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New Fractional Operators Including Wright Function in Their Kernels

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Abstract. In this paper, we defined new two-fractional derivative operators with a Wright function in their kernels. We also gave their Laplace and inverse Laplace transforms. Then, we presented some connections between the new fractional operators. Furthermore, as examples, we obtained solutions of differential equations involving new fractional operators. Finally, we examined the relations of the new fractional operators with the fractional operators, which can be found in the literature.

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1. Introduction

Fractional order derivatives and integrals are generalizations of classical derivatives and integrals studied in detail by Leibniz and Newton. What is meant to be expressed as a fractional-order derivative is actually any-order derivative. The concepts of fractional order derivative and integral are as old as the concepts of integer derivative and integral, and the expression of fractional derivative is first mentioned in letters between Leibniz and L'Hospital in 1695, as stated in many sources [15, 20, 26, 27].

In the last decade, researchers have been doing a lot of work on fractional operators that have become popular. Very recently, many studies on new fractional operators, which have various special functions in their kernels, can be found in literature (see for example [1, 3, 4, 10–12, 14, 16–19, 21, 22, 24, 29, 30], and the references therein).

In the third section of this paper, we defined two new fractional derivatives in the sense of Riemann-Liouville and Caputo fractional derivatives. These fractional derivatives includes a normalization function as a coefficient and a Wright function in the kernel. In the same section, we also obtained the Laplece transformations of new fractional derivatives. In the fourth section, we gave the solution of two differential equations using these fractional derivatives as examples.

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2. Preliminaries

In this section, we give relevant material which will be used throughout the paper.

Definition 2.1 (Gamma Function [5])**.** The gamma function is defined by

$$\Gamma(x)=\int_0^{\infty} t^{x-1}\exp(-t)dt, \quad (Re(x)>0)\,.$$

The above integral for $\Gamma(x)$ is sometimes called the Eulerian integral of the second kind. The gamma function has been the focus of attention of many researchers and has been presented in the literature by making various generalizations of this function. For some of these studies, see [2, 6–9, 13, 23, 25, 28].

Definition 2.2 (LT: Laplace Transform [15])**.** The LT of a function $f(t)$ is defined as:

$$\mathfrak{L}\{f(t)\}=F(s)=\int_0^{\infty} f(t)\exp(-st)dt, \quad (Re(s)>0).$$

Clearly, the LT is a generalized integral and is calculated as follows:

$$\int_0^{\infty} f(t)\exp(-st)dt=\lim_{A\to\infty}\int_0^{A} f(t)\exp(-st)dt.$$

The integral is convergent if the limit on the right hand side of the equation is present. In this case, the integral given on the left has a certain value. Otherwise, the LT of $f(t)$ does not exist.

Definition 2.3 (ILT: Inverse Laplace Transform [15])**.** The ILT of a function $F(s)$ is defined as:

$$\mathfrak{L}^{-1}\{F(s)\}=f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp(st)ds, \quad (c>0).$$

Remark 2.4. Note that, $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ are linear integral operators.

Definition 2.5 (Convolution [15])**.** The convolution of $f(t)$ and $g(t)$ is given by

$$f(t)*g(t)=\int_0^{t} f(t-\tau)g(\tau)d\tau. \tag{2.1}$$

Theorem 2.6 (Convolution Theorem [15])**.** *Let $\mathfrak{L}\{f(t)\}=F(s)$ and $\mathfrak{L}\{g(t)\}=G(s)$. Then, the following formulas holds true:*

$$\mathfrak{L}\{f(t)*g(t)\}=F(s)G(s), \tag{2.2}$$

and

$$\mathfrak{L}^{-1}\{F(s)G(s)\}=f(t)*g(t). \tag{2.3}$$

Definition 2.7 (Riemann-Liouville and Caputo Fractional Derivatives [20])**.** Let $Re(\alpha)>0$ and $n-1<Re(\alpha)<n$ for $n\in\mathbb{N}$. Then, the Riemann-Liouville and Caputo fractional derivatives $D^{\alpha}_{a^+}y$ and ${}^{C}D^{\alpha}_{a^+}y$ of order $\alpha\in\mathbb{C}$ are defined, respectively:

$$\left[D^{\alpha}_{a^+}y\right](x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_a^{x}(x-t)^{n-\alpha-1}y(t)dt, \quad (x>a)\,, \tag{2.4}$$

and

$$\left[{}^{C}D^{\alpha}_{a^+}y\right](x)=\frac{1}{\Gamma(n-\alpha)}\int_a^{x}(x-t)^{n-\alpha-1}y^{(n)}(t)dt, \quad (x>a)\,. \tag{2.5}$$

Definition 2.8 (Wright Function [20])**.** The Wright function is defined by the series

$${}_0\Psi_1(\alpha,\beta;z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{2.6}$$

where $\alpha,\beta\in\mathbb{C}$ and $Re(\alpha)>-1$.

3. New Fractional Derivatives and Their Properties

In this section, we give definitions of two new fractional derivatives. In one of them, the classical derivative operator is outside the integral as in the Riemann-Liouville fractional derivative, and in the other, it is included in the integral as in the Caputo fractional derivative. Therefore, we call these operators as ΨRL and ΨC fractional derivatives, respectively.

Definition 3.1. Let $0<\alpha,\varepsilon<1$, $0\le a\le t\le x<\infty$, $\gamma>-1,\beta>0$, $f\in H^1(a,b)$. Then, the new fractional derivatives of a function f defined respectively as:

$$\left[{}^{\Psi RL}D_{a^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x):=\frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}\frac{d}{dx}\int_a^x(x-t)^{\beta-1}f(t)\,{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha(x-t)^\gamma}{\varepsilon}\right)dt, \tag{3.1}$$

and

$$\left[{}^{\Psi C}D_{a^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x):=\frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}f'(t)\,{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha(x-t)^\gamma}{\varepsilon}\right)dt, \tag{3.2}$$

where $N(\alpha,\beta)$ is the normalization function and $N(0,0)=N(1,1)=1$.

If we take $\alpha=0$ and $\beta=\varepsilon=1$ in the ΨRL (3.1) and ΨC (3.2) fractional operators, we obtain the following equations respectively:

$$\left[{}^{\Psi RL}D_{a^+}^{0,1,\gamma,1}f\right](x)=N(0,1)f(x),$$

and

$$\left[{}^{\Psi C}D_{a^+}^{0,1,\gamma,1}f\right](x)=N(0,1)(f(x)-f(a)).$$

Now, let us give a lemma that will be used frequently in further theorems.

Lemma 3.2. *Let $\gamma>-1$, $s>0$. Then,*

$$\mathfrak{L}\left\{x^{\beta-1}{}_0\Psi_1(\gamma,\beta;\lambda x^\gamma)\right\}=s^{-\beta}\exp\left(\frac{\lambda}{s^\gamma}\right). \tag{3.3}$$

Proof. Using the definition of LT, we have

$$\begin{aligned}\mathfrak{L}\left\{x^{\beta-1}{}_0\Psi_1(\gamma,\beta;\lambda x^\gamma)\right\}&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\gamma n+\beta)}\frac{\lambda^n}{n!}\mathfrak{L}\left\{x^{\gamma n+\beta-1}\right\}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\gamma n+\beta)}\frac{\lambda^n}{n!}\frac{\Gamma(\gamma n+\beta)}{s^{\gamma n+\beta}}\\&=s^{-\beta}\sum_{n=0}^{\infty}\left(\frac{\lambda}{s^\gamma}\right)^n\frac{1}{n!}\\&=s^{-\beta}\exp\left(\frac{\lambda}{s^\gamma}\right).\end{aligned}$$

□

Theorem 3.3. *Let $0<\alpha,\varepsilon<1$, $\gamma>-1,\beta>0$, $s>0$. Then,*

$$\mathfrak{L}\left\{\left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x)\right\}=\frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}s^{1-\beta}F(s)\exp\left(-\frac{\alpha}{\varepsilon s^\gamma}\right). \tag{3.4}$$

Proof. Using the definition of ΨRL fractional operator and considering Eqs. (2.2) and (3.3), we get

$$\begin{aligned}
\mathfrak{L}\left\{\left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x)\right\} &= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}\mathfrak{L}\left\{\frac{d}{dx}\int_0^x (x-t)^{\beta-1}f(t)\,{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right)dt\right\}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}s\mathfrak{L}\left\{\int_0^x (x-t)^{\beta-1}f(t)\,{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right)dt\right\}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}sF(s)\mathfrak{L}\left\{x^{\beta-1}\,{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha x^{\gamma}}{\varepsilon}\right)\right\}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}sF(s)\sum_{n=0}^{\infty}\frac{1}{\Gamma(\gamma n+\beta)}\left(-\frac{\alpha}{\varepsilon}\right)^n\frac{1}{n!}\mathfrak{L}\left\{x^{\gamma n+\beta-1}\right\}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}sF(s)\sum_{n=0}^{\infty}\frac{1}{\Gamma(\gamma n+\beta)}\left(-\frac{\alpha}{\varepsilon}\right)^n\frac{1}{n!}\frac{\Gamma(\gamma n+\beta)}{s^{\gamma n+\beta}}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}s^{1-\beta}F(s)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right)^n\frac{1}{n!}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}s^{1-\beta}F(s)\exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right).
\end{aligned}$$

□

Theorem 3.4. *Let* $0<\alpha,\varepsilon<1$, $\gamma>-1$, $\beta>0$, $s>0$. *Then,*

$$\mathfrak{L}\left\{\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x)\right\} = \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}(sF(s)-f(0))s^{-\beta}\exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right). \tag{3.5}$$

Proof. Using the definition of ΨC fractional operator and considering Eqs. (2.2) and (3.3), we obtain

$$\begin{aligned}
\mathfrak{L}\left\{\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x)\right\} &= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}\mathfrak{L}\left\{\int_0^x (x-t)^{\beta-1}f'(t)\,{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right)dt\right\}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}\mathfrak{L}\left\{f'(t)\right\}\mathfrak{L}\left\{x^{\beta-1}\,{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha x^{\gamma}}{\varepsilon}\right)\right\}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}(sF(s)-f(0))\sum_{n=0}^{\infty}\frac{1}{\Gamma(\gamma n+\beta)}\left(-\frac{\alpha}{\varepsilon}\right)^n\frac{1}{n!}\mathfrak{L}\left\{x^{\gamma n+\beta-1}\right\}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}(sF(s)-f(0))\sum_{n=0}^{\infty}\frac{1}{\Gamma(\gamma n+\beta)}\left(-\frac{\alpha}{\varepsilon}\right)^n\frac{1}{n!}\frac{\Gamma(\gamma n+\beta)}{s^{\gamma n+\beta}}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}(sF(s)-f(0))s^{-\beta}\sum_{n=0}^{\infty}\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right)^n\frac{1}{n!}\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}(sF(s)-f(0))s^{-\beta}\exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right).
\end{aligned}$$

□

Theorem 3.5. *Let* $0<\alpha,\varepsilon<1$, $x\geq 0$, $\gamma>-1$, $\beta>0$. *Then,*

$$\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x) = \left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x) - \frac{N(\alpha,\beta)f(0)}{\varepsilon\,\Gamma(\beta)}x^{\beta-1}{}_0\Psi_1\left(\gamma,\beta;-\frac{\alpha x^{\gamma}}{\varepsilon}\right).$$

Proof. Using Eq. (3.5) and considering Eq. (3.4), we have

$$\begin{aligned}
\mathfrak{L}\left\{\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x)\right\} &= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}(sF(s)-f(0))s^{-\beta}\exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right)\\
&= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)}s^{1-\beta}F(s)\exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right) - \frac{N(\alpha,\beta)f(0)}{\varepsilon\,\Gamma(\beta)}s^{-\beta}\exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right)\\
&= \mathfrak{L}\left\{\left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon}f\right](x)\right\} - \frac{N(\alpha,\beta)f(0)}{\varepsilon\,\Gamma(\beta)}s^{-\beta}\exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right).
\end{aligned} \tag{3.6}$$

Application of the ILT to Eq. (3.6) gives

$$\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) = \left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) - \frac{N(\alpha,\beta) f(0)}{\varepsilon\,\Gamma(\beta)} \mathfrak{L}^{-1}\left\{ s^{-\beta} \exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right)\right\}. \tag{3.7}$$

Taking $\lambda = -\frac{\alpha}{\varepsilon}$ in Eq. (3.3) and applying the ILT to both sides, and then using in Eq. (3.7), we obtain

$$\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) = \left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) - \frac{N(\alpha,\beta) f(0)}{\varepsilon\,\Gamma(\beta)} x^{\beta-1}\,{}_0\Psi_1\left(\gamma,\beta; -\frac{\alpha x^{\gamma}}{\varepsilon}\right), \tag{3.8}$$

which completes the proof. □

Remark 3.6. If we take $f(0) = 0$ in Eq. (3.8), we get that the ΨRL and ΨC fractional operators are equal.

Theorem 3.7. *Let $0 < \alpha, \varepsilon < 1$, $x \geq 0$, $\gamma > -1$, $\beta > 0$. Then, the solution of fractional differential equation*

$$\left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) = g(x)$$

can be found as

$$f(x) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x g(t)(x-t)^{-\beta}\,{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right) dt. \tag{3.9}$$

Proof. Application of the LT to the fractional differential equation gives

$$G(s) = \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)} s^{1-\beta} F(s) \exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right).$$

Then,

$$F(s) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} G(s) s^{\beta-1} \exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right). \tag{3.10}$$

Taking $H(s) = s^{\beta-1}\exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right)$ in Eq. (3.10), we obtain

$$F(s) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} G(s) H(s). \tag{3.11}$$

Application of the ILT to Eq. (3.11) gives

$$f(x) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \mathfrak{L}^{-1}\left\{G(s)H(s)\right\}. \tag{3.12}$$

Writing $\beta \to 1-\beta$ and $\lambda \to \frac{\alpha}{\varepsilon}$ in Eq. (3.3), we have

$$\mathfrak{L}\left\{ x^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha x^{\gamma}}{\varepsilon}\right)\right\} = s^{\beta-1}\exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right). \tag{3.13}$$

Application of the ILT to Eq. (3.13) gives

$$\begin{aligned} x^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha x^{\gamma}}{\varepsilon}\right) &= \mathfrak{L}^{-1}\left\{ s^{\beta-1}\exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right)\right\} \\ &= \mathfrak{L}^{-1}\left\{H(s)\right\} \\ &= h(x). \end{aligned} \tag{3.14}$$

Using Eqs. (2.3) and (2.1) in Eq. (3.12) and considering Eq. (3.14), we have

$$\begin{aligned} f(x) &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} (g(x) * h(x)) \\ &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x g(t)h(x-t)dt \\ &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x g(t)(x-t)^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right) dt. \end{aligned}$$

□

Theorem 3.8. *Let* $0 < \alpha, \varepsilon < 1$*,* $x \geq 0$*,* $\gamma > -1$*,* $\beta > 0$*. Then, the solution of fractional differential equation*

$$\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) = g(x)$$

can be found as

$$f(x) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x g(t)(x-t)^{-\beta}\, {}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right) dt + f(0). \tag{3.15}$$

Proof. Application of the LT to the fractional differential equation gives

$$\begin{aligned} G(s) &= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)} (sF(s) - f(0)) s^{-\beta} \exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right) \\ &= \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)} s^{1-\beta} \exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right) F(s) - \frac{N(\alpha,\beta)}{\varepsilon\,\Gamma(\beta)} s^{-\beta} f(0) \exp\left(-\frac{\alpha}{\varepsilon s^{\gamma}}\right). \end{aligned}$$

Then,

$$F(s) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} G(s) s^{\beta-1} \exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right) + \frac{f(0)}{s}. \tag{3.16}$$

Application of the ILT to Eq. (3.16) gives

$$\mathfrak{L}^{-1}\{F(s)\} = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \mathfrak{L}^{-1}\left\{G(s) s^{\beta-1} \exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right)\right\} + f(0)\mathfrak{L}^{-1}\left\{\frac{1}{s}\right\}. \tag{3.17}$$

Taking $H(s) = s^{\beta-1} \exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right)$ and using Eqs. (2.3) and (2.1) in Eq. (3.17), we have

$$\begin{aligned} f(x) &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} (g(x) * h(x)) + f(0) \\ &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x g(t) h(x-t) dt + f(0) \\ &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x g(t)(x-t)^{-\beta}\, {}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right) dt + f(0). \end{aligned}$$

□

Remark 3.9. If we take $f(0) = 0$ in Eq. (3.15), we get that Eqs. (3.9) and (3.15) are equal.

Corollary 3.10. *Let* $0 < \alpha, \varepsilon < 1$*,* $x \geq 0$*,* $\gamma > -1$*,* $\beta > 0$*. Then, the functions* $f_1(x) = 0$ *and* $f_2(x) = f(0)$ *are the solutions of the following differential equations, respectively*:

$$\left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f_1\right](x) = 0,$$

and

$$\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f_2\right](x) = 0.$$

Proof. The desired results are easily obtained by choosing $g(x) = 0$ in Eqs. (3.9) and (3.15). □

Theorem 3.11. *Let g be a differentiable function and the integral* $\int_0^x g(t)dt$ *is valid. Then, the following equation holds true for* $0 < \alpha, \beta, \varepsilon < 1$*,* $x \geq 0$*,* $\gamma > -1$:

$$\int_0^x \left[{}^{\Psi RL}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} g\right](t) dt = \left[{}^{\Psi C}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} \int_0^x g(t) dt\right](x). \tag{3.18}$$

Proof. Rewriting Eq. (3.9), we have

$$f(x) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x g(t)(x-t)^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right) dt. \tag{3.19}$$

Taking the differential of Eq. (3.19), we get

$$\begin{aligned} f'(x) &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \frac{d}{dx} \int_0^x g(t)(x-t)^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right) dt \\ &= \frac{\varepsilon^2\,\Gamma(\beta)\,\Gamma(1-\beta)}{N(\alpha,\beta)N(-\alpha,1-\beta)} \left[{}^{\Psi RL}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} g\right](x). \end{aligned} \tag{3.20}$$

Taking the integral of Eq. (3.20) and considering the formula $\int_0^x f'(t)dt = f(x) - f(0)$, we obtain

$$f(x) = \frac{\varepsilon^2\,\Gamma(\beta)\,\Gamma(1-\beta)}{N(\alpha,\beta)N(-\alpha,1-\beta)} \int_0^x \left[{}^{\Psi RL}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} g\right](t)dt + f(0). \tag{3.21}$$

Let $v(x) = \int_0^x g(t)dt$ and $v'(x) = g(x)$. Then, substituting in Eq. (3.15), we have

$$\begin{aligned} f(x) &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x v'(t)(x-t)^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^\gamma}{\varepsilon}\right)dt + f(0) \\ &= \frac{\varepsilon^2\,\Gamma(\beta)\,\Gamma(1-\beta)}{N(\alpha,\beta)N(-\alpha,1-\beta)} \left[{}^{\Psi C}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} v\right](x) + f(0) \\ &= \frac{\varepsilon^2\,\Gamma(\beta)\,\Gamma(1-\beta)}{N(\alpha,\beta)N(-\alpha,1-\beta)} \left[{}^{\Psi C}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} \int_0^x g(t)dt\right](x) + f(0). \end{aligned} \tag{3.22}$$

Considering Eqs. (3.21) and (3.22) together, we get

$$\int_0^x \left[{}^{\Psi RL}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} g\right](t)dt = \left[{}^{\Psi C}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} \int_0^x g(t)dt\right](x).$$

□

Theorem 3.12. *Let* $0 < \alpha, \beta, \varepsilon < 1$, $x \geq 0$, $\gamma > -1$. *Then,*

$$\int_0^x \left[{}^{\Psi RL}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} g'\right](t)dt = \left[{}^{\Psi C}D_{0^+}^{-\alpha,1-\beta,\gamma,\varepsilon} g\right](x).$$

Proof. The desired result is obtained substituting g' for g in Eq. (3.18). □

4. Solutions of Differential Equations Involving New Fractional Derivative Operators

In this section, as examples, we find the solution of two fractional differential equations, for both of the fractional operators. The theorems given above will be used to obtain the solutions of differential equations.

Example 4.1. Let $\gamma, \rho > -1$, $0 < \alpha, \varepsilon < 1$, $\beta > 0$, $x \geq 0$. Assume that, the fractional differential equation

$$\left[{}^{\Psi RL}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) = x^\rho$$

is given. Taking $g(t) = t^\rho$ in Eq. (3.9), we have

$$f(x) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x t^\rho (x-t)^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^\gamma}{\varepsilon}\right)dt. \tag{4.1}$$

Application of the LT to Eq. (4.1) gives

$$\begin{aligned} \mathfrak{L}\{f(x)\} &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \mathfrak{L}\{x^\rho\}\, \mathfrak{L}\left\{x^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha x^\gamma}{\varepsilon}\right)\right\} \\ &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \frac{\Gamma(\rho+1)}{s^{\rho+1}} s^{\beta-1} \exp\left(\frac{\alpha}{\varepsilon s^\gamma}\right) \\ &= \frac{\varepsilon\,\Gamma(\beta)\,\Gamma(\rho+1)}{N(\alpha,\beta)} s^{\beta-\rho-2} \exp\left(\frac{\alpha}{\varepsilon s^\gamma}\right). \end{aligned} \tag{4.2}$$

Application of the ILT to Eq. (4.2) gives

$$\begin{aligned} f(x) &= \frac{\varepsilon\,\Gamma(\beta)\,\Gamma(\rho+1)}{N(\alpha,\beta)} \mathfrak{L}^{-1}\left\{s^{\beta-\rho-2} \exp\left(\frac{\alpha}{\varepsilon s^\gamma}\right)\right\} \\ &= \frac{\varepsilon\,\Gamma(\beta)\,\Gamma(\rho+1)}{N(\alpha,\beta)} x^{\rho-\beta+1}\, {}_0\Psi_1\left(\gamma, \rho-\beta+2; \frac{\alpha x^\gamma}{\varepsilon}\right). \end{aligned} \tag{4.3}$$

Remark 4.2. If we take $\rho = \beta$ in Eq. (4.3), we have

$$f(x) = \frac{\varepsilon x\,\Gamma(\beta)\,\Gamma(\beta+1)}{N(\alpha,\beta)}\, {}_0\Psi_1\left(\gamma, 2; \frac{\alpha x^\gamma}{\varepsilon}\right).$$

Example 4.3. Let $\gamma, \rho > -1$, $0 < \alpha, \varepsilon < 1$, $\beta > 0$, $x \geq 0$. We consider the fractional differential equation

$$\left[{}^{\Psi C}D_{0^+}^{\alpha,\beta,\gamma,\varepsilon} f\right](x) = x^{\rho},$$

with the initial condition

$$f(0) = c,$$

where c is constant. Taking $g(t) = t^{\rho}$ in Eq. (3.15), we get

$$f(x) = \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \int_0^x t^{\rho}(x-t)^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha(x-t)^{\gamma}}{\varepsilon}\right) dt + f(0). \tag{4.4}$$

Application of the LT to Eq. (4.4) gives

$$\begin{aligned}\mathfrak{L}\{f(x)\} &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \mathfrak{L}\{x^{\rho}\}\,\mathfrak{L}\left\{x^{-\beta}{}_0\Psi_1\left(\gamma, 1-\beta; \frac{\alpha x^{\gamma}}{\varepsilon}\right)\right\} + \mathfrak{L}\{c\} \\ &= \frac{\varepsilon\,\Gamma(\beta)}{N(\alpha,\beta)} \frac{\Gamma(\rho+1)}{s^{\rho+1}} s^{\beta-1} \exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right) + \frac{c}{s}.\end{aligned} \tag{4.5}$$

Application of the ILT to Eq. (4.5) gives

$$\begin{aligned}f(x) &= \frac{\varepsilon\,\Gamma(\beta)\,\Gamma(\rho+1)}{N(\alpha,\beta)} \mathfrak{L}^{-1}\left\{s^{\beta-\rho-2}\exp\left(\frac{\alpha}{\varepsilon s^{\gamma}}\right)\right\} + \mathfrak{L}^{-1}\left\{\frac{c}{s}\right\} \\ &= \frac{\varepsilon\,\Gamma(\beta)\,\Gamma(\rho+1)}{N(\alpha,\beta)} x^{\rho-\beta+1}\,{}_0\Psi_1\left(\gamma, \rho-\beta+2; \frac{\alpha x^{\gamma}}{\varepsilon}\right) + c.\end{aligned} \tag{4.6}$$

Remark 4.4. If we take $\rho = \beta$ in Eq. (4.6), we get

$$f(x) = \frac{\varepsilon x\,\Gamma(\beta)\,\Gamma(\beta+1)}{N(\alpha,\beta)}\,{}_0\Psi_1\left(\gamma, 2; \frac{\alpha x^{\gamma}}{\varepsilon}\right) + c.$$

5. Conclusion

In this paper, we defined fractional derivatives operators ΨRL (3.1), ΨC (3.2), which have a Wright function (2.6) in their kernels. Since the Wright function has a more general form then most of the special functions, many fractional derivatives becomes the special cases of the fractional derivatives introduced here.

Some of the popular definitions of fractional derivatives, which recently defined, are given below.

Caputo-Fabrizio [12]:

$$\left[\mathscr{D}_x^{(\alpha)} f\right](x) = \frac{M(\alpha)}{(1-\alpha)} \int_a^x f'(\tau) \exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right) d\tau. \tag{5.1}$$

Losada-Nieto [21]:

$$\left[{}^{CF}D^{\alpha} f\right](x) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^x f'(\tau) \exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right) d\tau. \tag{5.2}$$

Yang-Srivastava-Machado [30]:

$$\left[D_{a^+}^{(\alpha)} f\right](x) = \frac{R(\alpha)}{(1-\alpha)} \frac{d}{dx} \int_a^x f(\tau) \exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right) d\tau. \tag{5.3}$$

Atangana-Baleanu [10]:

$$\left[{}^{ABR}_{\ \ a}D_x^{\alpha} f\right](x) = \frac{B(\alpha)}{(1-\alpha)} \frac{d}{dx} \int_a^x f(\tau) E_{\alpha}\left(-\frac{\alpha(x-\tau)^{\alpha}}{1-\alpha}\right) d\tau, \tag{5.4}$$

$$\left[{}^{ABC}_{\ \ a}D_x^{\alpha} f\right](x) = \frac{B(\alpha)}{(1-\alpha)} \int_a^x f'(\tau) E_{\alpha}\left(-\frac{\alpha(x-\tau)^{\alpha}}{1-\alpha}\right) d\tau. \tag{5.5}$$

Gomez-Atangana [16]:

$$\left[{}^{GAR}_{\ a}D_x^{\alpha,\gamma} f\right](x) = \frac{M(\alpha)}{(n-\alpha)\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_a^x f(\tau)(x-\tau)^{n-\gamma-1} \exp\left(-\frac{\alpha(x-\tau)}{n-\alpha}\right) d\tau, \tag{5.6}$$

$$\left[{}^{GAC}_{\ a}D_x^{\alpha,\gamma} f\right](x) = \frac{M(\alpha)}{(n-\alpha)\Gamma(n-\gamma)} \int_a^x f^{(n)}(\tau)(x-\tau)^{n-\gamma-1} \exp\left(-\frac{\alpha(x-\tau)}{n-\alpha}\right) d\tau. \tag{5.7}$$

İlhan [17]:

$$\left[{}^{FR}D_{a^+}^{\alpha,\beta,\gamma} f\right](x) = \frac{N(\alpha,\beta)}{\Gamma(\beta)(1-\alpha)} \frac{d}{dx} \int_a^x f(\tau)(x-\tau)^{\beta-1} {}_1F_1\left(\gamma;\beta;-\frac{\alpha(x-\tau)}{1-\alpha}\right) d\tau, \tag{5.8}$$

$$\left[{}^{FC}D_{a^+}^{\alpha,\beta,\gamma} f\right](x) = \frac{N(\alpha,\beta)}{\Gamma(\beta)(1-\alpha)} \int_a^x f'(\tau)(x-\tau)^{\beta-1} {}_1F_1\left(\gamma;\beta;-\frac{\alpha(x-\tau)}{1-\alpha}\right) d\tau. \tag{5.9}$$

We present the relations between ΨRL and ΨC fractional operators and the fractional operators given above, in Table 1.

Table 1. Relationships of Fractional Operators

Riemann-Liouville	(2.4)	$\left[{}^{\Psi RL}D_{a^+}^{0,\beta,\gamma,\varepsilon} f\right](x) \overset{(n=1)}{=} \frac{N(0,\beta)}{\varepsilon\,\Gamma(\beta)} \left[D_{a^+}^{1-\beta} f\right](x)$
Caputo	(2.5)	$\left[{}^{\Psi C}D_{a^+}^{0,\beta,\gamma,\varepsilon} f\right](x) \overset{(n=1)}{=} \frac{N(0,\beta)}{\varepsilon\,\Gamma(\beta)} \left[{}^{C}D_{a^+}^{1-\beta} f\right](x)$
Caputo-Fabrizio	(5.1)	$\left[{}^{\Psi C}D_{a^+}^{0,1,\gamma,\varepsilon} f\right](x) = \frac{N(0,1)}{\varepsilon M(0)} \left[\mathscr{D}_x^{(0)} f\right](x)$
Losada-Nieto	(5.2)	$\left[{}^{\Psi C}D_{0}^{0,1,\gamma,\varepsilon} f\right](x) = \frac{N(0,1)}{\varepsilon M(0)} \left[{}^{CF}D^{0} f\right](x)$
Yang and et. al.	(5.3)	$\left[{}^{\Psi RL}D_{a^+}^{0,1,\gamma,\varepsilon} f\right](x) = \frac{N(0,1)}{\varepsilon R(0)} \left[D_{a^+}^{(0)} f\right](x)$
Atangana-Baleanu	(5.4)	$\left[{}^{\Psi RL}D_{a^+}^{0,1,\gamma,\varepsilon} f\right](x) = \frac{N(0,1)}{\varepsilon B(0)} \left[{}^{ABR}_{\ a}D_x^{0} f\right](x)$
Atangana-Baleanu	(5.5)	$\left[{}^{\Psi C}D_{a^+}^{0,1,\gamma,\varepsilon} f\right](x) = \frac{N(0,1)}{\varepsilon B(0)} \left[{}^{ABC}_{\ a}D_x^{0} f\right](x)$
Gomez-Atangana	(5.6)	$\left[{}^{\Psi RL}D_{a^+}^{0,\beta,\gamma,\varepsilon} f\right](x) \overset{(n=1)}{=} \frac{N(0,\beta)}{\varepsilon\Gamma(\beta)M(0)} \left[{}^{GAR}_{\ a}D_x^{0,1-\beta} f\right](x)$
Gomez-Atangana	(5.7)	$\left[{}^{\Psi C}D_{a^+}^{0,\beta,\gamma,\varepsilon} f\right](x) \overset{(n=1)}{=} \frac{N(0,\beta)}{\varepsilon\Gamma(\beta)M(0)} \left[{}^{GAC}_{\ a}D_x^{0,1-\beta} f\right](x)$
İlhan	(5.8)	$\left[{}^{\Psi RL}D_{a^+}^{0,\beta,\gamma,\varepsilon} f\right](x) = \frac{1}{\varepsilon\Gamma(\beta)} \left[{}^{FR}D_{a^+}^{0,\beta,\gamma} f\right](x)$
İlhan	(5.9)	$\left[{}^{\Psi C}D_{a^+}^{0,\beta,\gamma,\varepsilon} f\right](x) = \frac{1}{\varepsilon\Gamma(\beta)} \left[{}^{FC}D_{a^+}^{0,\beta,\gamma} f\right](x)$

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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