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Blow-up for a Generalized Dullin-Gottwald-Holm Equation

Nurhan Dündar

Abstract
In this paper, the blow up of solutions for a generalized version of the Dullin-Gottwald-Holm equation which is a nonlinear shallow water wave equation is studied. The precise blow-up scenario and a result of blow-up solutions are described. The blow-up occurs as wave breaking. This means the solution (representing the wave) remains bounded but its slope becomes infinite in finite time. We use an approach devised in [1].

Keywords: *Generalized Dullin-Gottwald-Holm equation; shallow water wave; blow-up.*

AMS Subject Classification (2020): *Primary: 35B44 ; Secondary: 35Q35; 35G20.*

1. Introduction

Dullin et al. in [3] presented the following the nonlinear dispersive evolution equation, then called the Dullin-Gottwald-Holm (DGH) equation:

$$u_t - \beta^2 u_{xxt} + k_0 u_x + 3uu_x + \Gamma u_{xxx} = \beta^2 \left(2u_x u_{xx} + u u_{xxx}\right), \qquad t > 0,\ x \in \Re. \tag{1.1}$$

The DGH equation is an equation modeling the unidirectional propagation of shallow water waves on a flat bottom. $u = u\,(t, x)$ is fluid velocity, where t and x are variables related to time and space respectively. β, Γ and k_0 are some physical positive parameters.

In equation (1.1), if $\beta = 0$ and $\Gamma \neq 0$, the Korteweg-de Vries (KdV) equation is obtained, and if $\beta = 1$ and $\Gamma = 0$, the Camassa-Holm (CH) equation is obtained. As can be seen, equation (1.1) contains two different integrable soliton equations for shallow water waves. The DGH equation (1.1) combines the linear dispersion of the KdV equation with the nonlinear/nonlocal dispersion of the CH equation. Equation (1.1) has important properties. Some of these important features are: It has the bi-Hamiltonian structure and soliton solutions and it is completely integrable [3]. For this equation, blow up occurs in the form of wave breaking: This means: while the solution u representing the wave remains bounded, u_x, which is its first derivative with respect to x becomes infinite in finite time [1, 12, 15].

Since the equation (1.1) was discovered, a great deal of space has been devoted to it in the literature and this equation has been the subject of intense research. Its mathematical behaviors such as local well-posedness, global

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strong solutions, global weak solutions, blow up solutions in finite time and stability of peakons have been studied in many works [8, 11, 12, 15–18].

In present paper, we study the following initial value problem for the generalized DGH equation:

$$\begin{cases} u_t - \beta^2 u_{xxt} + (P(u))_x + \Gamma u_{xxx} = \beta^2 \left(\frac{Q'(u)}{2} u_x^2 + Q(u) u_{xx}\right)_x, & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re, \end{cases} \tag{1.2}$$

where $P(u),\ Q(u)\ : \Re \to \Re$ are given C^3-functions. For $P(u) = 2\omega u + \frac{3}{2}u^2$ (where $2\omega = k_0$) and $Q(u) = u$, it is seen that the (1.2) turns into equation equation (1.1). Some mathematical behaviors of equation (1.2) have been studied by many authors before. In [13, 14], the authors established the well-posedness a finite time for (1.2) by using Kato's theory. Furthermore, the stability of peakons of (1.2) was discussed with $P(u) = 2\omega u + \frac{a+2}{2}u^{a+1}$ and $Q(u) = u^a$ in [13]. In [4], Dündar and Polat investigated the blow up of the solutions of (1.2) with $Q(u) = u$. Also in the same article, they proved stability of solitary waves by using the method in [7] for $P(u) = 2\omega u + \frac{a+2}{2}u^{a+1}$ and $Q(u) = u^a$.

In (1.2), if the weak dispersive term Γu_{xxx} is changed into the strong dispersive term $\Gamma\left(u - \beta^2 u_{xx}\right)_{xxx}$, we obtain

$$\begin{cases} u_t - \beta^2 u_{xxt} + (P(u))_x + \Gamma\left(u - \beta^2 u_{xx}\right)_{xxx} = \beta^2 \left(\frac{Q'(u)}{2} u_x^2 + Q(u) u_{xx}\right)_x, & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re. \end{cases} \tag{1.3}$$

Dündar and Polat studied the well-posedness for (1.3) a finite time in [6]. Also, they showed the existence of solitary waves and proved the stability of solitary wave solutions of (1.3) in [5].

The main aim of this paper is to investigate the blow up of the solutions of (1.2) in finite time. In [4], authors obtained the blow up of the strong solutions of (1.2) with $Q(u) = u$. In this paper, we remove this restriction and obtain more general results.

The content of this article is as follows: In Section 2, we will give the notations and some basic informations, and recall some necessary conclusions. In Section 3, we will examine the blow up of solutions of (1.2).

2. Preliminaries

We introduce by summarizing some notations. The convolution is denoted by $*$. $\|.\|_{\mathfrak{B}}$ denotes the norm of Banach space $\mathfrak{B}$. Since all space of functions are over $\Re$, for convenience, we will not use $\Re$ in our notations of function spaces if there is no equivocalness. We denote the norm in the Sobolev space H^s by

$$\|v\|_s = \|v\|_{H^s} = \left(\int_{\Re} \left(1 + |\xi|^2\right)^s |\widehat{v}(\xi)|^2 d\xi\right)^{1/2}$$

for $s \in \Re$. Here $\widehat{v}(\xi)$ is the Fourier transform of v. We use the $\|.\|_{L^p}$ for the norm of the space L^p, $1 \le p \le \infty$. We define the operator Λ^s by the formula $\Lambda^s = \left(1 - \partial_x^2\right)^{\frac{s}{2}}$, $s \in \Re$.

From now on, throughout this article, we assume $\beta = 1$ for convenience. Note that if $f(x) = \frac{1}{2}e^{-|x|}$, $x \in \Re$, then $\left(1 - \partial_x^2\right)^{-1} v = f * v$ for all $v \in L^2$. Then (1.2) can be rewritten as follows:

$$\begin{cases} u_t + (Q(u) - \Gamma) u_x = f * [Q(u) u_x] - \partial_x f * \left[\frac{Q'(u)}{2} u_x^2 + P(u) + \Gamma u\right], & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re. \end{cases} \tag{2.1}$$

Or in the equivalent form:

$$\begin{cases} u_t + (Q(u) - \Gamma) u_x = \left(1 - \partial_x^2\right)^{-1} [Q(u) u_x] - \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\frac{Q'(u)}{2} u_x^2 + P(u) + \Gamma u\right], & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re. \end{cases} \tag{2.2}$$

It can be seen that (1.2) is equivalent to (2.1) (or (2.2)) for $\beta = 1$. So, we will investigate the blow up of solutions of (2.1) (or (2.2)).

2.1 Local well-posedness for the Cauchy problem of (2.1)

Theorem 2.1. *[14]. Let $n \geq 2$ be a natural number, $s \in \left(\frac{3}{2}, n\right)$, and $P, Q \in C^{n+3}$, with $P(0) = 0$. If $u_0 \in H^s$, there exists a maximal $T = T(u_0) > 0$, and a unique solution u to (2.1) (or (2.2)) such that*

$$u = u(., u_0) \in C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \to u(., u_0) : H^s \to C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right)$$

is continuous.

In [13], Liu and Yin obtained the local well-posedness theorem of the Cauchy problem (2.1) with the constraint $Q(0) = 0$ by applying Kato's theory [10]. Later, in [14] (Theorem 1.2 and Corollary 1.1), the authors removed the limiting condition $Q(0) = 0$, which makes an improvement in the results in [13].

Theorem 2.2. *Let $n \geq 2$ be a natural number, $s \in \left(\frac{3}{2}, n\right)$, and $P, Q \in C^{n+3}$, with $P(0) = 0$. Then T in Theorem 2.1 may be chosen independent of s in the following sense. If*

$$u = u(., u_0) \in C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right)$$

to 2.1 (or 2.2), and if $u_0 \in H^{s'}$ for some $s' \neq s$, $\frac{3}{2} < s' < n$, then

$$u \in C\left([0, T); H^{s'}\right) \cap C^1\left([0, T); H^{s'-1}\right)$$

and with the same T. In particular, if $P,\ Q \in C^\infty$ and let $u_0 \in H^\infty = \cap_{s \geq 0} H^s$, $u \in C([0, T); H^\infty)$.

Proof. For $\beta = 1$, since (1.2) can be rewritten as

$$\frac{dw}{dt} + K(t)\, w + L(t)\, w = R(t), \qquad w(0) = \Lambda^2 u(0),$$

where

$$K(t)\, w = \partial_x((Q(u) - \Gamma)\, w), \qquad L(t)\, w = Q'(u)\, u_x w,$$

and

$$R(t) = u_x\left(\frac{1}{2}Q''(u)\, u_x^2 - P'(u) + 2Q'(u)\, u + Q(u) - \Gamma\right),$$

thus the proof of Theorem 2.2 is alike to the proof of Theorem 1.2 of [6]. The proof is completed with reference the proof of Theorem 1.2 in [6].

2.2 Some lemmas

We will now give some lemmas that we will use in this paper. We list below without proof.

Lemma 2.1. *[9]. Let $s > 0$. Then we have*

$$\|[\Lambda^s, y]\, z\|_{L^2} \leq K\left(\|\partial_x y\|_{L^\infty} \left\|\Lambda^{s-1} z\right\|_{L^2} + \|\Lambda^s y\|_{L^2} \|z\|_{L^\infty}\right).$$

Here K is constant depending only on s.

Lemma 2.2. *[9]. Let $s > 0$. Then $H^s \cap L^\infty$ is an algebra. Moreover*

$$\|yz\|_s \leq K\left(\|y\|_{L^\infty} \|z\|_s + \|y\|_s \|z\|_{L^\infty}\right)$$

where K is constant depending only on s.

Lemma 2.3. *[2]. Assume that $G \in C^{n+2}$ with $G(0) = 0$. Then for every $\frac{1}{2} < s \leq n$, we have that*

$$\|G(u)\|_s \leq \tilde{G}\left(\|u\|_{L^\infty}\right) \|u\|_s, \qquad u \in H^s,$$

where $\tilde{G}$ is a monotone increasing function depending only on G and s.

Lemma 2.4. *[1]. Let $T > 0$ and $u \in C^1\left([0,T);H^2\right)$. Then for every $t \in [0,T)$, there exist at least one pair points $\theta(t)$, $\Theta(t) \in \Re$, such that*

$$j(t) = \inf_{x\in\Re} u_x(t,x) = u_x(t,\theta(t)), \qquad J(t) = \sup_{x\in\Re} u_x(t,x) = u_x(t,\Theta(t)),$$

and $j(t)$, $J(t)$ are absolutely continuous on $(0,T)$. Furthermore,

$$\frac{dj(t)}{dt} = u_{tx}(t,\theta(t)), \qquad \frac{dJ(t)}{dt} = u_{tx}(t,\Theta(t)), \qquad a.e.on\,(0,T).$$

Lemma 2.5. *[13]. Let $u(t,x)$ be a solution of (1.2). Then the functionals*

$$\mathcal{E}(u) = \int_\Re \left(u^2 + \beta^2 u_x^2\right) dx,$$

$$\mathcal{F}(u) = \int_\Re \left(2\mathfrak{P}(u) + \beta^2 Q(u) u_x^2 - \Gamma u_x^2\right) dx$$

are constant with respect to t, where $\mathfrak{P}'(s) = P(s)$.

3. Blow-up analysis

In this section, we examine the blow-up phenomena of the (2.1) (or (2.2)).

Remark 3.1. Given in Lemma (2.5), $\mathcal{E}(u) = \int_\Re \left(u^2 + u_x^2\right) dx$ ($\beta = 1$) is an invariant for equation (2.1). So, we have that

$$\|u\|_{L^\infty}^2 \le \int_\Re \left(u^2 + u_x^2\right) = \mathcal{E}(u) = \mathcal{E}(u_0) = \|u_0\|_1^2.$$

Remark 3.2. Since $Q \in C^{n+3}$ with $n \ge 2$, by using $\|u\|_{L^\infty} \le \|u\|_1 = \|u_0\|_1$ which can be seen in Remark 3.1, a positive constant $a_1 > 0$ can be found such that

$$|Q'(u)| \le \sup_{|z|\le\|u_0\|_1} |Q'(z)| \le a_1. \tag{3.1}$$

We will first give the following theorem.

Theorem 3.1. *Let $P, Q \in C^{n+3}$, $n \ge 2$, $P(0) = 0$ and $u_0 \in H^s$, $\frac{3}{2} < s \le n$. Then the solution $u(t,x)$ of (2.2) blows up in finite time $T < \infty$ if and only if*

$$\lim_{t\to T} \sup_{0\le\tau\le t} \|u_x(\tau,x)\|_{L^\infty} = +\infty. \tag{3.2}$$

Moreover, if $T < \infty$, then

$$\int_0^T \left(\|u_x(t,x)\|_{L^\infty} + 1\right)^2 dt = +\infty.$$

Proof. Let $\Gamma = Q(0)$. We can rewrite (2.2) as

$$u_t + (Q(u) - Q(0)) u_x = \left(1 - \partial_x^2\right)^{-1} [Q(u) u_x] - \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\frac{Q'(u)}{2} u_x^2 + P(u) + Q(0) u\right]. \tag{3.3}$$

If we apply the operator Λ^s, then multiply by $2\Lambda^s u$ on both sides of (3.3) and finally integrate with respect to the variable x over $\Re$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Re (\Lambda^s u)^2 dx &= -2\int_\Re \Lambda^s u \Lambda^s [(Q(u) - Q(0)) u_x] dx + 2\int_\Re \Lambda^s u \Lambda^s \left(1 - \partial_x^2\right)^{-1} [Q(u) u_x] dx \\ &\quad -2\int_\Re \Lambda^s u \Lambda^s \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\frac{Q'(u)}{2} u_x^2 + P(u) + Q(0) u\right] = I_1 + I_2 + I_3. \end{aligned} \tag{3.4}$$

We now estimate I_1, I_2, I_3. By using Lemma 2.1, Lemma 2.3 with $G(u) = Q(u) - Q(0)$, Remark 3.1 and Cauchy-Schwartz inequality as well as (3.1), we obtain

$$\begin{aligned} I_1 &= -2\int_{\Re} \Lambda^s u \Lambda^s \left[(Q(u) - Q(0)) u_x\right] dx \\ &= -2\int_{\Re} \Lambda^s u \left[\Lambda^s \left[(Q(u) - Q(0)) u_x\right] - (Q(u) - Q(0)) \Lambda^s u_x\right] dx \\ &\quad -2\int_{\Re} (Q(u) - Q(0)) \Lambda^s u \Lambda^s u_x dx \\ &= -2\int_{\Re} \Lambda^s u \left[\Lambda^s, (Q(u) - Q(0))\right] u_x dx - \int_{\Re} Q'(u) u_x (\Lambda^s u)^2 dx \\ &\leq 2K \|u\|_s \left[\|\partial_x (Q(u) - Q(0))\|_{L^\infty} \left\|\Lambda^{s-1} u_x\right\|_{L^2} + \|\Lambda^s (Q(u) - Q(0))\|_{L^2} \|u_x\|_{L^\infty}\right] \\ &\quad + \|u\|_s^2 \|Q'(u) u_x\|_{L^\infty} \\ &\leq K \|u\|_s \left[\|Q'(u) u_x\|_{L^\infty} \|u\|_s + \|(Q(u) - Q(0))\|_s \|u_x\|_{L^\infty}\right] + \|u\|_s^2 \|Q'(u) u_x\|_{L^\infty} \\ &\leq K \|u\|_s^2 \left(2a_1 \|u_x\|_{L^\infty} + \tilde{G}(\|u_0\|_1) \|u_x\|_{L^\infty}\right) \\ &\leq K \|u\|_s^2 \|u_x\|_{L^\infty}. \end{aligned} \tag{3.5}$$

By using Lemma 2.3 with $G(u) = G_1(u) - G_1(0)$ and Remark 3.1, Cauchy-Schwartz inequality and Sobolev embedding ($H^s \hookrightarrow H^{s-1}$), we obtain

$$\begin{aligned} I_2 &= 2\int_{\Re} \Lambda^s u \Lambda^s \left(1 - \partial_x^2\right)^{-1} \left[Q(u) u_x\right] dx \\ &= 2\int_{\Re} \Lambda^s u \Lambda^s \partial_x \left(1 - \partial_x^2\right)^{-1} \left[G_1(u) - G_1(0)\right] dx \quad (\text{where } G_1'(u) = Q(u)) \\ &\leq K \|u\|_s \|G_1(u) - G_1(0)\|_{s-1} \\ &\leq K\tilde{G}(\|u_0\|_1) \|u\|_s \|u\|_{s-1} \\ &\leq K \|u\|_s^2. \end{aligned} \tag{3.6}$$

By using Lemma 2.2, Lemma 2.3 with $G(u) = Q'(u) - Q'(0)$ and Remark 3.1, Cauchy-Schwartz inequality, Sobolev embedding ($H^s \hookrightarrow H^{s-1}$) and (3.1), we obtain

$$\begin{aligned} I_3 &= -2\int_{\Re} \Lambda^s u \Lambda^s \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\frac{Q'(u)}{2} u_x^2 + P(u) + Q(0) u\right] \\ &\leq 2 \|u\|_s \left\|\frac{Q'(u)}{2} u_x^2 + P(u) + Q(0) u\right\|_{s-1} \\ &\leq K \|u\|_s \left[\left\|\left(\frac{Q'(u) - Q'(0) + Q'(0)}{2}\right) u_x^2\right\|_{s-1} + \|P(u)\|_{s-1} + |Q(0)| \|u\|_{s-1}\right] \\ &\leq K \|u\|_s \left[\left\|(Q'(u) - Q'(0)) u_x^2\right\|_{s-1} + |Q'(0)| \left\|u_x^2\right\|_{s-1} + \|u\|_{s-1} + |Q(0)| \|u\|_{s-1}\right] \\ &\leq K \|u\|_s \left[K\left(\|Q'(u) - Q'(0)\|_{L^\infty} \left\|u_x^2\right\|_{s-1} + \left\|u_x^2\right\|_{L^\infty} \|Q'(u) - Q'(0)\|_{s-1}\right) + K\left(\|u\|_s + \|u\|_s \|u_x\|_{L^\infty}\right)\right] \\ &\leq K \|u\|_s \left[\left(\sup_{|z| \leq \|u_0\|_1} |Q'(z)|\right) \|u_x\|_{L^\infty} \|u\|_s + \tilde{G}(\|u_0\|_1) \left\|u_x^2\right\|_{L^\infty} \|u\|_s + (1 + \|u_x\|_{L^\infty}) \|u\|_s\right] \\ &\leq K \|u\|_s^2 \left[a_1 \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + 1 + \|u_x\|_{L^\infty}\right] \\ &\leq K \|u\|_s^2 (1 + \|u_x\|_{L^\infty})^2. \end{aligned} \tag{3.7}$$

Combining (3.5)-(3.7) with (3.4), we get

$$\frac{d}{dt} \|u\|_s^2 \leq K \|u\|_s^2 (1 + \|u_x\|_{L^\infty})^2. \tag{3.8}$$

When we apply Gronwall's inequality to (3.8), we obtain

$$\|u\|_s^2 \leq e^{K\int_0^t \left(\|u_x\|_{L^\infty}+1\right)^2 d\tau} \|u_0\|_s^2 . \tag{3.9}$$

If the solution to (2.2) blows up in finite time, in other words,

$$\lim_{t\to T} \sup_{0\leq\tau\leq t} \|u\|_s = +\infty, \tag{3.10}$$

then from (3.9), we have

$$\lim_{t\to T} \sup_{0\leq\tau\leq t} \|u_x(\tau,x)\|_{L^\infty} = +\infty. \tag{3.11}$$

If (3.11) is valid, since $\|u\|_{L^\infty} \leq \|u\|_{s-1}$ with $s > \frac{3}{2}$, we have (3.10). When the maximal existence time $T < \infty$, if

$$\int_0^T \left(\|u_x(t,x)\|_{L^\infty}+1\right)^2 dt < +\infty,$$

from (3.9), we know that $\|u\|_s < \infty$ which contradicts with the fact that T is the maximal existence time. We get the same result for $\Gamma \neq Q(0)$. We complete the proof of Theorem 3.1.

Theorem 3.2. *Let $P, Q \in C^{n+3}$, $n \geq 3$, $P(0) = 0$. Given $u_0 \in H^s$, $3 \leq s \leq n$. If $Q'(u) \geq a_2 > 0$, then the corresponding $u(t,x)$ of (2.1) blows up in finite time $T < \infty$ if and only if*

$$\lim_{t\to T} \inf_{0\leq\tau\leq t} \inf_{x\in\Re} u_x(\tau,x) = -\infty. \tag{3.12}$$

Proof. If (3.12) is valid, then the corresponding solution $u(t,x)$ of (2.1) blows up in finite time $T < \infty$ since $\|u\|_{L^\infty} \leq \|u\|_{s-1}$ with $s > \frac{3}{2}$. We prove (3.12) by contradiction. Assume that (3.12) is invalid, then there exists $J > 0$ such that $\inf_{x\in\Re} u_x(t,x) > -J$, then we make inference that the solution will not blow up in finite time. Let's take the differentiate of (2.1) with respect to x, so we get

$$u_{tx} + Q'(u)\,u_x^2 + Q(u)\,u_{xx} - \Gamma u_{xx} = \partial_x f * [Q(u)\,u_x] - \partial_x^2 f * \left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right]. \tag{3.13}$$

Since $\partial_x^2(f*v) = f*v - v$ and $\partial_x(f*v) = f*v_x$, we have

$$\begin{aligned} u_{tx} + Q'(u)\,u_x^2 + Q(u)\,u_{xx} - \Gamma u_{xx} &= f*\left[Q'(u)\,u_x^2 + Q(u)\,u_{xx}\right] - f*\left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right] \\ &+ \frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u. \end{aligned} \tag{3.14}$$

From Lemma 2.4, we define

$$J(t) = u_x(t,\Theta(t)) = \sup_{x\in\Re}[u_x(t,x)]$$

and

$$j(t) = u_x(t,\theta(t)) = \inf_{x\in\Re}[u_x(t,x)].$$

Since we deal with a maximum, $u_{xx}(t,\Theta(t)) = 0$ for all $t \in [0,T)$, it follows that a.e. on $[0,T)$

$$\begin{aligned} J'(t) &= -\frac{Q'(u(t,\Theta(t)))}{2}J^2(t) + f*\left[Q'(u)\,u_x^2\right](t,\Theta(t)) + P(u(t,\Theta(t))) + \Gamma u(t,\Theta(t)) \\ &\quad -f*\left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right](t,\Theta(t)). \end{aligned} \tag{3.15}$$

By Young's inequality and $f(x) = \frac{1}{2}e^{-|x|}$, we have

$$\|f*v\|_{L^\infty} \leq \|f\|_{L^\infty}\|v\|_{L^1} \leq \frac{1}{2}\|v\|_{L^1}$$

and

$$\|f*v\|_{L^\infty} \le \|f\|_{L^1} \|v\|_{L^\infty} \le \|v\|_{L^\infty} .$$

By using these inequalities, (3.1) and Remark 3.1, we obtain

$$\begin{aligned} \left\|f*\left(Q'(u)\,u_x^2\right)\right\|_{L^\infty} &\le \|f\|_{L^\infty} \left\|Q'(u)\,u_x^2\right\|_{L^1} \\ &\le \frac{1}{2}\|Q'(u)\|_{L^\infty}\|u_x\|_{L^2}^2 \\ &\le \frac{a_1}{2}\|u\|_1^2 = \frac{a_1}{2}\|u_0\|_1^2 . \end{aligned} \tag{3.16}$$

Similarly, we have

$$\|f*P(u)\|_{L^\infty} \le \|P(u)\|_{L^\infty} \le \sup_{|z|\le\|u_0\|_1} |P(z)| \tag{3.17}$$

and

$$\|f*u\|_{L^\infty} \le \|u\|_{L^\infty} \le \|u_0\|_1 . \tag{3.18}$$

Using (3.16)-(3.18) and the assumption in lemma, it then follows from (3.15) that a.e. on $[0,T)$,

$$J'(t) \le -\frac{a_2}{2}J^2(t) + A, \tag{3.19}$$

where

$$A = 2\left(\sup_{|z|\le\|u_0\|_1}|P(z)| + \frac{3}{8}a_1\|u_0\|_1^2 + \|u_0\|_1\right) \tag{3.20}$$

If $J(t) > \sqrt{\frac{2A}{a_2}}$, then $J'(t) < 0$ and $J(t)$ is decreasing. Otherwise, $J(t) \le \sqrt{\frac{2A}{a_2}}$. Thus we obtain that

$$-J < j(t) \le u_x \le J(t) \le \max\left\{J(0), \sqrt{\frac{2A}{a_2}}\right\}, \qquad t\in[0,T).$$

From this inequality, we obtain the fact that u_x, that is, the slope of solution of (2.1) is bounded. When Theorem 3.1 is applied, the solution of (2.1) will not blow up in finite time. We finish the proof of Theorem 3.2.

Now, we present the following blow up result.

Theorem 3.3. *Assume that $P, Q \in C^{n+3}$, $n \ge 2$, $P(0) = 0$, $u_0 \in H^s$, $\frac{3}{2} < s \le n$, $Q'(u) \ge a_2 > 0$. If there exists a point $x_0 \in \Re$ such that $u_0'(x_0) < -\sqrt{\frac{2A}{a_2}}$, then the corresponding solution $u(t,x)$ of (2.1) blows up in finite time $T < \infty$ and*

$$T < \frac{1}{\sqrt{2Aa_2}} \ln\left(\frac{\sqrt{\frac{a_2}{2}}u_0'(x_0) - \sqrt{A}}{\sqrt{\frac{a_2}{2}}u_0'(x_0) + \sqrt{A}}\right),$$

where

$$A = 2\left(\sup_{|z|\le\|u_0\|_1}|P(z)| + \frac{3}{8}a_1\|u_0\|_1^2 + \|u_0\|_1\right).$$

Proof. By Theorem 2.1- Theorem 2.2 and a simple density argument, we only need to prove that theorem provides for $s = 3$. Let T be maximal existence time of the solution $u \in C([0,T); H^s) \cap C^1\left([0,T); H^{s-1}\right)$ of (2.1). Differentiating (2.1) with respect to x, since $\partial_x^2(f*v) = (f*v - v)$ and $\partial_x(f*v) = f*v_x$, we have

$$\begin{aligned} u_{tx} + Q'(u)\,u_x^2 + Q(u)\,u_{xx} - \Gamma u_{xx} &= f*\left[Q'(u)\,u_x^2 + Q(u)\,u_{xx}\right] - f*\left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right] \\ &+ \frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u. \end{aligned} \tag{3.21}$$

Now define $j(t) = \inf_{x \in R}[u_x(t,x)] = u_x(t,\theta(t))$ by Lemma 2.4 and let $\theta(t) \in \Re$ be a point where this infimum is attained. For $x = \theta(t)$, since $u_{xx}(t,\theta(t)) = 0$, we have

$$\begin{aligned} j'(t) &= -\frac{Q'(u(t,\theta(t)))}{2} j^2(t) + f * \left[Q'(u) u_x^2\right](t,\theta(t)) + P(u(t,\theta(t))) + \Gamma u(t,\theta(t)) \\ &\quad - f * \left[\frac{Q'(u)}{2} u_x^2 + P(u) + \Gamma u\right](t,\theta(t)). \end{aligned} \tag{3.22}$$

Using (3.16)-(3.18) and the assumption in lemma, it then follows from (3.22) that a.e. on $[0,T)$,

$$j'(t) \leq -\frac{a_2}{2} j^2(t) + A, \tag{3.23}$$

where

$$A = 2\left(\sup_{|z| \leq \|u_0\|_1} |P(z)| + \frac{3}{8} a_1 \|u_0\|_1^2 + \|u_0\|_1\right).$$

Note that if $j(0) \leq -\sqrt{\frac{2}{a_2}A}$, then $j(t) \leq -\sqrt{\frac{2}{a_2}A}$, fol all $t \in [0,T)$. By (3.23), we get

$$\frac{\sqrt{\frac{a_2}{2}} j(0) + \sqrt{A}}{\sqrt{\frac{a_2}{2}} j(0) - \sqrt{A}} e^{\sqrt{2a_2 A} t} - 1 \leq \frac{2\sqrt{A}}{\sqrt{\frac{a_2}{2}} j(t) - \sqrt{A}} \leq 0.$$

Due to $0 < \frac{\sqrt{\frac{a_2}{2}} j(0) + \sqrt{A}}{\sqrt{\frac{a_2}{2}} j(0) - \sqrt{A}} < 1$, there exists

$$0 < T < \frac{1}{\sqrt{2Aa_2}} \ln\left(\frac{\sqrt{\frac{a_2}{2}} j(0) - \sqrt{A}}{\sqrt{\frac{a_2}{2}} j(0) + \sqrt{A}}\right)$$

such that $\lim_{t \to T} j(t) = -\infty$. For this reason, the solution u does not exist globally in time. Thus, the proof of Theorem 3.3 is completed.

4. Conclusion

In this study, we investigated the blow up of solutions of the Cauchy problem (2.1) (or (2.2)), which we obtained by taking $\beta = 1$ in (1.2).

Our main results can be summarised as follows:

1. We give the precise blow up scenario for solutions of the Cauchy problem (2.1), see Theorem 3.2.

2. We also give a blow up result of solutions of (2.1), see Theorem 3.3.

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Affiliations

NURHAN DÜNDAR
ADDRESS: The Ministry of National Education, Diyarbakir, Turkey.
E-MAIL: dundarnurhan@gmail.com, nurhandundar@hotmail.com
ORCID ID: 0000-0003-0258-0953