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Research Article

Number of Subsets of the Set [n] Including No Three Consecutive Odd Integers

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Abstract

For every $n \in \mathbb{N}$, let a_n be the number of subsets S of the set $[n] = \{1, 2, \ . \ . \ . \ , n\}$ including no three consecutive odd integers. We give the generating function and the closed form formula of the sequence $(a_n)_{n\geq 0}$ obtaining sixth order linear homogeneous recurrence relation with constant coefficients of the integer sequence. The sequence is associated with the Tribonacci sequence. The combinatorial representation of the sequence $(a_n)_{n\geq 0}$ is obtained and limit of the ratios of consecutive terms of the sequence is found.

Keywords: Tribonacci numbers, Consecutive odd integers, Generating function, Combinatorial representation.

[n] Kümesinin Ardışık Üç Tam Sayı İçermeyen Alt Kümelerinin Sayısı

Öz

Her $n \in \mathbb{N}$ için a_n, $[n] = \{1, 2, \ . \ . \ . \ , n\}$ kümesinin ardışık üç tek tam sayı içermeyen S alt kümelerinin sayısı olsun. $(a_n)_{n\geq 0}$ dizisinin altıncı dereceden sabit katsayılı lineer homojen rekürans bağıntısını elde ederek dizinin üreteç fonksiyonunu ve kapalı form formülünü verdik. Dizi Tribonacci sayı dizisi ile ilişkilendirildi. $(a_n)_{n\geq 0}$ dizisinin kombinatoryal gösterimi elde edildi ve dizinin ardışık terimlerinin oranlarının limiti bulundu.

Anahtar Kelimeler: Tribonacci sayıları, Ardışık tek sayılar, Üreteç fonksiyon, Kombinatoryal gösterim.

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1. Introduction

The Tribonacci numbers are a generalization of the Fibonacci numbers. Some properties of Tribonacci numbers are given in [1, 3, 5, 6, 9, 10].

The Tribonacci sequence $(T_n)_{n\geq 0}$ is defined by the third-order recurrence relation:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3,}$$
$$T_0 = 0, T_1 = 1, T_2 = 1 \quad (1)$$

In [7] the Binet's formula for the Tribonacci sequence is given by

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha))(\gamma-\beta)} \quad (2)$$

where α, β and γ are roots of the cubic equation $x^3 - x^2 - x - 1 = 0$, ie.,

$$\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},$$
$$\beta = \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3},$$
$$\gamma = \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3},$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity.

"The number of subsets S of the set [n] = $\{1, 2, \ldots, n\}$ such that S contains no three consecutive integers." can be expressed in terms of the Tribonacci numbers. The answer is T_{n+2} by obtaining a recurrence by considering those subsets S which do or do not contain the first element '1'. By taking consecutive odd integers instead of consecutive integers, we consider the following counting problem:

What is the number of subsets S of the set [n] = $\{1, 2, \ldots, n\}$ such that S contains no three consecutive odd integers? In this paper, we denote the sequence by $(a_n)_{n\geq 0}$ corresponding to the counting problem.

After obtaining recursive definition of the sequence $(a_n)_{n\geq 0}$, we give the generating function, the closed form formula, the combinatorial representation and limit of the ratios of consecutive terms of the sequence.

2. Main Results

2.1. Recursive definition of the sequence

Let's write subsets S of the set [n] = $\{1, 2, \ldots, n\}$ such that S contains no three consecutive odd integers for some small n values as shown in Table 1.

Table 1. Subsets S of the set $\{1, 2, \ldots, n\}$ containing no three consecutive odd integers for some small n values

n	S	a_n
0	{ }	1
1	{ }, {1}	2
2	{ }, {1}, {2}, {1, 2}	4
3	{ }, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1,2, 3}	8
4	{ }, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}	16
5	{ }, {1}, {2}, {3}, {4}, {5}, {1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {4, 5}, {1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 4, 5}, {2, 3, 4}, {2, 3, 5}, {2, 4, 5}, {3, 4, 5}, {1, 2, 3, 4}, {1, 2, 4, 5}, {2, 3, 4, 5}	28

Hence, we get the initial conditions;

$$a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ a_4 = 16,\ a_5 = 28.$$

Consider subsets counted by a_n. Let's find a recurrence for the sequence $(a_n)_{n\geq 0}$. For $n > 5$ there are three cases for the subsets:

- The number of subsets not containing 1 as an element is $2a_{n-2}$.
- The number of subsets which contain 1, but don't contain 3, is $4a_{n-4}$.
- The number of subsets which contain 1 and 3, but don't contain 5, is $8a_{n-6}$.

This gives a recurrence

$$a_n = 2a_{n-2} + 4a_{n-4} + 8a_{n-6}. \quad (3)$$

2.2. Generating function and the Binet formula of the sequence

Let the generating function associated to the sequence $(a_n)_{n\geq 0}$ be the formal power series

$$F(x) = \sum_{n\geq 0} a_n x^n.$$

To find $F(x)$, multiply both sides of the recurrence relation (3) by x^n and sum over the values of n for which the recurrence is valid, namely, over $n \geq 6$. We get,

$$\sum_{n\geq 6} a_n x^n = \sum_{n\geq 6} 2a_{n-2}x^n + \sum_{n\geq 6} 4a_{n-4}x^n + \sum_{n\geq 6} 8a_{n-6}x^n \quad (4)$$

Then try to relate these sums to the unknown generating function $F(x)$. We have,

$$\sum_{n\geq 6} a_n x^n = F(x) - a_0 - a_1x - a_2x^2 - a_3x^3 - a_4x^4 - a_5x^5$$
$$= F(x) - 1 - 2x - 4x^2 - 8x^3 - 16x^4 - 28x^5$$

$$\sum_{n\geq 6} 2a_{n-2}x^n = 2x^2 \sum_{n\geq 6} a_{n-2}x^{n-2}$$
$$= 2x^2(F(x) - a_0 - a_1x - a_2x^2 - a_3x^3)$$
$$= 2x^2(F(x) - 1 - 2x - 4x^2 - 8x^3)$$
$$\sum_{n\geq 6} 4a_{n-4}x^n = 4x^4 \sum_{n\geq 6} a_{n-4}x^{n-4} = 4x^4(F(x) - a_0 - a_1x)$$
$$= 4x^4(F(x) - 1 - 2x)$$
$$\sum_{n\geq 6} 8a_{n-6}x^n = 8x^6 \sum_{n\geq 6} a_{n-6}x^{n-6} = 8x^6F(x)$$

If we write these results on the two sides of (4), we find

$$F(x) - 1 - 2x - 4x^2 - 8x^3 - 16x^4 - 28x^5$$
$$= 2x^2(F(x) - 1 - 2x - 4x^2 - 8x^3) + 4x^4(F(x) - 1 - 2x) + 8x^6F(x).$$

Which is trivial to solve for the unknown generating function $F(x)$ in the form

$$F(x) = \frac{1 + 2x + 2x^2 + 4x^3 + 4x^4 + 4x^5}{1 - 2x^2 - 4x^4 - 8x^6}. \quad (5)$$

Theorem 1. For $n \in \mathbb{N}$, let a_n be the number of subsets of S of the set [n] = $\{1, 2, \ldots, n\}$ containing no three consecutive odd integers. Then we have the following formulas for the subsequences of $(a_n)_{n\geq 0}$

$$a_{2n} = 2^n T_{n+2}, \quad (6)$$
$$a_{2n-1} = 2^{n-1} T_{n+2}\,. \quad (7)$$

where T_n is the nth Tribonacci number defined by (1).

Proof. If $A(x)$ is the generating function for even terms of the sequence $(a_n)_{n\geq 0}$, then it is clear that $A(x) = \frac{1}{2}(F(x) + F(-x))$. Substituting (5) we get,

$$A(x) = \frac{1 + 2x^2 + 4x^4}{1 - 2x^2 - 4x^4 - 8x^6} \quad (8)$$

Substituting $u = 2x^2$ in (8) we have,

$$A(u) = \frac{1 + u + u^2}{1 - u - u^2 - u^3}.$$

The generation function of the Tribonacci sequence with initial conditions $T_0 = 1, T_1 = 1, T_2 = 2$ is

$$\frac{1}{1 - x - x^2 - x^3}.$$

$$(1, 1, 2, 4, 7, 13, 24, \ldots) \leftrightarrow \frac{1}{1 - x - x^2 - x^3} \quad (9)$$

Now let's right- shift the sequence (9) by adding 1and 2 leading zeros respectively:

$$(0,1, 1, 2, 4, 7, 13, 24, \ldots) \leftrightarrow \frac{x}{1 - x - x^2 - x^3}$$

$$(0, 0,1, 1, 2, 4, 7, 13, 24, \ldots) \leftrightarrow \frac{x^2}{1 - x - x^2 - x^3}$$

Let's try to obtain the generating function $A(x)$ using the generating functions of the Tribonacci sequences given in terms of initial conditions:

$$A(u) = (1 + u + 2u^2 + 4u^3 + \ldots + T_{n+1}u^n + \ldots)$$
$$+(0 + u + u^2 + 2u^3 + \ldots + T_n u^n + \ldots)$$
$$+(0 + 0u + u^2 + u^3 + \ldots + T_{n-1}u^n + \ldots)$$
$$A(u) = (1 + 2u + 4u^2 + 7u^3 + \ldots + T_{n+2}u^n + \ldots)$$
$$A(x) = (1 + 2(2x^2) + 4(2x^2)^2 + 7(2x^2)^3 + \ldots$$
$$+T_{n+2}(2x^2)^n + \ldots)$$
$$A(x) = 1 + 2.2x^2 + 4.2^2x^4 + 7.2^3x^6 + \ldots$$
$$+T_{n+2}.2^n x^{2n} + \ldots)$$

Since $A(x)$ is the generating function for even terms of the sequence $(a_n)_{n\geq 0}$, we have

$$a_{2n} = 2^n T_{n+2}$$

where T_n is the Tribonacci numbers with initial conditions;

$$T_0 = 0, T_1 = 1, T_2 = 1.$$

If $B(x)$ is the generating function for odd terms of the sequence $(a_n)_{n\geq 0}$, then it is clear that $B(x) = \frac{1}{2}(F(x) - F(-x))$. Similarly using (5) and generating function method, for $n \geq 1$we have

$$a_{2n-1} = 2^{n-1} T_{n+2}.$$

The proof is completed.

Corollary 1. For $n \in \mathbb{N}$, let a_n be the number of subsets of S of the set [n] = $\{1, 2, \ldots, n\}$ including no three consecutive odd integers. Then we have the following closed form formula

$$a_n = 2^{\left\lfloor \frac{n}{2} \right\rfloor} T_{\left\lceil \frac{n+4}{2} \right\rceil}.$$

where T_n is the nth Tribonacci number, $\lfloor n \rfloor$ is the floor of n and $\lceil n \rceil$ is the ceiling of n.

Proof. Using Theorem 1, we can write piecewise defined sequence $(a_n)_{n\geq 0}$ asfollows:

$$a_n = \begin{cases} 2^{\frac{n}{2}} T_{\frac{n+4}{2}}\,, & \text{if n is even} \\ 2^{\frac{n-1}{2}} T_{\frac{n+5}{2}}\,, & \text{if n is odd} \end{cases}$$

When n is even, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$ and $\left\lceil \frac{n+4}{2} \right\rceil = \frac{n+4}{2}$. When n is odd, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$ and $\left\lceil \frac{n+4}{2} \right\rceil = \frac{n+5}{2}$.Then it is easy to see that

$$a_n = 2^{\left\lfloor \frac{n}{2} \right\rfloor} T_{\left\lceil \frac{n+4}{2} \right\rceil}.$$

2.3. Obtaining Binet formula of the sequence with combinatorial approach

Let's try to find formulas respectively for the subsequences $(a_{2n})_{n\geq 0}$ and $(a_{2n-1})_{n\geq 1}$ of the sequence $(a_n)_{n\geq 0}$. Let's consider the set, $M = \{1, 2, 3, \ \ldots, 2n\}$. For every $n \in \mathbb{N}$, let a_{2n} be the number of subsets of S of the set $M = \{1, 2, 3, \ \ldots, 2n\}$ containing no three consecutive odd integers. First, we separate the set M into two disjoint subset $S_1 = \{1, 3, 5, \ldots, 2n-1\}$ and $S_2 = \{2, 4, 6, \ldots, 2n\}$. First notice that, counting subsets from S_1 including no three consecutive odd integers is equivalent to counting subsets from $\{1, 2, \ \ldots, n\}$ including no three consecutive integers. Hence there are T_{n+2} subsets where T_n is the Tribonacci numbers defined by (1). The number of subsets of S_2 include no three consecutive odd integers is equal to 2^n since all elements of S_2 are even integers. Using multiplication principle, the total number of subsets of M containing no three consecutive odd integers is $2^n T_{n+2}$. Hence, we have

$$a_{2n} = 2^n T_{n+2}.$$

Considering the set, $M = \{1, 2, 3, \ \ldots, 2n-1\}$ and using the same counting technique we have

$$a_{2n-1} = 2^{n-1} T_{n+2}.$$

2.4. The combinatorial representation of the sequence

The explicit formula of Tribonacci sequence is given in [4] by the formula

$$T_n = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-j}{i}. \qquad (10)$$

Using (6), (7) and (10) we have the combinatorial representation of the sequence $(a_n)_{n\geq 0}$

$$a_{2n} = 2^n \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n+1-i-j}{i}, \ n \geq 0 \qquad (11)$$

$$a_{2n-1} = 2^{n-1} \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n+1-i-j}{i}, \ n \geq 1. \qquad (12)$$

Writing combinatorial identity for 2^n and using (11) and (12) we have,

$$a_{2n} = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n+1-i-j}{i}, \ n \geq 0$$

$$a_{2n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n+1-i-j}{i}, \ n \geq 1.$$

2.5. Limit of the ratios of consecutive terms of the sequence

It's well known that the limit of the ratio of two consecutive Fibonacci numbers is the Golden Ratio. A similar relationship occurs for the Tribonaccci numbers.

Define the sequence $x_n = \frac{T_{n+1}}{T_n}$ for $n \geq 1$ and $\lim_{n\to\infty} x_n = L$ exist. Using (1) for $n \geq 3$ we have

$$x_n = \frac{T_{n-2} + T_{n-1} + T_n}{T_n} = \frac{T_{n-2}}{T_n} + \frac{T_{n-1}}{T_n} + 1,$$

$$x_n = \frac{T_{n-1}}{T_{n-1}} \frac{T_{n-2}}{T_n} + \frac{T_{n-1}}{T_n} + 1,$$

$$x_n = \frac{1}{\frac{T_{n-1}}{T_{n-2}}} \frac{1}{\frac{T_n}{T_{n-1}}} + \frac{1}{\frac{T_n}{T_{n-1}}} + 1,$$

$$x_n = \frac{1}{x_{n-2}} \frac{1}{x_{n-1}} + \frac{1}{x_{n-1}} + 1. \qquad (13)$$

Taking the limit of both sides of (1), we obtain $L = \frac{1}{L^2} + \frac{1}{L} + 1$. Then $L^3 - L^2 - L - 1 = 0$. We know that the terms of values of T_n are real-valued and positive. From (13) we know that

$$L = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}$$

is the only real-valued root of the equation $L^3 - L^2 - L - 1 = 0$. Therefore,

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}$$

$$\approx 1.839286755. \qquad (14)$$

For any positive integer k and $\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}$ the following limit is obtained in [1].

$$\lim_{n\to\infty} \frac{T_{n+k}}{T_n} = \alpha^k \qquad (15)$$

Corollary 2. For $n \in \mathbb{N}$, let a_n be the number of subsets of S of the set $[n] = \{1, 2, \ \ldots, n\}$ including no three consecutive odd integers. Then we have the following results:

$$\lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3} \qquad (16)$$

$$\lim_{n\to\infty} \frac{a_{2n}}{a_{2n-1}} = 2, \qquad (17)$$

Proof. (16) is implied by (6), (7) and (14). (17) is an immediate consequence of (6) and (7).

Corollary 3. For $n \in \mathbb{N}$, let a_n be the number of subsets of S of the set [n] $= \{1, 2, \ . \ . \ . \ , n\}$ including no three consecutive odd integers. Then we have the following limit:

$$\lim_{n\to\infty} \frac{a_{2n+2k-1}}{a_{2n}} = \alpha^{k} \qquad (18)$$

where k is a positive integer and $\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}$.

Proof. (18) is an immediate consequence of (6), (7) and (15).

3. Conclusions

In this paper, we first obtained recursive formula of the sequence $(\boldsymbol{a_n})_{\boldsymbol{n\geq 0}}$ which counts the number of subsets of S of the set [n] $= \{\mathbf{1}, \mathbf{2}, \ . \ . \ . \ , \boldsymbol{n}\}$ including no three consecutive odd integers. Then we had the closed form formula of the sequence $(\boldsymbol{a_n})_{\boldsymbol{n\geq 0}}$ using the generating function method and combinatorial approach. The combinatorial representation and limit of the ratio of consecutive terms of the sequence are obtained.

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