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Characterizations of Adjoint Curves According to Alternative Moving Frame

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Abstract

In this paper, the adjoint curve is defined by using the alternative moving frame of a unit speed space curve in 3-dimensional Euclidean space. The relationships between Frenet vectors and alternative moving frame vectors of the curve are used to offer various characterizations. Besides, ruled surfaces are constructed with the curve and its adjoint curve, and their properties are examined. In the last section, there are examples of the curves and surfaces defined in the previous sections.

1. Introduction

In differential geometry, the theory of curves in the 3-dimensional Euclidean space E^3 is one of the leading fields of study. In terms of curves, the most interesting curves in recent years are helices and slant helices[1, 2]. However, curves associated with a given curve are also widely studied. Among these curves, the most studied are Bertrand curve pairs, Mannheim curve pairs and involute-evolute curve pairs [3, 4]. In addition to the aforementioned pairs of curves, there are associated curves that have gained a lot of popularity. We can list some of them as the principal normal-direction curve, binormal-direction curve, principal-donor curve and binormal-donor curve, which were defined by Choi and Kim in 2012 with the help of integral curves [5]. With the adjoint curves discussed in 2019, a new definition of binormal-direction curves has been introduced. Also in this study, characterizations of adjoint curves and ruled and tube surfaces associated with adjoint curves were studied [6]. The W-direction curves in Macit and Düldül's paper are another reference for integral curves. Here, the relationships between a curve and the integral curve of the vector W of this curve are given. The relationships between the curvatures of the associated curves are explained and the characterizations of the curves are studied [7].
The curves are generally characterized by a moving Frenet frame. However, it may not be possible to obtain characterizations by using this frame or it is difficult to characterize them in some cases. For this reason, it will be useful to examine the curves with the help of another moving frame. In 2016, Yaylı et al. defined an alternative moving frame on the curve in their study [8]. The ruled surfaces introduced by Monge is one of the most frequently studied topics in differential geometry. The ruled surfaces have application areas especially in kinematics, computer-aided geometric design, architecture and many other fields. Any ruled surface occurs as a result of the continuous movement of a line along a curve. The ruled surfaces have been studied in differential geometry in different spaces, different dimensions and different frames [9]-[14].
In this study, a new definition is given to the W-direction curve of a curve in an alternative moving frame. This curve, which is direction curve considered with an alternative moving frame, is defined and characterized as $W-$adjoint curve in E^3. The significant relationships are founded between alternative moving frame apparatus and Frenet frame apparatus of curve pairs occuring. Then, the ruled surfaces associated with these direction curves are studied. The ruled surfaces obtained by different

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variations of the associated curves of the base curve and the direction curve are given. Moreover, Maple software is applied to model the data in this paper.

2. Preliminaries

In this section, let's remember the basic concepts in differential geometry:
A curve α is defined by coordinate neighborhood (I,α) in E^n, where $I \subseteq R$ is an open interval and $\alpha : I \to E^n$ $(t \to \alpha(t))$ is differentiable function. A curve whose velocity vector at each point is nonzero is called a regular curve. That is, $\alpha'(s) \neq 0$ for $\forall s \in I$.
Let's the curve α be given with neighborhood (I,α). If $\|\alpha'(s)\| = 1$, for $\forall s \in I$. α is called a unit speed curve according to (I,α). In this case, the parameter $s \in I$ of the curve is called the arc length parameter.
The orthonormal basis vectors T,N,B, also known as the Frenet frame or TNB frame, correspond to each point of a unit speed curve in three-dimensional Euclidean space. Here, $T = \alpha^{'}$ is the unit tangent vector field, $N = \frac{\alpha''}{\|\alpha''\|}$ is the principal normal vector field, and $B = T \times N$ is the binormal vector field. Furthermore, the Frenet formulas $T^{'}(s) = \kappa(s)N(s)$, $N^{'}(s) = -\kappa(s)T(s) + \tau(s)B(s)$, and $B^{'}(s) = -\tau(s)N(s)$, where $\kappa(s) > 0$ and $\tau(s)$ are curvature and torsion at the point $\alpha(s)$, respectively. In terms of the Frenet-Serret apparatus, the Darboux vector w can be expressed as $w = \tau T + \kappa B$. Here, we can write

$$\kappa = \|w\|\cos\phi, \tau = \|w\|\sin\phi,$$

where ϕ is the angle between B and w. If the unit vector in the direction w is W, $W = \frac{\tau}{\|w\|}T + \frac{\kappa}{\|w\|}B$, where $\|w\| = \sqrt{\kappa^2 + \tau^2} \geq 0$ [15].
In Euclidean 3-space, the alternative moving frame along the curve α is given by $\{N,C,N \times C = W\}$. Here, the unit principal normal vector, the derivative of the principal normal vector, and the Darboux vector, respectively, are N, $C = \frac{N^{'}}{\|N'\|}$ and $W = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}$. The following equations produce the derivative vectors of these vectors:

$$\begin{aligned} N^{'}(s) &= f(s)C(s), \\ C^{'}(s) &= -f(s)N(s) + g(s)W(s), \\ W^{'}(s) &= -g(s)C(s), \end{aligned}$$

where

$$f = \kappa\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}$$

and

$$g = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'$$

are the first and second curvature of the curve $\alpha(s)$ with respect to alternative moving frame, respectively [8].
A curve is called a general helix or cylindrical helix if its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio of curvature to torsion is constant [15].

Definition 2.1. *Let α be a unit speed curve in E^3 with non-zero torsion and the Frenet frame of α be $\{T_\alpha, N_\alpha, B_\alpha\}$. The adjoint curve of α is defined as [6]*

$$\beta(s) = \int_{s_0}^{s} B_\alpha(s)\,ds.$$

Theorem 2.2. *If α is a curve with arc length parameter s, then the arc length parameter of adjoint curve of α is also s [6].*

Theorem 2.3. *Let α be a curve with arc length s and β be the adjoint curve of α. If the Frenet vectors of α and β are $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$, the curvature and torsion are κ_α, τ_α and κ_β, τ_β respectively, then the following relations hold [6]:*

$$\begin{aligned} T_\beta &= B_\alpha \\ N_\beta &= -N_\alpha \\ B_\beta &= T_\alpha \end{aligned}$$

and

$$\begin{aligned} \kappa_\beta &= \tau_\alpha \\ \tau_\beta &= \kappa_\alpha. \end{aligned}$$

Corollary 2.4. *If a is a general helix parametrized by arc length parameter s, then the adjoint curve β of α is a general helix.*

Definition 2.5. *Let α be a Frenet curve in E^3 and W be the unit Darboux vector field of α. We call an integral curve of $W(s)$ the W-direction curve of α. Namely, if $\beta(s)$ is the W-direction curve of α, then*

$$W(s)=\beta'(s),$$

where $W=\frac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}}$ [7].

Theorem 2.6. *Let α be a Frenet curve in E^3 with the curvature κ and the torsion τ, and β be W-direction curve of α. If α is not a general helix, then the curvature κ_β and the torsion τ_β of β are given by [7]*

$$\kappa_\beta=\frac{|\tau\kappa'-\tau'\kappa|}{\kappa^2+\tau^2}$$

$$\tau_\beta=\sqrt{\kappa^2+\tau^2}.$$

Theorem 2.7. *Let β be the W-direction curve of a nonplanar curve α. Then α is a general helix if and only if β is a straight line [7].*

Definition 2.8. *A ruled surface in E^3 may therefore be represented in the form*

$$\varphi(\alpha,d):I\times E\to E^3,(s,v)\to\varphi(\alpha,d)(s,v)=\alpha(s)+vd(s)$$

such that $\alpha:I\to E^3, d:I\to E^3\setminus\{0\}$ are differentiable transformations. Here, α is called base curve and d is called the director curve [16].

The distribution parameter of a ruled surface parameterized by

$$\varphi(s,v)=\alpha(s)+vX(s),$$

where α is the base curve and X is the director curve, is the function D_X defined by

$$D_X=\frac{\det(\alpha',X,X')}{\|X'\|^2}.$$

A developable ruled surface is characterized by $D_X=0$ [15].

Definition 2.9. *Let $\alpha(s)$ be a curve with arc length in E^3 and $\{N,C,W\}$ be the alternative moving frame of α. The $C-$ruled surface can be given by the following parameterization as [17]*

$$\varphi(s,v)=\alpha(s)+vC(s),$$

and the $W-$ruled surface can be given by the following parameterization as [18]

$$\varphi(s,v)=\alpha(s)+vW(s).$$

3. $W-$adjoint curve

Let α be a unit speed curve in E^3 with non-zero torsion and the alternative moving frame of α be $\{N_\alpha,C_\alpha,W_\alpha\}$. The $W-$adjoint curve of α can be write as

$$\beta(s)=\int_{s_0}^{s}W_\alpha(s)\,ds, \tag{3.1}$$

where $C_\alpha(s)=\frac{N'_\alpha(s)}{\left\|N'_\alpha(s)\right\|}$, $W_\alpha(s)=\frac{\tau_\alpha(s)T_\alpha(s)+\kappa_\alpha(s)B_\alpha(s)}{\sqrt{(\kappa_\alpha(s))^2+(\tau_\alpha(s))^2}}$. We know that $\{T_\alpha,N_\alpha,B_\alpha\}$ is Frenet frame of α and κ_α, τ_α are curvature and torsion of α, respectively. The derivative vectors of $\{N_\alpha,C_\alpha,W_\alpha\}$ can also be given as:

$$N'_\alpha(s)=f_\alpha(s)C_\alpha(s)$$

$$C'_\alpha(s)=-f_\alpha(s)N_\alpha(s)+g_\alpha(s)W_\alpha(s),$$

$$W'_\alpha(s)=-g_\alpha(s)C_\alpha(s), \tag{3.2}$$

where

$$f_\alpha(s)=\sqrt{\kappa_\alpha(s)^2+\tau_\alpha(s)^2},$$

$$g_\alpha(s)=\frac{\kappa_\alpha(s)^2}{\sqrt{(\kappa_\alpha(s))^2+(\tau_\alpha(s))^2}}\left(\frac{\tau_\alpha(s)}{\kappa_\alpha(s)}\right)'. \tag{3.3}$$

Theorem 3.1. *If α is a curve with arc length parameter s, then the arc length parameter of $W-$adjoint curve of α is also s.*

Proof. By differentiating both sides of (3.1), we have

$$\frac{d}{ds}\beta(s) = W_\alpha(s).$$

Here, if we take the norm of both sides and we use $\|W_\alpha(s)\| = 1$, we obtain $\|\beta'(s)\| = 1$. This means that β is a unit speed curve and

$$T_\beta(s) = W_\alpha(s), \tag{3.4}$$

where $T_\beta(s)$ is unit tangent vector of β. □

Theorem 3.2. *Let α be a curve with arc length s and β be the $W-$adjoint curve of α. If the alternative moving frame vectors of α and β are $\{N_\alpha, C_\alpha, W_\alpha\}$ and $\{N_\beta, C_\beta, W_\beta\}$, curvatures according to the alternative moving frame of α and β are $\{f_\alpha, g_\alpha\}$ and $\{f_\beta, g_\beta\}$ respectively, then the following relations hold:*

$$N_\beta = -C_\alpha, \tag{3.5}$$

$$W_\beta = \frac{f_\alpha W_\alpha + g_\alpha N_\alpha}{\sqrt{{f_\alpha}^2 + {g_\alpha}^2}}, \tag{3.6}$$

and

$$C_\beta = \frac{f_\alpha N_\alpha - g_\alpha W_\alpha}{\sqrt{{f_\alpha}^2 + {g_\alpha}^2}}. \tag{3.7}$$

Proof. If we take derivative both sides of (3.4) and divide by their norms, we get

$$N_\beta(s) = \frac{W'_\alpha(s)}{\|W'_\alpha(s)\|}.$$

Considering (3.2), we have (3.5). We know that we can write

$$W_\beta = \frac{\tau_\beta T_\beta + \kappa_\beta B_\beta}{\sqrt{{\kappa_\beta}^2 + {\tau_\beta}^2}}. \tag{3.8}$$

Now let's write W_β in terms of alternative moving frame apparatus of α. In that case from (3.4), (3.5) and the equation $B_\beta = T_\beta \times N_\beta$, we obtain $B_\beta = W_\alpha \times (-C_\alpha)$ and

$$B_\beta = N_\alpha. \tag{3.9}$$

If we take derivative both sides of (3.4) and we use (3.2), we obtain $T'_\beta = -g_\alpha C_\alpha$. In last equation, if we take the norm of both sides, we get $\left\|T'_\beta\right\| = g_\alpha$. This means that

$$\kappa_\beta = g_\alpha. \tag{3.10}$$

If we take derivative both sides of (3.9) and we use Frenet and alternative moving frame formulae, we have

$$-\tau_\beta N_\beta = f_\alpha C_\alpha.$$

From (3.5), we get

$$\tau_\beta = f_\alpha. \tag{3.11}$$

Hence, if we use (3.4), (3.9), (3.10) and (3.11) in (3.8), we obtain (3.6).
On the other hand, it is known that

$$C_\beta = \frac{-\kappa_\beta T_\beta + \tau_\beta B_\beta}{\sqrt{{\kappa_\beta}^2 + {\tau_\beta}^2}}. \tag{3.12}$$

If we use (3.4), (3.9), (3.10) and (3.11) in (3.12), we get (3.7). □

Theorem 3.3. *The relationships between alternative moving frame curvatures $\{f_\alpha, g_\alpha\}$ and $\{f_\beta, g_\beta\}$ with respect to α and β are*

$$f_\beta = \sqrt{{f_\alpha}^2 + {g_\alpha}^2} \tag{3.13}$$

and

$$g_\beta = \frac{g_\alpha^2}{\sqrt{(g_\alpha)^2 + (f_\alpha)^2}} \left(\frac{f_\alpha}{g_\alpha}\right)'. \tag{3.14}$$

Proof. The relationships between the curvatures with respect to Frenet frame and alternative moving frame of the unit speed curve β are

$$f_\beta = \sqrt{\kappa_\beta^2 + \tau_\beta^2}, \tag{3.15}$$

$$g_\beta = \frac{\kappa_\beta^2}{\sqrt{(\kappa_\beta)^2 + (\tau_\beta)^2}} \left(\frac{\tau_\beta}{\kappa_\beta}\right)'. \tag{3.16}$$

If we use (3.10), (3.11) in the equations (3.15) and (3.16), we easily get (3.13) and (3.14). □

Theorem 3.4. *Let α be a nonplanar curve with arc length s in E^3. α is a helix if and only if $g_\alpha = 0$, where g_α is second curvature with respect to alternative moving frame of α.*

Proof. It is known that if α is helix, $\frac{\tau_\alpha}{\kappa_\alpha} = c$ ($c = constant$). Here, if we take derivative of both sides we have

$$\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = 0.$$

Considering (3.3), we write $g_\alpha = 0$.
Conversely, if $g_\alpha = 0$, from (3.3) we get $\frac{\kappa_\alpha^2}{\sqrt{(\kappa_\alpha)^2 + (\tau_\alpha)^2}} \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = 0$. Since α is a nonplanar curve, $\kappa \neq 0$ and $\tau \neq 0$. Then we can say $\frac{\tau_\alpha}{\kappa_\alpha} = c$ ($c = constant$). Thus, α is a helix. □

Theorem 3.5. *Let α be a curve with arc length s in E^3 and β be W-adjoint curve of α. β is helix if and only if the ratio $\frac{f_\alpha}{g_\alpha}$ is constant.*

Proof. If β is helix, $\frac{\tau_\beta}{\kappa_\beta} = c$. From (3.10) ve (3.11), we obtain

$$\frac{f_\alpha}{g_\alpha} = c. \tag{3.17}$$

Conversely, given by (3.17). From (3.10) and (3.11), we have $\frac{\tau_\beta}{\kappa_\beta} = c$. Hence, β is a helix. This completes the proof. □

4. Ruled surfaces associated with $W-$adjoint curve

4.1. Ruled surface with base curve α and director curve β

We examine ruled surface created by a curve and $W-$adjoint curve of this curve under this heading. Let α be a curve with arc length s and β be $W-$adjoint curve of α. The ruled surface with the base curve α and the director curve β can be defined by

$$\phi(s,v) = \alpha(s) + v\beta(s). \tag{4.1}$$

Theorem 4.1. *Let α be a curve with arc length s and β be $W-$adjoint curve of α. Given by $\{T_\beta, N_\beta, B_\beta\}$ is the Frenet frame of β and $\{N_\alpha, C_\alpha, W_\alpha\}$ is alternative moving frame of α. Distribution parameter of the ruled surface given by (4.1) is*

$$D_\phi = \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \langle \beta, B_\beta \rangle, \tag{4.2}$$

where β is position vector of the curve β.

Proof. If we calculate distribution parameter of the ruled surface given by (4.1), we have

$$D_{\phi}=\frac{\det\left(\frac{d\alpha}{ds},\beta,\frac{d\beta}{ds}\right)}{\left\|\frac{d\beta}{ds}\right\|^{2}}.$$

From (3.4), we get

$$D_{\phi}=\frac{\det\left(T_{\alpha},\beta,W_{\alpha}\right)}{\left\|T_{\beta}\right\|^{2}}.$$

$$D_{\phi}=\det\left(T_{\alpha},\beta,\frac{\tau_{\alpha}T_{\alpha}+\kappa_{\alpha}B_{\alpha}}{\sqrt{\kappa_{\alpha}^{2}+\tau_{\alpha}^{2}}}\right)$$

$$D_{\phi}=\frac{\tau_{\alpha}}{\sqrt{\kappa_{\alpha}^{2}+\tau_{\alpha}^{2}}}\det\left(T_{\alpha},\beta,T_{\alpha}\right)+\frac{\kappa_{\alpha}}{\sqrt{\kappa_{\alpha}^{2}+\tau_{\alpha}^{2}}}\det\left(T_{\alpha},\beta,B_{\alpha}\right)$$

Since $\det(T_{\alpha},\beta,T_{\alpha})=0$, we can write

$$D_{\phi}=\frac{\kappa_{\alpha}}{\sqrt{\kappa_{\alpha}^{2}+\tau_{\alpha}^{2}}}\det\left(T_{\alpha},\beta,B_{\alpha}\right).$$

Using determinant and mixed product properties, we have

$$D_{\phi}=\frac{\kappa_{\alpha}}{\sqrt{\kappa_{\alpha}^{2}+\tau_{\alpha}^{2}}}\left\langle\beta,N_{\alpha}\right\rangle. \tag{4.3}$$

If we substitute the equation (3.9) in (4.3), we have (4.2). □

Corollary 4.2. *The ruled surface given by (4.1) is developable if and only if the position vector β and the binormal vector of β are orthogonal. In this case $D_{\phi}=0$.*

Proof. The ruled surface is developable if and only if $D_{\phi}=0$. Then, let us consider the equation (4.2). Since $\kappa_{\alpha}\neq 0$, $\left\langle\beta,B_{\beta}\right\rangle=0$. Then, surface given by (4.1) is developable. □

4.2. Ruled surface with base curve β and director curve α

Let α be a curve with arc length s and β be $W-$adjoint curve of α. The ruled surface with the base curve β and the director curve α can be defined by

$$\psi(s,v)=\beta(s)+v\alpha(s). \tag{4.4}$$

Theorem 4.3. *Let α be a curve with arc length s and β be $W-$adjoint curve of α. Given by $\left\{T_{\beta},N_{\beta},B_{\beta}\right\}$ is the Frenet frame of β and $\{N_{\alpha},C_{\alpha},W_{\alpha}\}$ is alternative moving frame of α. Distribution parameter of the ruled surface given by (4.4) is*

$$D_{\psi}=-\frac{\kappa_{\alpha}}{\sqrt{\kappa_{\alpha}^{2}+\tau_{\alpha}^{2}}}\left\langle\alpha,N_{\alpha}\right\rangle,$$

where α is position vector of the curve α.

Corollary 4.4. *The ruled surface given by (4.4) is developable if and only if the position vector α and the normal vector of α are orthogonal.*

5. Applications

Example 5.1. *Let us consider the curve $\alpha(s)$ with arc length s in E^3 given by*

$$\alpha(s)=(-\frac{1}{12}\cos(4s)-\frac{1}{3}\cos(2s),\frac{1}{12}\sin(4s)+\frac{1}{3}\sin(2s),-\frac{2\sqrt{2}}{3}\cos(s))$$

(see Figure 5.1 and Figure 5.3). $W-$adjoint curve of α is

$$\beta(s)=(-\frac{1}{9}\sin(3s),\frac{1}{9}\cos(3s),-\frac{2\sqrt{2}}{3}s)$$

(see Figure 5.2 and Figure 5.3).

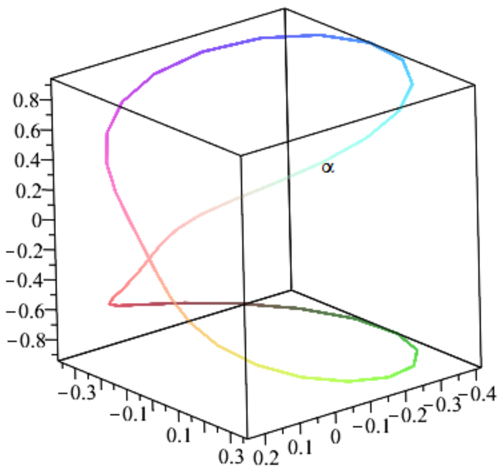

Figure 5.1: The curve $\alpha(s)$

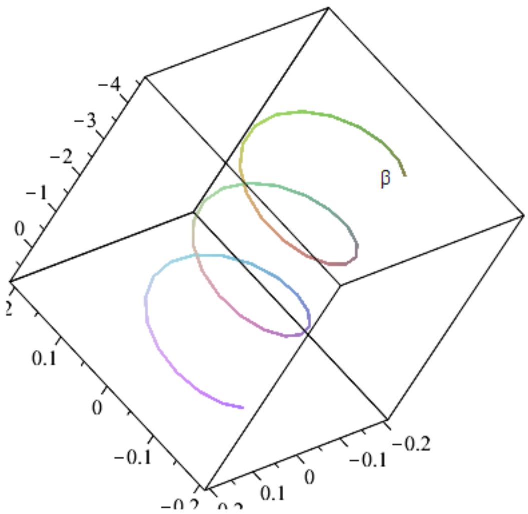

Figure 5.2: $W-$Adjoint curve of $\alpha(s)$

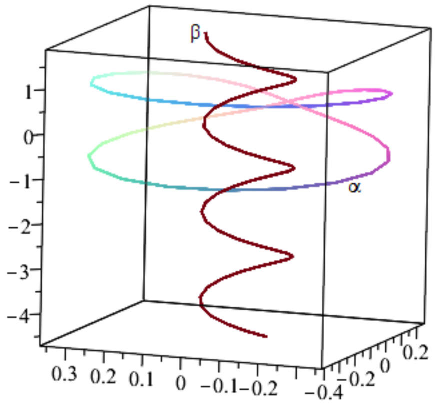

Figure 5.3: The curves $\alpha(s)$ and $\beta(s)$

Example 5.2. *Let's exemplify the ruled surfaces associated with the α and $W-$adjoint curve of α that we took in Example 5.1. First, let's write the ruled surface with base curve is α and director curve is $W-$adjoint curve of α*

$$\phi_{\alpha}(s,v)=(-\frac{1}{12}\cos(4s)-\frac{1}{3}\cos(2s),\frac{1}{12}\sin(4s)+\frac{1}{3}\sin(2s),-\frac{2\sqrt{2}}{3}\cos(s))+v(-\frac{1}{9}\sin(3s),\frac{1}{9}\cos(3s),-\frac{2\sqrt{2}}{3}s)$$

(see Figure 5.4). On the other hand, the ruled surface with the base curve W and the director curve α is

$$\phi(s,v)=(-\frac{1}{9}\sin(3s),\frac{1}{9}\cos(3s),-\frac{2\sqrt{2}}{3}s)+v(-\frac{1}{12}\cos(4s)-\frac{1}{3}\cos(2s),\frac{1}{12}\sin(4s)+\frac{1}{3}\sin(2s),-\frac{2\sqrt{2}}{3}\cos(s))$$

(see Figure 5.5).

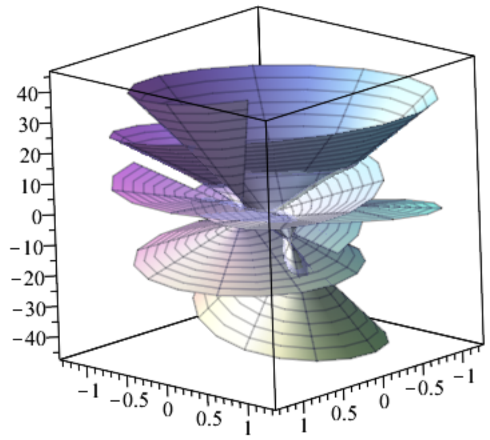

Figure 5.4: $\phi(s,v)=\alpha(s)+v\beta(s)$

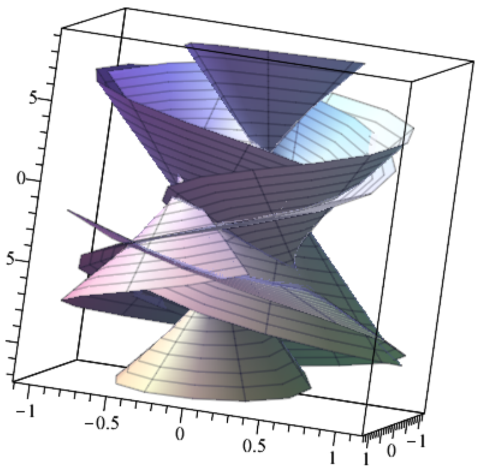

Figure 5.5: $\phi(s,v)=\beta(s)+v\alpha(s)$

6. Conclusion

In this study, the curve β is defined as the W-adjoint curve of the curve α with respect to alternative moving frame. The relationships are established between the alternative moving frame vectors of the curves α and β. In addition, connections between the curvatures defined in the alternative moving frame are constructed. The results relating to the helix curve are collected at this point. The ruled surfaces created with the curves α and β are obtained.It is found under which conditions the acquired ruled surfaces may be developable. In the last section, it is reinforced with examples.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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