m + i(P_{n+1} \pm P_{m+1}) + j(P_{n+2} \pm P_{m+2}) + k(P_{n+3} \pm P_{m+3})$. Multiplication by a scalar is as follows: $\lambda\widehat{P}_n = \lambda P_n + i\lambda P_{n+1} + j\lambda P_{n+2} + k\lambda P_{n+3}$, $\lambda \in \mathbb{R}$. Also, multiplication of $\widehat{P}_n$ and $\widehat{P}_m$ is given by using equation (1.4):

$$\widehat{P}_n\widehat{P}_m = P_nP_m + i\,(P_nP_{m+1} + P_{n+1}P_m) + j\,(P_nP_{m+2} + P_{n+2}P_m) + k\,(P_nP_{m+3} + P_{n+3}P_m)\,. \tag{3.4}$$

Theorem 3.2. *The recurrence relations for Padovan and Perrin dual quaternions with non-negative and negative subscripts are given as follows:*

$$\widehat{P}_{n+3} = \widehat{P}_n + \widehat{P}_{n+1}, \quad \forall n \in \mathbb{N}, \tag{3.5}$$

$$\widehat{R}_{n+3} = \widehat{R}_n + \widehat{R}_{n+1}, \quad \forall n \in \mathbb{N}, \tag{3.6}$$

$$\widehat{P}_{-n} = \widehat{P}_{-(n-3)} - \widehat{P}_{-(n-1)}, \quad \forall n \in \mathbb{Z}^+, \tag{3.7}$$

$$\widehat{R}_{-n} = \widehat{R}_{-(n-3)} - \widehat{R}_{-(n-1)}, \quad \forall n \in \mathbb{Z}^+.$$

Proof. Using equations (1.1) and (3.1), we have:

$$\begin{aligned}\widehat{P}_n + \widehat{P}_{n+1} &= P_n + P_{n+1} + i\,(P_{n+1} + P_{n+2}) + j\,(P_{n+2} + P_{n+3}) + k(P_{n+3} + P_{n+4})\\ &= P_{n+3} + iP_{n+4} + jP_{n+5} + kP_{n+6}\\ &= \widehat{P}_{n+3}.\end{aligned}$$

The others can be obtained similarly. □

It should be noted that, equations (3.1), (3.2), (3.5) and (3.6) are valid $\forall n \in \mathbb{Z}$.

Inspired by the study [21], we find a way to compute nth and $-(n+1)$th term of Padovan dual quaternions in the following Theorem 3.3. The proof is clear using equations (3.5) (for part (i)) and (3.7) (for part (ii)) via the study [21].

Theorem 3.3. $\forall n \in \mathbb{N}$, *the following determinant equalities can be given.*

(i) $$\widehat{P}_n = \begin{vmatrix} \widehat{P}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widehat{P}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widehat{P}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \end{vmatrix}_{(n+1)\times(n+1)},$$

(ii) $$\widehat{P}_{-(n+1)} = \begin{vmatrix} \widehat{P}_{-1} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widehat{P}_{-2} & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widehat{P}_{-3} & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

Example 3.4. Using the above theorem, the 5th term of the Padovan dual quaternion can be obtained as:

$$\begin{vmatrix} \widehat{P}_0 & -1 & 0 & 0 & 0 & 0 \\ \widehat{P}_1 & 0 & -1 & 0 & 0 & 0 \\ \widehat{P}_2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}_{6\times 6} = 3 + i4 + j5 + k7 = \widehat{P}_5.$$

Theorem 3.5. $\forall n \in \mathbb{Z}$, *the following properties are satisfied:*

(i) $\widehat{P}_n - i\widehat{P}_{n+1} - j\widehat{P}_{n+2} - k\widehat{P}_{n+3} = P_n$,
(ii) $\widehat{P}_n + i\widehat{P}_{n+1} + j\widehat{P}_{n+2} + k\widehat{P}_{n+3} = 2\widehat{P}_n - P_n$,
(iii) $\widehat{P}_n\overline{\overline{\widehat{P}}}_n = P_n^2$,
(iv) $\widehat{P}_n + \overline{\overline{\widehat{P}}}_n = 2P_n$,
(v) $\widehat{P}_n - \overline{\overline{\widehat{P}}}_n = 2\widehat{P}_n - 2P_n$,
(vi) $\widehat{P}_n^2 = P_n^2 + i2P_nP_{n+1} + j2P_nP_{n+2} + k2P_nP_{n+3} = 2P_n\widehat{P}_n - P_n^2$,
(vii) $\overline{\overline{\widehat{P}}}_n^{\,2} = P_n^2 - i2P_nP_{n+1} - j2P_nP_{n+2} - k2P_nP_{n+3} = 3P_n^2 - 2P_n\widehat{P}_n$.

Proof. (i) Using equations (1.4) and (3.1), we have:

$$\begin{aligned} \widehat{P}_n - i\widehat{P}_{n+1} - j\widehat{P}_{n+2} - k\widehat{P}_{n+3} =& P_n + iP_{n+1} + jP_{n+2} + kP_{n+3} - i(P_{n+1} + iP_{n+2} + jP_{n+3} + kP_{n+4}) \\ & - j(P_{n+2} + iP_{n+3} + jP_{n+4} + kP_{n+5}) - k(P_{n+3} + iP_{n+4} + jP_{n+5} + kP_{n+6}) \\ =& P_n. \end{aligned}$$

(ii) Using equations (1.4) and (3.1), we get:

$$\begin{aligned} \widehat{P}_n + i\widehat{P}_{n+1} + j\widehat{P}_{n+2} + k\widehat{P}_{n+3} =& P_n + iP_{n+1} + jP_{n+2} + kP_{n+3} + i(P_{n+1} + iP_{n+2} + jP_{n+3} + kP_{n+4}) \\ & + j(P_{n+2} + iP_{n+3} + jP_{n+4} + kP_{n+5}) + k(P_{n+3} + iP_{n+4} + jP_{n+5} + kP_{n+6}) \\ =& 2\widehat{P}_n - P_n. \end{aligned}$$

(iii) Considering equations (1.4), (3.1) and (3.3), we obtain:

$$\widehat{P}_n\overline{\overline{\widehat{P}}}_n = (P_n + iP_{n+1} + jP_{n+2} + kP_{n+3})(P_n - iP_{n+1} - jP_{n+2} - kP_{n+3}) = P_n^2.$$

□

Theorem 3.6. *$\forall n \in \mathbb{Z}$, the following properties are satisfied:*

(i) $\widehat{P}_n\widehat{P}_m - \overline{\overline{\widehat{P}}}_n\overline{\overline{\widehat{P}}}_m = 2(P_n\widehat{P}_m + P_m\widehat{P}_n - 2P_nP_m)$,

(ii) $\widehat{P}_n\widehat{P}_m + \overline{\overline{\widehat{P}}}_n\overline{\overline{\widehat{P}}}_m = 2P_nP_m$,

(iii) $\widehat{P}_n\overline{\widehat{P}}_m - \overline{\widehat{P}}_n\widehat{P}_m = 2(P_m\widehat{P}_n - P_n\widehat{P}_m)$,

(iv) $\widehat{P}_n\overline{\widehat{P}}_m + \overline{\widehat{P}}_n\widehat{P}_m = 2P_nP_m$.

Proof. (i) Considering equations (1.4), (3.1), (3.3) and (3.4), we have:

$$\begin{aligned}\widehat{P}_n\widehat{P}_m - \overline{\overline{\widehat{P}}}_n\overline{\overline{\widehat{P}}}_m =& (P_n + iP_{n+1} + jP_{n+2} + kP_{n+3})(P_m + iP_{m+1} + jP_{m+2} + kP_{m+3}) \\ & - (P_n - iP_{n+1} - jP_{n+2} - kP_{n+3})(P_m - iP_{m+1} - jP_{m+2} - kP_{m+3}) \\ =& 2\left[P_n(iP_{m+1} + jP_{m+2} + kP_{m+3}) + P_m(iP_{n+1} + jP_{n+2} + kP_{n+3})\right] \\ =& 2\left(P_n\widehat{P}_m + P_m\widehat{P}_n - 2P_nP_m\right).\end{aligned}$$

(ii) Using equations (1.4), (3.1), (3.3) and (3.4), we obtain:

$$\begin{aligned}\widehat{P}_n\widehat{P}_m + \overline{\overline{\widehat{P}}}_n\overline{\overline{\widehat{P}}}_m =& (P_n + iP_{n+1} + jP_{n+2} + kP_{n+3})(P_m + iP_{m+1} + jP_{m+2} + kP_{m+3}) \\ & + (P_n - iP_{n+1} - jP_{n+2} - kP_{n+3})(P_m - iP_{m+1} - jP_{m+2} - kP_{m+3}) \\ =& 2P_nP_m.\end{aligned}$$

□

Thanks to the study [4], we obtain some recurrence relations for Padovan dual quaternions:

Theorem 3.7. *$\forall n \in \mathbb{Z}$, the recurrence relation*

$$\widehat{P}_n = \rho_a\widehat{P}_{n-a} + \sigma_a\widehat{P}_{n-2a} + \widehat{P}_{n-3a}$$

is obtained with (ρ_a, σ_a) such that $\rho_a, \sigma_a \in \mathbb{Z}$, $1 \leq a \leq 8$, $a \in \mathbb{N}$ (see in Table 3).

TABLE 3. Some recurrence relations for Padovan dual quaternions

a	(ρ_a, σ_a)	$\widehat{P}_n = \rho_a\widehat{P}_{n-a} + \sigma_a\widehat{P}_{n-2a} + \widehat{P}_{n-3a}$
1	$(0, 1)$	$\widehat{P}_n = \widehat{P}_{n-2} + \widehat{P}_{n-3}$
2	$(2, -1)$	$\widehat{P}_n = 2\widehat{P}_{n-2} - \widehat{P}_{n-4} + \widehat{P}_{n-6}$
3	$(3, -2)$	$\widehat{P}_n = 3\widehat{P}_{n-3} - 2\widehat{P}_{n-6} + \widehat{P}_{n-9}$
4	$(2, 3)$	$\widehat{P}_n = 2\widehat{P}_{n-4} + 3\widehat{P}_{n-8} + \widehat{P}_{n-12}$
5	$(5, -4)$	$\widehat{P}_n = 5\widehat{P}_{n-5} - 4\widehat{P}_{n-10} + \widehat{P}_{n-15}$
6	$(5, 2)$	$\widehat{P}_n = 5\widehat{P}_{n-6} + 2\widehat{P}_{n-12} + \widehat{P}_{n-18}$
7	$(7, 1)$	$\widehat{P}_n = 7\widehat{P}_{n-7} + \widehat{P}_{n-14} + \widehat{P}_{n-21}$
8	$(10, -5)$	$\widehat{P}_n = 10\widehat{P}_{n-8} - 5\widehat{P}_{n-16} + \widehat{P}_{n-24}$

Proof. For $a = 4$, using equation (3.1) and the fourth row of Table 6, we obtain:

$$\begin{aligned}2\widehat{P}_{n-4} + 3\widehat{P}_{n-8} + \widehat{P}_{n-12} =& 2P_{n-4} + 3P_{n-8} + P_{n-12} + i(2P_{n-3} + 3P_{n-7} + P_{n-11}) \\ & + j(2P_{n-2} + 3P_{n-6} + P_{n-10}) + k(2P_{n-1} + 3P_{n-5} + P_{n-9}) \\ =& P_n + iP_{n+1} + jP_{n+2} + kP_{n+3} \\ =& \widehat{P}_n.\end{aligned}$$

Considering the other values a, the proof is conducted by the same way. Hence, this concludes the proof. □

Theorem 3.8. *$\forall n \in \mathbb{Z}$, the relations between Padovan and Perrin dual quaternions hold:*

(i) $\widehat{R}_n = 3\widehat{P}_{n-5} + 2\widehat{P}_{n-4}$,
(ii) $\widehat{R}_n = \widehat{P}_{n-5} + 2\widehat{P}_{n-2}$,
(iii) $\widehat{P}_{n-1} = \frac{1}{23}\left(\widehat{R}_{n-3} + 8\widehat{R}_{n-2} + 10\widehat{R}_{n-1}\right)$.

Proof. The proofs are clear, so we omit them. □

Theorem 3.9. *$\forall m, n \in \mathbb{N}$, the following relations can be given:*

(i) $\widehat{P}_{m-3}\widehat{P}_{n-3} + \widehat{P}_{m-1}\widehat{P}_{n-2} + \widehat{P}_{m-2}\widehat{P}_{n-1} = 2\widehat{P}_{m+n-1} - P_{m+n-1}$,
(ii) $\widehat{P}_{m-3}\widehat{R}_{n-3} + \widehat{P}_{m-1}\widehat{R}_{n-2} + \widehat{P}_{m-2}\widehat{R}_{n-1} = 2\widehat{R}_{m+n-1} - R_{m+n-1}$,
(iii) $\widehat{R}_{m-3}\widehat{R}_{n-3} + \widehat{R}_{m-1}\widehat{R}_{n-2} + \widehat{R}_{m-2}\widehat{R}_{n-1} = 8\widehat{P}_{m+n-5} + 8\widehat{P}_{m+n-8} + 2\widehat{P}_{m+n-11} - 4P_{m+n-5} - 4P_{m+n-8} - P_{m+n-10}$,
(iv) $\widehat{R}_{m-3}\widehat{R}_{n-3} + \widehat{R}_{m-1}\widehat{R}_{n-2} + \widehat{R}_{m-2}\widehat{R}_{n-1} = 4\widehat{R}_{m+n-3} + 2\widehat{R}_{m+n-6} - 2R_{m+n-3} - R_{m+n-6}$.

Proof. (i) Using equations (1.4), (A.4) and (3.1), we have:

$$\begin{aligned}\widehat{P}_{m-3}\widehat{P}_{n-3} + \widehat{P}_{m-1}\widehat{P}_{n-2} + \widehat{P}_{m-2}\widehat{P}_{n-1} =& P_{m-3}P_{n-3} + P_{m-1}P_{n-2} + P_{m-2}P_{n-1} \\ &+ i\left(P_{m-2}P_{n-3} + P_{m-3}P_{n-2} + P_{m-1}P_{n-1} + P_mP_{n-2} + P_{m-2}P_n + P_{m-1}P_{n-1}\right) \\ &+ j\left(P_{m-1}P_{n-3} + P_{m-3}P_{n-1} + P_{m-1}P_n + P_{m+1}P_{n-2} + P_{m-2}P_{n+1} + P_mP_{n-1}\right) \\ &+ k\left(P_mP_{n-3} + P_{m-3}P_n + P_{m-1}P_{n+1} + P_{m+2}P_{n-2} + P_{m-2}P_{n+2} + P_{m+1}P_{n-1}\right) \\ =& \widehat{P}_{m+n-1} + iP_{m+n} + jP_{m+n+1} + kP_{m+n+2} \\ =& 2\widehat{P}_{m+n-1} - P_{m+n-1}.\end{aligned}$$

The other parts can be seen using equation (A.4). □

Theorem 3.10. *$\forall m, n \in \mathbb{N}$, the following relations are satisfied:*

(i) $\widehat{P}_{m-3}\widehat{P}_{-n-3} + \widehat{P}_{m-1}\widehat{P}_{-n-2} + \widehat{P}_{m-2}\widehat{P}_{-n-1} = 2\widehat{P}_{m-n-1} - P_{m-n-1}$,
(ii) $\widehat{P}_{m-3}\widehat{R}_{-n-3} + \widehat{P}_{m-1}\widehat{R}_{-n-2} + \widehat{P}_{m-2}\widehat{R}_{-n-1} = 2\widehat{R}_{m-n-1} - R_{m-n-1}$,
(iii) $\widehat{R}_{m-3}\widehat{R}_{-n-3} + \widehat{R}_{m-1}\widehat{R}_{-n-2} + \widehat{R}_{m-2}\widehat{R}_{-n-1} = 4\widehat{R}_{m-n-3} + 2\widehat{R}_{m-n-6} - 2R_{m-n-3} - R_{m-n-6}$,
(iv) $\widehat{P}_{-m-3}\widehat{P}_{-n-3} + \widehat{P}_{-m-1}\widehat{P}_{-n-2} + \widehat{P}_{-m-2}\widehat{P}_{-n-1} = 2\widehat{P}_{-m-n-1} - P_{-m-n-1}$,
(v) $\widehat{P}_{-m-3}\widehat{R}_{-n-3} + \widehat{P}_{-m-1}\widehat{R}_{-n-2} + \widehat{P}_{-m-2}\widehat{R}_{-n-1} = 2\widehat{R}_{-m-n-1} - R_{-m-n-1}$,
(vi) $\widehat{R}_{-m-3}\widehat{R}_{-n-3} + \widehat{R}_{-m-1}\widehat{R}_{-n-2} + \widehat{R}_{-m-2}\widehat{R}_{-n-1} = 8\widehat{P}_{-m-n-5} + 8\widehat{P}_{-m-n-8} + 2\widehat{P}_{-m-n-11} - 4P_{-m-n-5} - 4P_{-m-n-8} - P_{-m-n-10}$,
(vii) $\widehat{R}_{-m-3}\widehat{R}_{-n-3} + \widehat{R}_{-m-1}\widehat{R}_{-n-2} + \widehat{R}_{-m-2}\widehat{R}_{-n-1} = 4\widehat{R}_{-m-n-3} + 2\widehat{R}_{-m-n-6} - 2R_{-m-n-3} - R_{-m-n-6}$.

Proof. The proofs are straightforward to obtain by using Definition 3.1 and equations (1.4), (A.4). □

Theorem 3.11. *$\forall m, n \in \mathbb{Z}^+$ such that $m < n$, the following relations are given:*

(i) $\widehat{P}_{m-1}\widehat{P}_{n-m} + \widehat{P}_{m+1}\widehat{P}_{n-m+1} + \widehat{P}_m\widehat{P}_{n-m+2} = 2\widehat{P}_n - P_n$,
(ii) $\widehat{P}_{m-1}\widehat{R}_{n-m} + \widehat{P}_{m+1}\widehat{R}_{n-m+1} + \widehat{P}_m\widehat{R}_{n-m+2} = 2\widehat{R}_n - R_n$.

Proof. Definition 3.1 and equations (1.4), (A.6) can be used for the proof. □

Theorem 3.12. *$\forall m, n \in \mathbb{Z}^+$ such that $m \le n$, we have the following relations:*

(i) $\left.\begin{array}{l} \star\ \widehat{P}_{2m-1}\widehat{P}_{2(n-m)} + \widehat{P}_{2m}\widehat{P}_{2(n-m)+2} + \widehat{P}_{2m+1}\widehat{P}_{2(n-m)+1} \\ *\ \widehat{P}_{2m}\widehat{P}_{2(n-m)-1} + \widehat{P}_{2m+1}\widehat{P}_{2(n-m)+1} + \widehat{P}_{2m+2}\widehat{P}_{2(n-m)} \end{array}\right\} = 2\widehat{P}_{2n} - P_{2n}$,

(ii) $\left.\begin{array}{l} \star\ \widehat{P}_{2m-1}\widehat{P}_{2(n-m)+1} + \widehat{P}_{2m}\widehat{P}_{2(n-m)+3} + \widehat{P}_{2m+1}\widehat{P}_{2(n-m)+2} \\ *\ \widehat{P}_{2m}\widehat{P}_{2(n-m)} + \widehat{P}_{2m+1}\widehat{P}_{2(n-m)+2} + \widehat{P}_{2m+2}\widehat{P}_{2(n-m)+1} \end{array}\right\} = 2\widehat{P}_{2n+1} - P_{2n+1}$.

Proof. The proofs are obvious from Definition 3.1 and equations (1.4), (A.7). Alternatively, Theorem 3.11 can be used. □

Theorem 3.13. *$\forall m, n \in \mathbb{Z}^+$ such that $m < n$, the following relations can be given:*

(i) $\widehat{P}_{2m-1}\widehat{R}_{2(n-m)} + \widehat{P}_{2m}\widehat{R}_{2(n-m)+2} + \widehat{P}_{2m+1}\widehat{R}_{2(n-m)+1} = 2\widehat{R}_{2n} - R_{2n}$,

(ii) $\left.\begin{array}{l} \star\ \widehat{P}_{2m-1}\widehat{R}_{2(n-m)+1} + \widehat{P}_{2m}\widehat{R}_{2(n-m)+3} + \widehat{P}_{2m+1}\widehat{R}_{2(n-m)+2} \\ *\ \widehat{P}_{2m}\widehat{R}_{2(n-m)} + \widehat{P}_{2m+1}\widehat{R}_{2(n-m)+2} + \widehat{P}_{2m+2}\widehat{R}_{2(n-m)+1} \end{array}\right\} = 2\widehat{R}_{2n+1} - R_{2n+1}$,

(iii) $\widehat{P}_{2m}\widehat{R}_{2(n-m)+1} + \widehat{P}_{2m+1}\widehat{R}_{2(n-m)+3} + \widehat{P}_{2m+2}\widehat{R}_{2(n-m)+2} = 2\widehat{R}_{2n+2} - R_{2n+2}$,
(iv) $\widehat{P}_{2m}\widehat{P}_{2(n-m)+1} + \widehat{P}_{2m+1}\widehat{P}_{2(n-m)+3} + \widehat{P}_{2m+2}\widehat{P}_{2(n-m)+2} = 2\widehat{P}_{2n+2} - P_{2n+2}$,
(v) $\widehat{P}_{m-1}\widehat{P}_{n-m+1} + \widehat{P}_{m}\widehat{P}_{n-m+3} + \widehat{P}_{m+1}\widehat{P}_{n-m+2} = 2\widehat{P}_{n+1} - P_{n+1}$,
(vi) $\widehat{P}_{m-1}\widehat{R}_{n-m+1} + \widehat{P}_{m}\widehat{R}_{n-m+3} + \widehat{P}_{m+1}\widehat{R}_{n-m+2} = 2\widehat{R}_{n+1} - R_{n+1}$,
(vii) $\widehat{P}_{m}\widehat{P}_{n-m+1} + \widehat{P}_{m+1}\widehat{P}_{n-m+3} + \widehat{P}_{m+2}\widehat{P}_{n-m+2} = 2\widehat{P}_{n+2} - P_{n+2}$,
(viii) $\widehat{P}_{m}\widehat{R}_{n-m+1} + \widehat{P}_{m+1}\widehat{R}_{n-m+3} + \widehat{P}_{m+2}\widehat{R}_{n-m+2} = 2\widehat{R}_{n+2} - R_{n+2}$.

Proof. Using Definition 3.1 and equations (1.4), (A.8), the proofs are clear. Also, Theorem 3.11 can be used. □

Theorem 3.14. $\forall n \in \mathbb{Z}^+$, *the followings are obtained:*

(i) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{P}_1 \\ \widehat{P}_0 \\ \widehat{P}_2 \end{pmatrix} = \begin{pmatrix} \widehat{P}_{n+1} \\ \widehat{P}_n \\ \widehat{P}_{n+2} \end{pmatrix}$,

(ii) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{P}_1 & \widehat{P}_0 & \widehat{P}_{-1} \\ \widehat{P}_2 & \widehat{P}_1 & \widehat{P}_0 \\ \widehat{P}_0 & \widehat{P}_{-1} & \widehat{P}_{-2} \end{pmatrix} = \begin{pmatrix} \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} \\ \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n \\ \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} \end{pmatrix}$,

(iii) $\begin{pmatrix} \widehat{P}_0 & \widehat{P}_1 & \widehat{P}_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{P}_{n+1} & \widehat{P}_{n+2} & \widehat{P}_n \end{pmatrix}$,

(iv) $\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{P}_{-2} \\ \widehat{P}_{-1} \\ \widehat{P}_0 \end{pmatrix} = \begin{pmatrix} \widehat{P}_{-n-2} \\ \widehat{P}_{-n-1} \\ \widehat{P}_{-n} \end{pmatrix}$,

(v) $\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{P}_0 & \widehat{P}_{-1} & \widehat{P}_{-2} \\ \widehat{P}_1 & \widehat{P}_0 & \widehat{P}_{-1} \\ \widehat{P}_2 & \widehat{P}_1 & \widehat{P}_0 \end{pmatrix} = \begin{pmatrix} \widehat{P}_{-n} & \widehat{P}_{-n-1} & \widehat{P}_{-n-2} \\ \widehat{P}_{-n+1} & \widehat{P}_{-n} & \widehat{P}_{-n-1} \\ \widehat{P}_{-n+2} & \widehat{P}_{-n+1} & \widehat{P}_{-n} \end{pmatrix}$,

(vi) $\begin{pmatrix} \widehat{P}_{-2} & \widehat{P}_{-1} & \widehat{P}_0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{P}_{-n-2} & \widehat{P}_{-n-2} & \widehat{P}_{-n} \end{pmatrix}$.

Proof. The proofs are clear through mathematical induction. □

For the construction of the following formulas and binomial properties for Padovan and Perrin dual quaternions, we utilize the studies [45–48].

Theorem 3.15. $\forall n \in \mathbb{Z}$, *the* ***Binet-like formulas*** *for Padovan and Perrin dual quaternions can be given as:*

$$\widehat{P}_n = \underline{a}ar_1^n + \underline{b}br_2^n + \underline{c}cr_3^n, \tag{3.8}$$
$$\widehat{R}_n = \underline{a}r_1^n + \underline{b}r_2^n + \underline{c}r_3^n, \tag{3.9}$$

where $\underline{a} = 1 + ir_1 + jr_1^2 + kr_1^3$, $\underline{b} = 1 + ir_2 + jr_2^2 + kr_2^3$, $\underline{c} = 1 + ir_3 + jr_3^2 + kr_3^3$.

Proof. Considering equations (2.1) and (3.1), we have:

$$\begin{aligned} \widehat{P}_n &= ar_1^n + br_2^n + cr_3^n + i\left(ar_1^{n+1} + br_2^{n+1} + cr_3^{n+1}\right) + j\left(ar_1^{n+2} + br_2^{n+2} + cr_3^{n+2}\right) + k\left(ar_1^{n+3} + br_2^{n+3} + cr_3^{n+3}\right) \\ &= \left(1 + ir_1 + jr_1^2 + kr_1^3\right)ar_1^n + \left(1 + ir_2 + jr_2^2 + kr_2^3\right)br_2^n + \left(1 + ir_3 + jr_3^2 + kr_3^3\right)cr_3^n \\ &= \underline{a}ar_1^n + \underline{b}br_2^n + \underline{c}cr_3^n. \end{aligned}$$

Equation (3.9) can be obtained similarly. □

Theorem 3.16. *The generating functions for Padovan and Perrin dual quaternions with non-negative and negative subscripts are as follows:*

$$\sum_{n=0}^{\infty} \widehat{P}_n x^n = \frac{\widehat{P}_0 + \widehat{P}_1 x + \left(\widehat{P}_2 - \widehat{P}_0\right) x^2}{1 - x^2 - x^3}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} \widehat{R}_n x^n = \frac{\widehat{R}_0 + \widehat{R}_1 x + \left(\widehat{R}_2 - \widehat{R}_0\right) x^2}{1 - x^2 - x^3}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} \widehat{P}_{-n} x^n = \frac{\widehat{P}_0 + \left(\widehat{P}_{-1} + \widehat{P}_0\right) x + \left(\widehat{P}_{-2} + \widehat{P}_{-1}\right) x^2}{1 + x - x^3},$$

$$\sum_{n=0}^{\infty} \widehat{R}_{-n} x^n = \frac{\widehat{R}_0 + \left(\widehat{R}_{-1} + \widehat{R}_0\right) x + \left(\widehat{R}_{-2} + \widehat{R}_{-1}\right) x^2}{1 + x - x^3}. \tag{3.12}$$

Proof. Suppose that, $\sum_{n=0}^{\infty} \widehat{P}_n x^n = \widehat{P}_0 + \widehat{P}_1 x + \widehat{P}_2 x^2 + ... + \widehat{P}_n x^n + ...$ is generating function of $\widehat{P}_n$. Then, we can write:

$$\left(1 - x^2 - x^3\right) \sum_{n=0}^{\infty} \widehat{P}_n x^n = \widehat{P}_0 + \widehat{P}_1 x + \left(\widehat{P}_2 - \widehat{P}_0\right) x^2 + \left(\widehat{P}_3 - \widehat{P}_1 - \widehat{P}_0\right) x^3 + \ldots + \left(\widehat{P}_{n+3} - \widehat{P}_{n+1} - \widehat{P}_n\right) x^{n+3} + \ldots$$

By the recurrence relation (3.5), we have equation (3.10). The proof of equation (3.11) is similar.

With similar thought, assume that $\sum_{n=0}^{\infty} \widehat{P}_{-n} x^n = \widehat{P}_0 + \widehat{P}_{-1} x + \widehat{P}_{-2} x^2 + \ldots + \widehat{P}_{-n} x^n + \ldots$ is the generating function of $\widehat{P}_{-n}$. By the recurrence relation (3.7), we obtain the following equality:

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{P}_{-n} x^n &= \widehat{P}_0 + \widehat{P}_{-1} x + \widehat{P}_{-2} x^2 + \sum_{n=3}^{\infty} \widehat{P}_{-(n-3)} x^n - \sum_{n=3}^{\infty} \widehat{P}_{-(n-1)} x^n \\
&= \widehat{P}_0 + \widehat{P}_{-1} x + \widehat{P}_{-2} x^2 + x^3 \sum_{n=0}^{\infty} \widehat{P}_{-n} x^n - x \sum_{n=2}^{\infty} \widehat{P}_{-n} x^n \\
&= \widehat{P}_0 + \widehat{P}_{-1} x + \widehat{P}_{-2} x^2 + x^3 \sum_{n=0}^{\infty} \widehat{P}_{-n} x^n - x \left(\sum_{n=0}^{\infty} \widehat{P}_{-n} x^n - \widehat{P}_0 - \widehat{P}_{-1} x \right) \\
&= \widehat{P}_0 + \left(\widehat{P}_{-1} + \widehat{P}_0\right) x + \left(\widehat{P}_{-2} + \widehat{P}_{-1}\right) x^2 + x^3 \sum_{n=0}^{\infty} \widehat{P}_{-n} x^n - x \sum_{n=0}^{\infty} \widehat{P}_{-n} x^n.
\end{aligned}$$

Then, we have:

$$\sum_{n=0}^{\infty} \widehat{P}_{-n} x^n = \frac{\widehat{P}_0 + \left(\widehat{P}_{-1} + \widehat{P}_0\right) x + \left(\widehat{P}_{-2} + \widehat{P}_{-1}\right) x^2}{1 + x - x^3}.$$

Equation (3.12) can be seen in the same way. □

Theorem 3.17. *The exponential generating functions for Padovan and Perrin dual quaternions with non-negative and negative subscripts are given by:*

$$\sum_{n=0}^{\infty} \widehat{P}_n \frac{y^n}{n!} = \underline{a}ae^{r_1 y} + \underline{b}be^{r_2 y} + \underline{c}ce^{r_3 y},$$

$$\sum_{n=0}^{\infty} \widehat{R}_n \frac{y^n}{n!} = \underline{a}e^{r_1 y} + \underline{b}e^{r_2 y} + \underline{c}e^{r_3 y},$$

$$\sum_{n=0}^{\infty} \widehat{P}_{-n} \frac{y^n}{n!} = \underline{a}ae^{\frac{y}{r_1}} + \underline{b}be^{\frac{y}{r_2}} + \underline{c}ce^{\frac{y}{r_3}},$$

$$\sum_{n=0}^{\infty} \widehat{R}_{-n} \frac{y^n}{n!} = \underline{a}e^{\frac{y}{r_1}} + \underline{b}e^{\frac{y}{r_2}} + \underline{c}e^{\frac{y}{r_3}}.$$

Proof. By using the equation (3.8), we get as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{P}_n \frac{y^n}{n!} &= \sum_{n=0}^{\infty} \left(\underline{a}ar_1^n + \underline{b}br_2^n + \underline{c}cr_3^n\right) \frac{y^n}{n!} \\
&= \underline{a}a \sum_{n=0}^{\infty} r_1^n \frac{y^n}{n!} + \underline{b}b \sum_{n=0}^{\infty} r_2^n \frac{y^n}{n!} + \underline{c}c \sum_{n=0}^{\infty} r_3^n \frac{y^n}{n!} \\
&= \underline{a}a \sum_{n=0}^{\infty} \frac{(r_1 y)^n}{n!} + \underline{b}b \sum_{n=0}^{\infty} \frac{(r_2 y)^n}{n!} + \underline{c}c \sum_{n=0}^{\infty} \frac{(r_3 y)^n}{n!} \\
&= \underline{a}ae^{r_1 y} + \underline{b}be^{r_2 y} + \underline{c}ce^{r_3 y}.
\end{aligned}$$

The other equalities can be shown in the same way. □

Theorem 3.18. $\forall n \in \mathbb{N}$, $\forall k, m, t \in \mathbb{Z}^+$ *and,* $t \geq m$, *the following summation formula is obtained:*

$$\sum_{n=0}^{k-1} \widehat{P}_{mn+t-1} = \frac{\widehat{P}_{mk+m+t-1} + \widehat{P}_{mk-m+t-1} - \widehat{P}_{mk+t-1}(R_m - 1) - \widehat{P}_{m+t-1} - \widehat{P}_{t-m-1} + \widehat{P}_{t-1}(R_m - 1)}{R_m - R_{-m}}. \tag{3.13}$$

Proof. By utilizing equation (3.8), we have:

$$\begin{aligned}
\sum_{n=0}^{k-1} \widehat{P}_{mn+t-1} &= \sum_{n=0}^{k-1} \left(\underline{a}ar_1^{mn+t-1} + \underline{b}br_2^{mn+t-1} + \underline{c}cr_3^{mn+t-1}\right) \\
&= \underline{a}ar_1^{t-1}\left(\frac{r_1^{mk} - 1}{r_1^m - 1}\right) + \underline{b}br_2^{t-1}\left(\frac{r_2^{mk} - 1}{r_2^m - 1}\right) + \underline{c}cr_3^{t-1}\left(\frac{r_3^{mk} - 1}{r_3^m - 1}\right).
\end{aligned}$$

Then, we obtain:

$$\sum_{n=0}^{k-1} \widehat{P}_{mn+t-1} = \frac{\underline{a}aA_1 + \underline{b}bA_2 + \underline{c}cA_3}{R_m - R_{-m}},$$

where

$$A_i = \left(r_i^{mk+t-m-1} + r_i^{mk+t+m-1} + r_i^{mk+t-1}(1 - R_m) - r_i^{t+m-1} - r_i^{t-m-1} + r_i^{t-1}(R_m - 1)\right)$$

for $i = 1, 2, 3$. Finally, using equation (3.8), we have equation (3.13). □

Theorem 3.19. $\forall k \in \mathbb{Z}^+$, *the following summation formula hold:*

$$\sum_{n=0}^{k-1} \widehat{P}_{-mn-t-1} = \frac{\widehat{P}_{-mk+m-t-1} + \widehat{P}_{-mk-m-t-1} - \widehat{P}_{-mk-t-1}(R_{-m} - 1) - \widehat{P}_{m-t-1} - \widehat{P}_{-t-m-1} + \widehat{P}_{-t-1}(R_{-m} - 1)}{R_{-m} - R_m}.$$

Proof. The proof can be shown similarly to proof of the Theorem 3.18. □

Theorem 3.20. *$\forall k, n \in \mathbb{N}$, the following binomial properties hold:*

(i) $\sum_{n=0}^{k} \binom{k}{n} \widehat{P}_n = \widehat{P}_{3k}$,

(ii) $\sum_{n=0}^{k+1} \binom{k+1}{n} \widehat{P}_n = \sum_{n=0}^{k} \binom{k}{n} \left(\widehat{P}_n + \widehat{P}_{n+1}\right)$.

Proof. (i) From the equation (3.8), we have:

$$\begin{aligned}\sum_{n=0}^{k} \binom{k}{n} \widehat{P}_n &= \sum_{n=0}^{k} \binom{k}{n} \left(\underline{a}ar_1^n + \underline{b}br_1^n + \underline{c}cr_1^n\right)\\ &= \underline{a}a \sum_{n=0}^{k} \binom{k}{n} r_1^n + \underline{b}b \sum_{n=0}^{k} \binom{k}{n} r_2^n + \underline{c}c \sum_{n=0}^{k} \binom{k}{n} r_3^n\\ &= \underline{a}a(1+r_1)^k + \underline{b}b(1+r_2)^k + \underline{c}c(1+r_3)^k\\ &= \underline{a}ar_1^{3k} + \underline{b}br_2^{3k} + \underline{c}cr_3^{3k}\\ &= \widehat{P}_{3k}.\end{aligned}$$

□

Corollary 3.21. *$\forall k, n \in \mathbb{N}$, the following binomial properties are satisfied:*

(i) $\widehat{P}_{3(k+1)} = \widehat{P}_{3k} + \sum_{n=0}^{k} \binom{k}{n} \widehat{P}_{n+1}$,

(ii) $\widehat{R}_{3(k+1)} = \widehat{R}_{3k} + \sum_{n=0}^{k} \binom{k}{n} \widehat{R}_{n+1}$.

Theorem 3.22. *$\forall n \in \mathbb{N}$ and, $\forall k \in \mathbb{Z}^+$, the following property can be given:*

$$\widehat{P}_{3(k+2)} = 3\widehat{P}_{3(k+1)} - 2\widehat{P}_{3k} + \widehat{P}_{3(k-1)}.$$

Proof. Using Theorem 3.20, equation (3.5) and the binomial properties, the proof is clear. □

Considering the summation formulas given in the study [38], the following summation formulas for Padovan dual quaternions can be given:

Theorem 3.23. *$\forall k, n \in \mathbb{N}$, the following summation formulas are obtained:*

(i) $\sum_{n=0}^{k} \widehat{P}_n = \widehat{P}_{k+5} - \widehat{P}_4$,

(ii) $\sum_{n=0}^{k} \widehat{P}_{2n} = \widehat{P}_{2k+3} - \widehat{P}_1$,

(iii) $\sum_{n=0}^{k} \widehat{P}_{2n-1} = \widehat{P}_{2k+2} - \widehat{P}_0$,

(iv) $\sum_{n=0}^{k} \widehat{P}_{2n+1} = \widehat{P}_{2k+4} - \widehat{P}_2$,

(v) $\sum_{n=0}^{k} \widehat{P}_{3n} = \widehat{P}_{3k+2} - \widehat{P}_{-1}$,

(vi) $\sum_{n=0}^{k} \widehat{P}_{3n+1} = \widehat{P}_{3k+3} - \widehat{P}_0$,

(vii) $\sum_{n=0}^{k} \widehat{P}_{3n+2} = \widehat{P}_{3k+4} - \widehat{P}_1$,

(viii) $\sum_{n=0}^{k} \widehat{P}_{5n} = \widehat{P}_{5k+1} - \widehat{P}_{-4}$,

(ix) $\sum_{n=0}^{k} \widehat{P}_{5n+1} = \widehat{P}_{5k+2} - \widehat{P}_{-3}$,

(x) $\sum_{n=0}^{k} \widehat{P}_{5n+2} = \widehat{P}_{5k+3} - \widehat{P}_{-2}$.

Proof. The proof is clear. □

Theorem 3.24. *$\forall k, n \in \mathbb{Z}^+$, the following summation formulas are satisfied:*

(i) $\sum_{n=1}^{k} \widehat{P}_{-n} = -\widehat{P}_{-k+4} + \widehat{P}_4$,

(ii) $\sum_{n=1}^{k} \widehat{P}_{-2n} = -\widehat{P}_{-2k+1} + \widehat{P}_1$,

(iii) $\sum_{n=1}^{k} \widehat{P}_{-2n-1} = -\widehat{P}_{-2k} + \widehat{P}_0$,

(iv) $\sum_{n=1}^{k} \widehat{P}_{-2n+1} = -\widehat{P}_{-2k+2} + \widehat{P}_2$,

(v) $\sum_{n=1}^{k} \widehat{P}_{-3n} = -\widehat{P}_{-3k-1} + \widehat{P}_{-1}$,

(vi) $\sum_{n=1}^{k} \widehat{P}_{-3n+1} = -\widehat{P}_{-3k} + \widehat{P}_{0}$,

(vii) $\sum_{n=1}^{k} \widehat{P}_{-3n+2} = -\widehat{P}_{-3k+1} + \widehat{P}_{1}$,

(viii) $\sum_{n=1}^{k} \widehat{P}_{-5n} = -\widehat{P}_{-5k-4} + \widehat{P}_{-4}$,

(ix) $\sum_{n=1}^{k} \widehat{P}_{-5n+1} = -\widehat{P}_{-5k-3} + \widehat{P}_{-3}$,

(x) $\sum_{n=1}^{k} \widehat{P}_{-5n+2} = -\widehat{P}_{-5k-2} + \widehat{P}_{-2}$.

Proof. (i) Via the recurrence relation (3.7), we get:

$$\begin{aligned}
\widehat{P}_{-1} &= \widehat{P}_{2} - \widehat{P}_{0} \\
\widehat{P}_{-2} &= \widehat{P}_{1} - \widehat{P}_{-1} \\
&\vdots \\
\widehat{P}_{-(k-1)} &= \widehat{P}_{-(k-4)} - \widehat{P}_{-(k-2)} \\
\widehat{P}_{-k} &= \widehat{P}_{-(k-3)} - \widehat{P}_{-(k-1)}.
\end{aligned}$$

Then, we have:

$$\sum_{n=1}^{k} \widehat{P}_{-n} = -\widehat{P}_{-k+2} - \widehat{P}_{-k+1} + \widehat{P}_{2} + \widehat{P}_{1} = -\widehat{P}_{-k+4} + \widehat{P}_{4}.$$

The other parts can be seen similarly. □

4. Pell-Padovan Dual Quaternions

In this section, we investigate Pell-Padovan dual quaternions with non-negative and negative subscripts using similar methods adopted in the previous section.

Definition 4.1. $\forall n \in \mathbb{N}$, the nth Pell-Padovan dual quaternion is defined as:

$$\widehat{T}_{n} = T_{n} + iT_{n+1} + jT_{n+2} + kT_{n+3}. \tag{4.1}$$

Besides, $\forall n \in \mathbb{Z}^{+}$, the $-n$th Pell-Padovan dual quaternion with negative subscripts is defined as:

$$\widehat{T}_{-n} = T_{-n} + iT_{-n+1} + jT_{-n+2} + kT_{-n+3},$$

where T_n is the nth Pell-Padovan number, T_{-n} is the $-n$th Pell-Padovan number and i, j, k are quaternionic units that satisfy the rules in equation (1.4). The set of all the Pell-Padovan dual quaternions[2] is represented by $\widehat{\mathcal{T}}_{\mathcal{D}}$.

It is quite obvious to realize that algebraic operations for Pell-Padovan dual quaternions are familiar with the previous section.

$\forall n \in \mathbb{Z}$, we give some examples of values of the Pell-Padovan dual quaternions in Table 4.

Table 4. Some values of $\widehat{T}_n$ and $\widehat{T}_{-n}$

n	$\widehat{T}_n$	$\widehat{T}_{-n}$
0	$1 + i + j + k3$	$1 + i + j + k3$
1	$1 + i + j3 + k3$	$-1 + i + j + k$
2	$1 + i3 + j3 + k7$	$3 - i + j + k$
3	$3 + i3 + j7 + k9$	$-5 + i3 - j + k$
$\vdots$	$\vdots$	$\vdots$

[2]It should be noted that, equation (4.1) is valid $\forall n \in \mathbb{Z}$.

Theorem 4.2. *The following recurrence relation for Pell-Padovan dual quaternions holds:*

$$\widehat{T}_{n+3} = 2\widehat{T}_{n+1} + \widehat{T}_n, \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Also, the recurrence relation for the Pell-Padovan dual quaternions with negative subscripts is satisfied:

$$\widehat{T}_{-n} = \widehat{T}_{-(n-3)} - 2\widehat{T}_{-(n-1)}, \quad \forall n \in \mathbb{Z}^+. \tag{4.3}$$

Proof. Taking into account equations (1.3) and (4.1), we obtain:

$$\begin{aligned} 2\widehat{T}_{n+1} + \widehat{T}_n &= 2T_{n+1} + T_n + i(2T_{n+2} + T_{n+1}) + j(2T_{n+3} + T_{n+2}) + k(2T_{n+4} + T_{n+3}) \\ &= T_{n+3} + iT_{n+4} + jT_{n+5} + kT_{n+6} \\ &= \widehat{T}_{n+3}. \end{aligned}$$

Recurrence relation (4.3) can be shown in the same manner. □

The recurrence relation (4.2) is valid $\forall n \in \mathbb{Z}$, as well.

Theorem 4.3. $\forall n \in \mathbb{N}$, *the following determinant equalities can be given.*

$$\text{(i)}\ \widehat{T}_n = \begin{vmatrix} \widehat{T}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widehat{T}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widehat{T}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 2 & 0 \end{vmatrix}_{(n+1)\times(n+1)},$$

$$\text{(ii)}\ \widehat{T}_{-(n+1)} = \begin{vmatrix} \widehat{T}_{-1} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widehat{T}_{-2} & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widehat{T}_{-3} & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -2 \end{vmatrix}_{(n+1)\times(n+1)}.$$

Proof. The proof is obvious with equations (4.2) (for part (i)), (4.3) (for part (ii)) and the study [21]. □

Example 4.4. Let us consider the −8th term of the Pell-Padovan dual quaternion by using the method presented in Theorem 4.3.

$$\begin{vmatrix} \widehat{T}_{-1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widehat{T}_{-2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \widehat{T}_{-3} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{vmatrix}_{8\times 8} = 74 + i(-41) + j25 + k(-15) = \widehat{T}_{-8}.$$

Theorem 4.5. $\forall n \in \mathbb{Z}$, *the following is satisfied:*

$$\widehat{T}_{n+2} = \widehat{T}_{n+1} + \widehat{T}_n - (-1)^n(1 - i + j - k).$$

Proof. Using equations (A.9) and (4.1), the proof is completed. □

Theorem 4.6. *$\forall n \in \mathbb{Z}$, the following properties hold:*

$$\begin{cases} \widehat{T}_n = -T_{-n+2} + 2 + iT_{-n+1} + j\left(-T_{-n} + 2\right) + kT_{-n-1}; & \text{if } n \text{ is odd,} \\ \widehat{T}_n = T_{-n+2} + i\left(-T_{-n+1} + 2\right) + jT_{-n} + k\left(-T_{-n-1} + 2\right); & \text{if } n \text{ is even.} \end{cases}$$

Proof. By using the equation (4.1) and (A.10), the proof can be completed. □

Theorem 4.7. *$\forall n \in \mathbb{Z}$, the below properties hold:*

(i) $\widehat{T}_n - i\widehat{T}_{n+1} - j\widehat{T}_{n+2} - k\widehat{T}_{n+3} = T_n$,
(ii) $\widehat{T}_n + i\widehat{T}_{n+1} + j\widehat{T}_{n+2} + k\widehat{T}_{n+3} = 2\widehat{T}_n - T_n$,
(iii) $\widehat{T}_n\overline{\widehat{T}}_n = T_n^2$,
(iv) $\widehat{T}_n + \overline{\widehat{T}}_n = 2T_n$,
(v) $\widehat{T}_n - \overline{\widehat{T}}_n = 2\widehat{T}_n - 2T_n$,
(vi) $\widehat{T}_n^2 = T_n^2 + i2T_nT_{n+1} + j2T_nT_{n+2} + k2T_nT_{n+3} = 2T_n\widehat{T}_n - T_n^2$,
(vii) $\overline{\widehat{T}}_n^{\,2} = T_n^2 - i2T_nT_{n+1} - j2T_nT_{n+2} - k2T_nT_{n+3} = 3T_n^2 - 2T_n\widehat{T}_n$.

Proof. (iv) Taking into account equation (4.1) and the conjugate of $\widehat{T}_n$, we have:

$$\begin{aligned} \widehat{T}_n + \overline{\widehat{T}}_n =& (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3}) + (T_n - iT_{n+1} - jT_{n+2} - kT_{n+3}) \\ =& 2T_n. \end{aligned}$$

(v) From equation (4.1) and conjugate of $\widehat{T}_n$, we obtain:

$$\begin{aligned} \widehat{T}_n - \overline{\widehat{T}}_n =& (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3}) - (T_n - iT_{n+1} - jT_{n+2} - kT_{n+3}) \\ =& 2\widehat{T}_n - 2T_n. \end{aligned}$$

(vi) From equations (1.4) and (4.1), we have:

$$\begin{aligned} \widehat{T}_n^2 =& (T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k)(T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k) \\ =& T_n^2 + i2T_nT_{n+1} + j2T_nT_{n+2} + k2T_nT_{n+3} \\ =& 2T_n\widehat{T}_n - T_n^2. \end{aligned}$$

(vii) Considering equation (1.4) and conjugate of $\widehat{T}_n$, we get:

$$\begin{aligned} \overline{\widehat{T}}_n^{\,2} =& (T_n - iT_{n+1} - jT_{n+2} - kT_{n+3})(T_n - iT_{n+1} - jT_{n+2} - kT_{n+3}) \\ =& T_n^2 - i2T_nT_{n+1} - j2T_nT_{n+2} - k2T_nT_{n+3} \\ =& 3T_n^2 - 2T_n\widehat{T}_n. \end{aligned}$$

□

Theorem 4.8. *$\forall n \in \mathbb{Z}$, the following properties hold:*

(i) $\widehat{T}_n\widehat{T}_m - \overline{\widehat{T}}_n\overline{\widehat{T}}_m = 2\left(T_n\widehat{T}_m + T_m\widehat{T}_n - 2T_nT_m\right)$,
(ii) $\widehat{T}_n\widehat{T}_m + \overline{\widehat{T}}_n\overline{\widehat{T}}_m = 2T_nT_m$,
(iii) $\widehat{T}_n\overline{\widehat{T}}_m - \overline{\widehat{T}}_n\widehat{T}_m = 2\left(T_m\widehat{T}_n - T_n\widehat{T}_m\right)$,
(iv) $\widehat{T}_n\overline{\widehat{T}}_m + \overline{\widehat{T}}_n\widehat{T}_m = 2T_nT_m$.

Proof. (iii) Using equations (1.4), (4.1), and conjugate of $\widehat{T}_n$, we have:

$$\begin{aligned} \widehat{T}_n\overline{\widehat{T}}_m - \overline{\widehat{T}}_n\widehat{T}_m =& (T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})(T_m - iT_{m+1} - jT_{m+2} - kT_{m+3}) \\ & - (T_n - iT_{n+1} - jT_{n+2} - kT_{n+3})(T_m + iT_{m+1} + jT_{m+2} + kT_{m+3}) \\ =& 2(T_m\widehat{T}_n - T_n\widehat{T}_m). \end{aligned}$$

(iv) Considering equations (1.4), (4.1), and conjugate of $\widehat{T}_n$, we obtain:

$$\begin{aligned}\widehat{T}_n\overline{\widehat{T}}_m + \overline{\widehat{T}}_n\widehat{T}_m =&(T_n + iT_{n+1} + jT_{n+2} + kT_{n+3})(T_m - iT_{m+1} - jT_{m+2} - kT_{m+3})\\ &+ (T_n - iT_{n+1} - jT_{n+2} - kT_{n+3})(T_m + iT_{m+1} + jT_{m+2} + kT_{m+3})\\ =&2T_nT_m.\end{aligned}$$

□

Inspired by the study [4], we obtain some recurrence relations as follows:

Theorem 4.9. $\forall n \in \mathbb{Z}$, *the recurrence relation*

$$\widehat{T}_n = \rho_a\widehat{T}_{n-a} + \sigma_a\widehat{T}_{n-2a} + \widehat{T}_{n-3a}$$

is obtained with (ρ_a, σ_a) *such that* $\rho_a, \sigma_a \in \mathbb{Z}$; $1 \le a \le 10$; $a \in \mathbb{N}$ *(see in Table 5).*

Table 5. Some recurrence relations for Pell-Padovan dual quaternions

a	(ρ_a, σ_a)	$\widehat{T}_n = \rho_a\widehat{T}_{n-a} + \sigma_a\widehat{T}_{n-2a} + \widehat{T}_{n-3a}$
1	$(0, 2)$	$\widehat{T}_n = 2\widehat{T}_{n-2} + \widehat{T}_{n-3}$
2	$(4, -4)$	$\widehat{T}_n = 4\widehat{T}_{n-2} - 4\widehat{T}_{n-4} + \widehat{T}_{n-6}$
3	$(3, 5)$	$\widehat{T}_n = 3\widehat{T}_{n-3} + 5\widehat{T}_{n-6} + \widehat{T}_{n-9}$
4	$(8, -8)$	$\widehat{T}_n = 8\widehat{T}_{n-4} - 8\widehat{T}_{n-8} + \widehat{T}_{n-12}$
5	$(10, 12)$	$\widehat{T}_n = 10\widehat{T}_{n-5} + 12\widehat{T}_{n-10} + \widehat{T}_{n-15}$
6	$(19, -19)$	$\widehat{T}_n = 19\widehat{T}_{n-6} - 19\widehat{T}_{n-12} + \widehat{T}_{n-18}$
7	$(28, 30)$	$\widehat{T}_n = 28\widehat{T}_{n-7} + 30\widehat{T}_{n-14} + \widehat{T}_{n-21}$
8	$(48, -48)$	$\widehat{T}_n = 48\widehat{T}_{n-8} - 48\widehat{T}_{n-16} + \widehat{T}_{n-24}$
9	$(75, 77)$	$\widehat{T}_n = 75\widehat{T}_{n-9} + 77\widehat{T}_{n-18} + \widehat{T}_{n-27}$
10	$(124, -124)$	$\widehat{T}_n = 124\widehat{T}_{n-10} - 124\widehat{T}_{n-20} + \widehat{T}_{n-30}$

Proof. For $a = 3$, by using equation (4.1) and the third row of Table 7, we obtain:

$$\begin{aligned}3\widehat{T}_{n-3} + 5\widehat{T}_{n-6} + \widehat{T}_{n-9} =&3T_{n-3} + 5T_{n-6} + T_{n-9} + i(3T_{n-2} + 5T_{n-5} + T_{n-8}) + j(3T_{n-1} + 5T_{n-4} + T_{n-7})\\ &+ k(3T_n + 5T_{n-3} + T_{n-6})\\ =&T_n + iT_{n+1} + jT_{n+2} + kT_{n+3}\\ =&\widehat{T}_n.\end{aligned}$$

The other recurrence relations can be obtained similarly. □

Theorem 4.10. $\forall n \in \mathbb{Z}^+$, *the followings are satisfied:*

(i) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{T}_0 \\ \widehat{T}_1 \\ \widehat{T}_2 \end{pmatrix} = \begin{pmatrix} \widehat{T}_n \\ \widehat{T}_{n+1} \\ \widehat{T}_{n+2} \end{pmatrix},$$

(ii) $$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}^n \begin{pmatrix} \widehat{T}_{-1} \\ \widehat{T}_0 \\ \widehat{T}_{-2} \end{pmatrix} = \begin{pmatrix} \widehat{T}_{-n-1} \\ \widehat{T}_{-n} \\ \widehat{T}_{-n-2} \end{pmatrix},$$

(iii) $$\begin{pmatrix} \widehat{T}_1 & \widehat{T}_2 & \widehat{T}_0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{T}_{n+1} & \widehat{T}_{n+2} & \widehat{T}_n \end{pmatrix},$$

(iv) $$\begin{pmatrix} \widehat{T}_0 & \widehat{T}_{-1} & \widehat{T}_{-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}^n = \begin{pmatrix} \widehat{T}_{-n} & \widehat{T}_{-n-1} & \widehat{T}_{-n-2} \end{pmatrix},$$

(v) $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{T}_1 & \widehat{T}_0 & \widehat{T}_{-1} \\ \widehat{T}_2 & \widehat{T}_1 & \widehat{T}_0 \\ \widehat{T}_0 & \widehat{T}_{-1} & \widehat{T}_{-2} \end{pmatrix} = \begin{pmatrix} \widehat{T}_{n+1} & \widehat{T}_n & \widehat{T}_{n-1} \\ \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_n \\ \widehat{T}_n & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{pmatrix}$,

(vi) $\begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{T}_0 & \widehat{T}_{-1} & \widehat{T}_{-2} \\ \widehat{T}_1 & \widehat{T}_0 & \widehat{T}_{-1} \\ \widehat{T}_2 & \widehat{T}_1 & \widehat{T}_0 \end{pmatrix} = \begin{pmatrix} \widehat{T}_{-n} & \widehat{T}_{-n-1} & \widehat{T}_{-n-2} \\ \widehat{T}_{-n+1} & \widehat{T}_{-n} & \widehat{T}_{-n-1} \\ \widehat{T}_{-n+2} & \widehat{T}_{-n+1} & \widehat{T}_{-n} \end{pmatrix}$.

Theorem 4.11. *$\forall n \in \mathbb{Z}$, the **Binet-like formula** for Pell-Padovan dual quaternions is given as:*

$$\widehat{T}_n = \underline{d} w_1 \widetilde{r}_1^n + \underline{e} w_2 \widetilde{r}_2^n - \underline{f} \widetilde{r}_3^n, \tag{4.4}$$

where $\underline{d} = 1 + i\widetilde{r}_1 + j\widetilde{r}_1^2 + k\widetilde{r}_1^3$, $\underline{e} = 1 + i\widetilde{r}_2 + j\widetilde{r}_2^2 + k\widetilde{r}_2^3$, $\underline{f} = 1 + i\widetilde{r}_3 + j\widetilde{r}_3^2 + k\widetilde{r}_3^3$ and $\widetilde{r}_1$, $\widetilde{r}_2$, $\widetilde{r}_3$ are the roots of the characteristic equation $x^3 - 2x - 1 = 0$ of Pell-Padovan numbers.

Proof. From equations (2.2) and (4.1), we obtain:

$$\begin{aligned}\widehat{T}_n =& w_1\widetilde{r}_1^n + w_2\widetilde{r}_2^n - \widetilde{r}_3^n + i\left(w_1\widetilde{r}_1^{n+1} + w_2\widetilde{r}_2^{n+1} - \widetilde{r}_3^{n+1}\right) + j\left(w_1\widetilde{r}_1^{n+2} + w_2\widetilde{r}_2^{n+2} - \widetilde{r}_3^{n+2}\right) + k\left(w_1\widetilde{r}_1^{n+3} + w_2\widetilde{r}_2^{n+3} - \widetilde{r}_3^{n+3}\right) \\ =& \left(1 + i\widetilde{r}_1 + j\widetilde{r}_1^2 + k\widetilde{r}_1^3\right) w_1\widetilde{r}_1^n + \left(1 + i\widetilde{r}_2 + j\widetilde{r}_2^2 + k\widetilde{r}_2^3\right) w_2\widetilde{r}_2^n - \left(1 + i\widetilde{r}_3 + j\widetilde{r}_3^2 + k\widetilde{r}_3^3\right)\widetilde{r}_3^n \\ =& \underline{d} w_1\widetilde{r}_1^n + \underline{e} w_2\widetilde{r}_2^n - \underline{f}\widetilde{r}_3^n.\end{aligned}$$

□

Theorem 4.12. *The generating functions for Pell-Padovan dual quaternions with non-negative and negative subscripts are given as follows, respectively:*

$$\sum_{n=0}^{\infty} \widehat{T}_n x^n = \frac{\widehat{T}_0 + \widehat{T}_1 x + \left(\widehat{T}_2 - 2\widehat{T}_0\right) x^2}{1 - 2x^2 - x^3}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \widehat{T}_{-n} x^n = \frac{\widehat{T}_0 + \left(2\widehat{T}_0 + \widehat{T}_{-1}\right) x + \left(2\widehat{T}_{-1} + \widehat{T}_{-2}\right) x^2}{1 + 2x - x^3}. \tag{4.6}$$

Proof. Let presume that $\sum_{n=0}^{\infty} \widehat{T}_n x^n = \widehat{T}_0 + \widehat{T}_1 x + \widehat{T}_2 x^2 + \cdots + \widehat{T}_n x^n + \cdots$ is generating function of the Pell-Padovan dual quaternions. By performing the necessary calculations, we obtain:

$$\left(1 - 2x^2 - x^3\right) \sum_{n=0}^{\infty} \widehat{T}_n x^n = \widehat{T}_0 + \widehat{T}_1 x + \left(\widehat{T}_2 - 2\widehat{T}_0\right) x^2 + \left(\widehat{T}_3 - 2\widehat{T}_1 - \widehat{T}_0\right) x^3 + \cdots + \left(\widehat{T}_{n+3} - 2\widehat{T}_{n+1} - \widehat{T}_n\right) x^{n+3} + \cdots$$

Eventually, utilizing equation (4.2), we obtain equation (4.5). With similar thought, equation (4.6) is obtained. □

Theorem 4.13. *The exponential generating function of Pell-Padovan dual quaternions with non-negative and negative subscripts are as follows:*

$$\sum_{n=0}^{\infty} \widehat{T}_n \frac{y^n}{n!} = \underline{d} w_1 e^{\widetilde{r}_1 y} + \underline{e} w_2 e^{\widetilde{r}_2 y} - \underline{f} e^{\widetilde{r}_3 y},$$

$$\sum_{n=0}^{\infty} \widehat{T}_{-n} \frac{y^n}{n!} = \underline{d} w_1 e^{\frac{y}{\widetilde{r}_1}} + \underline{e} w_2 e^{\frac{y}{\widetilde{r}_2}} - \underline{f} e^{\frac{y}{\widetilde{r}_3}}.$$

Proof. Using equation (4.4), we have:

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{T}_n \frac{y^n}{n!} =& \sum_{n=0}^{\infty} \left(\underline{d} w_1\widetilde{r}_1^n + \underline{e} w_2\widetilde{r}_2^n - \underline{f}\widetilde{r}_3^n\right) \frac{y^n}{n!} \\ =& \underline{d} w_1 \sum_{n=0}^{\infty} \widetilde{r}_1^n \frac{y^n}{n!} + \underline{e} w_2 \sum_{n=0}^{\infty} \widetilde{r}_2^n \frac{y^n}{n!} - \underline{f} \sum_{n=0}^{\infty} \widetilde{r}_3^n \frac{y^n}{n!} \\ =& \underline{d} w_1 e^{\widetilde{r}_1 y} + \underline{e} w_2 e^{\widetilde{r}_2 y} - \underline{f} e^{\widetilde{r}_3 y}.\end{aligned}$$

The other equality can be shown by using the similar way. □

The proofs of the following theorems can be conducted by mathematical induction. By utilizing [38], we give the following summation formulas in the following theorems.

Theorem 4.14. *$\forall n, m \in \mathbb{N}$, the following summation properties hold:*

(i) $\sum_{n=0}^{m} \widehat{T}_n = \frac{1}{2}\Big(\widehat{T}_{m+2} + \widehat{T}_{m+1} + \widehat{T}_m + \widehat{T}_0 - \widehat{T}_1 - \widehat{T}_2\Big)$,

(ii) $\sum_{n=0}^{m} \widehat{T}_{2n} = \widehat{T}_{2m+1} + m(\widehat{T}_2 - \widehat{T}_1 - \widehat{T}_0) + \widehat{T}_0 - \widehat{T}_1$,

(iii) $\sum_{n=0}^{m} \widehat{T}_{2n+1} = \frac{1}{2}\Big(\widehat{T}_{2m+3} + \widehat{T}_{2m+2} - \widehat{T}_{2m+1} + 2m(-\widehat{T}_2 + \widehat{T}_1 + \widehat{T}_0) - \widehat{T}_2 + \widehat{T}_1 - \widehat{T}_0\Big)$.

Theorem 4.15. *$\forall n, m \in \mathbb{Z}^+$, the following summation properties hold:*

(i) $\sum_{n=1}^{m} \widehat{T}_{-n} = \frac{1}{2}\Big(-3\widehat{T}_{-m-1} - 3\widehat{T}_{-m-2} - \widehat{T}_{-m-3} + \widehat{T}_2 + \widehat{T}_1 - \widehat{T}_0\Big)$,

(ii) $\sum_{n=1}^{m} \widehat{T}_{-2n} = -\widehat{T}_{-2m+1} + \widehat{T}_{-2m} + m(\widehat{T}_2 - \widehat{T}_1 - \widehat{T}_0) + \widehat{T}_1 - \widehat{T}_0$,

(iii) $\sum_{n=1}^{m} \widehat{T}_{-2n+1} = \frac{1}{2}\Big(\widehat{T}_{-2m+1} - 3\widehat{T}_{-2m} - \widehat{T}_{-2m-1} + 2m(-\widehat{T}_2 + \widehat{T}_1 + \widehat{T}_0) + \widehat{T}_2 - \widehat{T}_1 + \widehat{T}_0\Big)$.

5. Conclusions

In this article, we bring together the properties of dual quaternions and Padovan, Perrin and Pell-Padovan numbers. We identify the Padovan, Perrin and Pell-Padovan dual quaternions with non-negative and negative subscripts by examining well-known relations and identities.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

References

[1] Atanassov, K., Dimitrov, D., Shannon, A., *A remark on ψ-function and Pell-Padovan's sequence*, Notes on Number Theory and Discrete Mathematics, **15**(2009), 1–44.

[2] Bilgici, G., *Generalized order-k Pell-Padovan-like numbers by matrix methods*, Pure and Applied Mathematics Journal, **2**(6)(2013), 174–178.

[3] Cerda-Morales, G., *Dual third order Jacobsthal quaternions*, Proyecciones Journal of Mathematics, **37**(4)(2018), 731–747.

[4] Cerda-Morales, G., *New identities for Padovan numbers*, https://arxiv.org/abs/1904.05492, (2019).

[5] Cerda-Morales, G., *On a generalization for Tribonacci quaternions*, Mediterr. J. Math., **14**(6)(2017), Article number: 239.

[6] Çimen, C.B., İpek, A., *On Pell quaternions and Pell-Lucas quaternions*, Advances in Applied Clifford Algebras, **26**(1)(2016), 39–51.

[7] Deveci, Ö., *The Pell-Padovan sequences and the Jacobsthal-Padovan sequences in finite groups*, Utilitas Mathematica, **98**(2015), 257–270.

[8] Deveci, Ö., Shannon, A.G., *Pell-Padovan-circulant sequences and their applications*, Notes on Number Theory and Discrete Mathematics, **23**(2017), 100–114.

[9] Deveci, Ö., Aküzüm, Y., Karaduman, E., *The Pell-Padovan p-sequences and its applications*, Utilitas Mathematica, **98**(2015), 327–347.

[10] Dişkaya, O., Menken, H., *On the (s,t)-Padovan and (s,t)-Perrin quaternions*, J. Adv. Math. Stud., **12**(2)(2019), 186–192.

[11] Dişkaya, O., Menken, H., *On the split (s,t)-Padovan and (s,t)-Perrin quaternions*, International Journal of Applied Mathematics and Informatics, **13**(2019), 25–28.

[12] Ercan, Z., Yüce, S., *On properties of the dual quaternions*, European Journal of Pure and Applied Mathematics, **4**(2)(2011), 142–146.

[13] Günay, H., Taşkara, N. *Some properties of Padovan quaternion*, Asian-European Journal of Mathematics, **12**(06)(2019), Art. No. 2040017.

[14] Halıcı, S., Karataş, A., *On a generalization for Fibonacci quaternions*, Chaos, Solitons & Fractals, **98**(2017), 178–182.

[15] Hamilton, W.R., *XI. On quaternions; or on a new system of imaginaries in algebra*, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, **33**(219)(1848), 58–60.

[16] Horadam, A.F., *Complex Fibonacci numbers and Fibonacci quaternions*, Amer. Math. Monthly, **70**(1963), 289–291.

[17] Işbilir, Z., Gürses, N., *Pell-Padovan generalized quaternions*, Notes on Number Theory and Discrete Mathematics, **27**(1)(2021), 171–187.

[18] Kalman, D., *Generalized Fibonacci numbers by matrix methods*, The Fibonacci Quarterly, **20**(1)(1982), 73–76.

[19] Kaygısız, K., Bozkurt, D., *k-generalized order-k Perrin number presentation by matrix method*, Ars Combinatoria, **105**(2012), 95–101.

[20] Khompungson, K., Rodjanadid, B., Sompong, S., *Some matrices in terms of Perrin and Padovan sequences*, Thai Journal of Mathematics, **17**(3)(2019), 767–774.
[21] Kızılateş, C., Catarino, P.M.M.C., Tuğlu, N., *On the bicomplex generalized Tribonacci quaternions*, Mathematics, **7**(1)(2019), 80.
[22] Lucas, E., *Théorie des fonctions numériques simplement périodiques*, Am. J. Math., **1**(3)(1878), 197–240.
[23] Majernik, V., *Quaternion formulation of the Galilean space-time transformation*, Acta Physica Slovaca, **56**(1)(2006), 9–14.
[24] Mangueira, M.C. dos S., Vieira, R.P.M., Alves, F.R.V., Catarino, P.M.M.C., *A generalização da forma matricial da sequência de Perrin*, Revista Sergipana de Matemática e Educação Matemática, **5**(1)(2020), 384–392.
[25] Padovan, R., Dom Hans Van der Laan: Modern Primitive, Architectura & Natura Press, Amsterdam, 1994.
[26] Padovan, R., *Dom Hans Van der Laan and the plastic number*, Nexus Network Journal, **4**(3)(2002), 181–193.
[27] Perrin, R., *Query 1484*, J. Intermed. Math., **6**(1899), 76–77.
[28] Seenukul, P., Netmanee, S., Panyakhun, T., Auiseekaen, R., Muangchan, S.-A., *Matrices which have similar properties to Padovan Q-matrix and its generalized relations*, SNRU Journal of Science and Technology, **7**(2)(2015), 90–94.
[29] Shannon, A.G., Horadam, A.F. *Some properties of third-order recurrence relations*, The Fibonacci Quarterly, **10**(2)(1972), 135–146.
[30] Shannon, A.G., Anderson, P.G., Horadam, A.F., *Properties of Cordonnier, Perrin and Van der Laan numbers*, International Journal of Mathematical Education in Science and Technology, **37**(7)(2006), 825–831.
[31] Shannon, A.G., Horadam, A.F., Anderson, P.G., *The auxiliary equation associated with the plastic number*, Notes on Number Theory and Discrete Mathematics, **12**(1)(2006), 1–12.
[32] Shannon, A.G., Wong, C.K., *Some properties of generalized third order Pell numbers*, Notes on Number Theory and Discrete Mathematics, 14(4)(2008), 16–24.
[33] Sloane, N.J.A., *The on-line encyclopedia of integer sequences*, (1964). Available online at: http://oeis.org/
[34] Sokhuma, K., *Matrices formula for Padovan and Perrin sequences*, Applied Mathematical Sciences, **7**(142)(2013), 7093–7096.
[35] Sokhuma, K., *Padovan Q-matrix and the generalized relations*, Applied Mathematical Sciences, **7**(56)(2013), 2777–2780.
[36] Sompong, S., Wora-Ngon, N., Piranan, A., Wongkaentow, N., *Some matrices with Padovan Q-matrix property*, Proceedings of the 13-th IMT-GT International Conference on Mathematics, Statistics and their Applications (ICMSA2017), 4 − 7 December 2017, Kedah, Malaysia, AIP Publishing LLC., 1905, 1, 030035, (2017), 6 pages.
[37] Soykan, Y., *Generalized Pell-Padovan numbers*, Asian Journal of Advanced Research and Reports, **11**(2)(2020), 8–28.
[38] Soykan, Y., *Summing formulas for generalized Tribonacci numbers*, Universal Journal of Mathematics and Applications, **3**(1)(2020), 1–11.
[39] Stewart, I., Math Hysteria: Fun and Games with Mathematics, Oxford University Press, New York, 2004.
[40] Stewart, I., *Tales of a neglected number*, Scientific American, **274**(1996), 102–103.
[41] Szynal-Liana, A., Wloch, I., *A note on Jacobsthal quaternions*, Advances in Applied Clifford Algebras, **26**(19)(2016), 441–447.
[42] Taşcı, D., *Padovan and Pell-Padovan quaternions*, Journal of Science and Arts, **42**(1)(2018), 125–132.
[43] Waddill, M.E., Sacks, L., *Another generalized Fibonacci sequence*, The Fibonacci Quarterly, **5**(3)(1967), 209–222.
[44] Yılmaz, F., Bozkurt, D., *Some properties of Padovan sequence by matrix method*, Ars Combinatoria, **104**(2012), 149–160.
[45] Yılmaz, N., *The matrix representations of Padovan and Perrin numbers*, Selçuk University, Graduate School of Natural and Applied Sciences, PhD Thesis, Konya, (2015).
[46] Yılmaz, N., Taşkara, N., *Binomial transforms of the Padovan and Perrin matrix sequences*, Abstr. Appl. Anal., **2013**, 497418, (2013), 7 pages.
[47] Yılmaz, N., Taşkara, N., *Matrix sequences in terms of Padovan and Perrin numbers*, J. App. Math., **2013**(2013), Article ID: 941673.
[48] Yılmaz, N., Taşkara, N., *On the negatively subscripted Padovan and Perrin matrix sequences*, Communications in Mathematics and Applications, **5**(2)(2014), 59–72.
[49] Yüce, S., Aydın, F.T., *Generalized dual Fibonacci quaternions*, Applied Mathematics E-Notes, **16**(309)(2016), 276–289.

Appendix A. Basic Concepts for Padovan, Perrin and Pell-Padovan Numbers

$\forall n \in \mathbb{Z}$, with *initial values* $P_0 = P_1 = P_2 = 1$, we have $P_n = \rho_a P_{n-a} + \sigma_a P_{n-2a} + P_{n-3a}$, where $\rho_a, \sigma_a \in \mathbb{Z}$, (ρ_a, σ_a) for $0 \le a \le 8, a \in \mathbb{N}$, [4], (see in Table 6).

Table 6. Recurrence relations for Padovan numbers, [4]

a	(ρ_a, σ_a)	$P_n = \rho_a P_{n-a} + \sigma_a P_{n-2a} + P_{n-3a}$
1	$(0, 1)$	$P_n = P_{n-2} + P_{n-3}$
2	$(2, -1)$	$P_n = 2P_{n-2} - P_{n-4} + P_{n-6}$
3	$(3, -2)$	$P_n = 3P_{n-3} - 2P_{n-6} + P_{n-9}$
4	$(2, 3)$	$P_n = 2P_{n-4} + 3P_{n-8} + P_{n-12}$
5	$(5, -4)$	$P_n = 5P_{n-5} - 4P_{n-10} + P_{n-15}$
6	$(5, 2)$	$P_n = 5P_{n-6} + 2P_{n-12} + P_{n-18}$
7	$(7, 1)$	$P_n = 7P_{n-7} + P_{n-14} + P_{n-21}$
8	$(10, -5)$	$P_n = 10P_{n-8} - 5P_{n-16} + P_{n-24}$

Table 6 is also valid for $\forall n \in \mathbb{Z}$ and Perrin numbers. $\forall n \in \mathbb{Z}$, we have (see [45, 47, 48]):

$$\begin{cases} R_n = 3P_{n-5} + 2P_{n-4}, \\ R_n = 2P_{n-2} + P_{n-5}, \\ P_{n-1} = \dfrac{1}{23}(R_{n-3} + 8R_{n-2} + 10R_{n-1}). \end{cases} \tag{A.1}$$

Also, the nth power of third-order Padovan matrices Q and $\widehat{Q}$ are as (see [45–48]):

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad Q^n = \begin{pmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{pmatrix}, \quad \forall n \in \mathbb{N}, \tag{A.2}$$

$$\widehat{Q} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \widehat{Q}^n = \begin{pmatrix} P_{-n-5} & P_{-n-3} & P_{-n-4} \\ P_{-n-4} & P_{-n-2} & P_{-n-3} \\ P_{-n-3} & P_{-n-1} & P_{-n-2} \end{pmatrix}, \quad \forall n \in \mathbb{N}. \tag{A.3}$$

$\forall m, n \in \mathbb{N}$, the relations regarding the Padovan and Perrin numbers are obtained, [45, 47, 48]:

$$\begin{cases} P_{m-3}P_{n-3} + P_{m-1}P_{n-2} + P_{m-2}P_{n-1} = P_{m+n-1}, \\ P_{m-3}R_{n-3} + P_{m-1}R_{n-2} + P_{m-2}R_{n-1} = R_{m+n-1}, \\ R_{m-3}R_{n-3} + R_{m-1}R_{n-2} + R_{m-2}R_{n-1} = 4P_{m+n-5} + 4P_{m+n-8} + P_{m+n-11}, \\ R_{m-3}R_{n-3} + R_{m-1}R_{n-2} + R_{m-2}R_{n-1} = 2R_{m+n-3} + R_{m+n-6}, \\ P_{m-3}P_{-n-3} + P_{m-1}P_{-n-2} + P_{m-2}P_{-n-1} = P_{m-n-1}, \\ P_{m-3}R_{-n-3} + P_{m-1}R_{-n-2} + P_{m-2}R_{-n-1} = R_{m-n-1}, \\ R_{m-3}R_{-n-3} + R_{m-1}R_{-n-2} + R_{m-2}R_{-n-1} = 2R_{m-n-3} + R_{m-n-6}, \\ P_{-m-3}P_{-n-3} + P_{-m-1}P_{-n-2} + P_{-m-2}P_{-n-1} = P_{-m-n-1}, \\ P_{-m-3}R_{-n-3} + P_{-m-1}R_{-n-2} + P_{-m-2}R_{-n-1} = R_{-m-n-1}, \\ R_{-m-3}R_{-n-3} + R_{-m-1}R_{-n-2} + R_{-m-2}R_{-n-1} = 4P_{-m-n-5} + 4P_{-m-n-8} + P_{-m-n-11}, \\ R_{-m-3}R_{-n-3} + R_{-m-1}R_{-n-2} + R_{-m-2}R_{-n-1} = 2R_{-m-n-3} + R_{-m-n-6}. \end{cases} \tag{A.4}$$

Using the initial values $P_0 = 0, P_1 = 0, P_2 = 1$, the Padovan matrix is identical to the matrix which is presented in (A.2). Q^n is defined as (see [35]):

$$Q^n = \begin{pmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{pmatrix}, \quad \forall n \geq 3. \tag{A.5}$$

From equation (A.5), $\forall m, n \in \mathbb{Z}^+$ such that $m < n$, the following are obtained (see [34, 35]):

$$\begin{cases} P_n = P_{m-1}P_{n-m} + P_{m+1}P_{n-m+1} + P_m P_{n-m+2}, \\ R_n = P_{m-1}R_{n-m} + P_{m+1}R_{n-m+1} + P_m R_{n-m+2}. \end{cases} \tag{A.6}$$

With $P_0 = 0, P_1 = 0, P_2 = 1$, the new third-order square matrices are examined in [28] and (A.6) is obtained by using these new matrices. $\forall n, m \in \mathbb{Z}^+$ such that $m \leq n$, we have ([36]):

$$\begin{cases} P_{2n} = P_{2m-1}P_{2(n-m)} + P_{2m}P_{2(n-m)+2} + P_{2m+1}P_{2(n-m)+1}, \\ P_{2n} = P_{2m}P_{2(n-m)-1} + P_{2m+1}P_{2(n-m)+1} + P_{2m+2}P_{2(n-m)}, \\ P_{2n+1} = P_{2m-1}P_{2(n-m)+1} + P_{2m}P_{2(n-m)+3} + P_{2m+1}P_{2(n-m)+2}, \\ P_{2n+1} = P_{2m}P_{2(n-m)} + P_{2m+1}P_{2(n-m)+2} + P_{2m+2}P_{2(n-m)+1}, \end{cases} \tag{A.7}$$

by using

$$Q_1^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad (Q_1^*)^n = \begin{pmatrix} P_{2n-1} & P_{2n} & P_{2n+1} \\ P_{2n+1} & P_{2n+2} & P_{2n+3} \\ P_{2n} & P_{2n+1} & P_{2n+2} \end{pmatrix}.$$

$\forall n, m \in \mathbb{Z}^+$ such that $m < n$ the following relations are given, [20]:

$$\begin{cases} R_{2n} = P_{2m-1}R_{2(n-m)} + P_{2m}R_{2(n-m)+2} + P_{2m+1}R_{2(n-m)+1}, \\ R_{2n+1} = P_{2m-1}R_{2(n-m)+1} + P_{2m}R_{2(n-m)+3} + P_{2m+1}R_{2(n-m)+2}, \\ R_{2n+1} = P_{2m}R_{2(n-m)} + P_{2m+1}R_{2(n-m)+2} + P_{2m+2}R_{2(n-m)+1}, \\ P_{2n+2} = P_{2m}P_{2(n-m)+1} + P_{2m+1}P_{2(n-m)+3} + P_{2m+2}P_{2(n-m)+2}, \\ R_{2n+2} = P_{2m}R_{2(n-m)+1} + P_{2m+1}R_{2(n-m)+3} + P_{2m+2}R_{2(n-m)+2}, \\ P_{n+1} = P_{m-1}P_{n-m+1} + P_{m}P_{n-m+3} + P_{m+1}P_{n-m+2}, \\ R_{n+1} = P_{m-1}R_{n-m+1} + P_{m}R_{n-m+3} + P_{m+1}R_{n-m+2}, \\ P_{n+2} = P_{m}P_{n-m+1} + P_{m+1}P_{n-m+3} + P_{m+2}P_{n-m+2}, \\ R_{n+2} = P_{m}R_{n-m+1} + P_{m+1}R_{n-m+3} + P_{m+2}R_{n-m+2}. \end{cases} \tag{A.8}$$

The Pell-Padovan matrix is examined in [7,37], utilizing Kalman's matrix formula in [18], such that (see [29,43]):

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The other type of Pell-Padovan matrix can be seen in [7]. Summation formulas of these special numbers are also examined in the study [38]. Then, $\forall n \in \mathbb{Z}$, we have (see in Table 1), [33]:

$$T_{n+2} = T_{n+1} + T_n - (-1)^n, \tag{A.9}$$

and

$$\begin{cases} T_n = -T_{-n+2} + 2; & \text{if } n \text{ is odd}, \\ T_n = T_{-n+2}; & \text{if } n \text{ is even}. \end{cases} \tag{A.10}$$

Additionally, if T_n is the nth Pell-Padovan number, and then $T_n = \rho_a T_{n-a} + \sigma_a T_{n-2a} + T_{n-3a}$ is obtained with (ρ_a, σ_a) such that $\rho_a, \sigma_a \in \mathbb{Z}; 1 \le a \le 10; a \in \mathbb{N}; n \in \mathbb{Z}$, (see in Table 7), [17].

Table 7. Some recurrence relations for Pell-Padovan numbers, [17]

a	(ρ_a, σ_a)	$T_n = \rho_a T_{n-a} + \sigma_a T_{n-2a} + T_{n-3a}$
1	$(0, 2)$	$T_n = 2T_{n-2} + T_{n-3}$
2	$(4, -4)$	$T_n = 4T_{n-2} - 4T_{n-4} + T_{n-6}$
3	$(3, 5)$	$T_n = 3T_{n-3} + 5T_{n-6} + T_{n-9}$
4	$(8, -8)$	$T_n = 8T_{n-4} - 8T_{n-8} + T_{n-12}$
5	$(10, 12)$	$T_n = 10T_{n-5} + 12T_{n-10} + T_{n-15}$
6	$(19, -19)$	$T_n = 19T_{n-6} - 19T_{n-12} + T_{n-18}$
7	$(28, 30)$	$T_n = 28T_{n-7} + 30T_{n-14} + T_{n-21}$
8	$(48, -48)$	$T_n = 48T_{n-8} - 48T_{n-16} + T_{n-24}$
9	$(75, 77)$	$T_n = 75T_{n-9} + 77T_{n-18} + T_{n-27}$
10	$(124, -124)$	$T_n = 124T_{n-10} - 124T_{n-20} + T_{n-30}$