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On Quaternions with Higher Order Jacobsthal Numbers Components

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Highlights

- This paper focuses on higher order Jacobsthal quaternions.
- We examined the basic properties of these numbers in this study.
- We have given some identities of these numbers.

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Abstract

In this study, we present higher order Jacobsthal numbers. Then we define higher order Jacobsthal quaternions by using higher order Jacobsthal numbers. We give the concept of the norm and conjugate for these quaternions. We express and prove some propositions related to these quaternions. Also, we find the recurrence relation, the Binet formula and the generating function for these quaternions. Finally, we calculate Cassini, Catalan, Vajda and d'Ocagne identities for higher order Jacobsthal quaternions.

1. INTRODUCTION

Hamilton defined a quaternion as the division of two vectors orientated in a three-dimensional space, or the division of two equivalent vectors in 1843 [1]. It is possible to see the effects of Hamilton's discovery today in astronautics, robotics, navigation, computer visualization, animation and special effects in movies, and many other areas. Quaternions are also vital to the control systems that guide airplanes and rockets. Quaternions have generated a growing interest in algebra. Now, many studies have emerged by combining quaternions with algebra. Horadam defined Fibonacci quaternions in 1963 and gave a generalization of these numbers [2]. Halıcı examined the basic properties of Fibonacci quaternions as number sequences [3]. In [4-10], some applications about quaternions were made with the Fibonacci and Lucas numbers.
Now let's give some basic properties about quaternions.

Quaternions are defined in the following form. Let p be a quaternion. p is written as:

$$p = p_0 + p_1\mathbb{i} + p_2\mathbb{j} + p_3\mathbb{k}$$

where p_0, p_1, p_2 and p_3 are real numbers, and $\mathbb{i}, \mathbb{j}, \mathbb{k}$ are the main quaternions which are satisfy rules in Table 1.

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Table 1. *The main multiplications*

.	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
$\mathbb{i}$	-1	$\mathbb{k}$	$-\mathbb{j}$
$\mathbb{j}$	$-\mathbb{k}$	-1	$\mathbb{i}$
$\mathbb{k}$	$\mathbb{j}$	$-\mathbb{i}$	-1

We can write $p = p_0 + u$ where $u = p_1\mathbb{i} + p_2\mathbb{j} + p_3\mathbb{k}$.

Let p^* and $\|p\|$ show conjugate and norm of the quaternion p, respectively. Then we give them

$p^* = p_0 - u$ and $\|p\| = \sqrt{{p_0}^2 + {p_1}^2 + {p_2}^2 + {p_3}^2}$.

Note that, $\|p\|^2 = pp^*$.

Many scientists have been interested in number sequences for many years, as they find application in nature and in many sciences [11-15]. Many generalizations of number sequences were then described and studied [15-18]. One of the most well-known number sequences is the Jacobsthal numbers [19-23]. Now, let's give them now.

The Jacobsthal numbers J_n are defined by

$$J_{n+2} = J_{n+1} + 2J_n,\ n \geq 0$$

with $J_0 = 0$ and $J_1 = 1$ [23].

Similarly, the Jacobsthal-Lucas numbers j_n are defined by

$$j_{n+2} = j_{n+1} + 2j_n, n \geq 0$$

with $j_0 = 2$ and $j_1 = 1$ [23].

Binet formulas for J_n and j_n are given by, respectively:

$$J_n = \frac{a^n - b^n}{a-b}$$

and

$$j_n = a^n + b^n \tag{1}$$

where $a = 2$ and $b = -1$ are roots of the equation $x^2 - x - 2 = 0$.

In [24], Jacobsthal and Jacobsthal-Lucas quaternions are presented and given many principal identities about the quaternions.

The Jacobsthal quaternions JQ_n and the Jacobsthal-Lucas quaternions JLQ_n are defined as

$$JQ_n = J_n + \mathbb{i}J_{n+1} + \mathbb{j}J_{n+2} + \mathbb{k}J_{n+3}$$

$$JLQ_n = j_n + \mathbb{i}j_{n+1} + \mathbb{j}j_{n+2} + \mathbb{k}j_{n+3}$$

respectively [25].

Recently, many generalizations of quaternions have been studied [24-29].

One of the recent studies in this field is [5] where higher order Fibonacci quaternions are introduced and given their basic properties.

In this paper, we introduce higher order Jacobsthal numbers. Then we define higher order Jacobsthal quaternions by using higher order Jacobsthal numbers. We present the concept of the norm and conjugate for these quaternions. We give some propositions related to these quaternions. In addition, we find the recurrence relation, the Binet formula and the generating function which are basic concepts in number sequences for these quaternions. Finally, we calculate Cassini, Catalan, Vajda and d'Ocagne identities which are the main identities in the literature for higher order Jacobsthal quaternions.

2. MAIN RESULTS

2.1. Higher Order Jacobsthal Numbers

Definition 2.1.1 The higher order Jacobsthal numbers described by

$$J_n^{(s)}=\frac{J_{ns}}{J_s}=\frac{\frac{a^{ns}-b^{ns}}{a-b}}{\frac{a^s-b^s}{a-b}}=\frac{a^{ns}-b^{ns}}{a-b}\frac{a-b}{a^s-b^s}=\frac{(a^s)^n-(b^s)^n}{a^s-b^s}. \tag{2}$$

Since J_{ns} is divisible by J_s , the ratio $\frac{J_{ns}}{J_s}$ is an integer. So, all higher order Jacobsthal numbers are integer. Note that for $s=1$, higher order Jacobsthal number $J_n^{(1)}$ is the ordinary Jacobsthal numbers.

Proposition 2.1.2. The higher order Jacobsthal numbers provide the following identity.

$$J_{n+1}^{(s)}=j_sJ_n^{(s)}-(-2)^sJ_{n-1}^{(s)}.$$

Proof. $j_sJ_n^{(s)}-(-2)^sJ_{n-1}^{(s)}=(a^s+b^s)\left(\frac{a^{sn}-b^{sn}}{a^s-b^s}\right)-(-2)^s\left(\frac{a^{sn-s}-b^{sn-s}}{a^s-b^s}\right).$

Since $(-2)^s=(ab)^s$,

$$j_sJ_n^{(s)}-(-2)^sJ_{n-1}^{(s)}=(a^s+b^s)\left(\frac{a^{sn}-b^{sn}}{a^s-b^s}\right)-(ab)^s\left(\frac{a^{sn-s}-b^{sn-s}}{a^s-b^s}\right)$$

$$=\left(\frac{a^{sn+s}-a^sb^{sn}+b^sa^{sn}-b^{sn+s}-a^{sn}b^s+a^sb^{sn}}{a^s-b^s}\right)$$

$$=\left(\frac{a^{sn+s}-b^{sn+s}}{a^s-b^s}\right)=\left(\frac{a^{s(n+1)}-b^{s(n+1)}}{a^s-b^s}\right)=J_{n+1}^{(s)}.$$

Thus, the proof is completed.

2.2. Higher Order Jacobsthal Quaternions

Definition 2.2.1. The higher order Jacobsthal quaternions is denoted by $OJ_n^{(s)}$ and defined by

$$OJ_n^{(s)}=J_n^{(s)}+J_{n+1}^{(s)}\mathbb{i}+J_{n+2}^{(s)}\mathbb{j}+J_{n+3}^{(s)}\mathbb{k} \tag{3}$$

where $\mathbb{i},\mathbb{j}$ and $\mathbb{k}$ are quaternions unit and $J_n^{(s)}$ is higher order Jacobsthal numbers.
If we take $s=1$ in (3) then we get the Jacobsthal quaternions. The real and imaginary parts of the higher order Jacobsthal quaternions in (3) are as follows

$$Re(OJ_n^{(s)}) = J_n^{(s)}$$

and

$$Im\left(OJ_n^{(s)}\right) = v = J_{n+1}^{(s)}\mathbb{i} + J_{n+2}^{(s)}\mathbb{j} + J_{n+3}^{(s)}\mathbb{k}\,.$$

Thus, we have

$$OJ_n^{(s)} = J_n^{(s)} + v.$$

The conjugate of $OJ_n^{(s)}$ is denoted by $OJ_n^{(s)^*}$ and given as follows

$$OJ_n^{(s)^*} = J_n^{(s)} - J_{n+1}^{(s)}\mathbb{i} - J_{n+2}^{(s)}\mathbb{j} - J_{n+3}^{(s)}\mathbb{k} = J_n^{(s)} - v. \quad (4)$$

Norm of the higher order Jacobsthal quaternions is denoted by $N(OJ_n^{(s)})$ and given as follows

$$\left(N(OJ_n^{(s)})\right)^2 = OJ_n^{(s)}OJ_n^{(s)^*} = \left(J_n^{(s)}\right)^2 + \left(J_{n+1}^{(s)}\right)^2 + \left(J_{n+2}^{(s)}\right)^2 + \left(J_{n+3}^{(s)}\right)^2. \quad (5)$$

Proposition 2.2.2. We have

$$OJ_n^{(s)} + OJ_n^{(s)^*} = 2J_n^{(s)}.$$

Proof. By using (3) and (4), the result is easily seen.

Proposition 2.2.3. We have the following identity

$$\left(OJ_n^{(s)}\right)^2 = -OJ_n^{(s)}OJ_n^{(s)^*} + 2J_n^{(s)}OJ_n^{(s)}.$$

Proof. By using (3) and (4), we obtain that

$$\left(OJ_n^{(s)}\right)^2 = \left(J_n^{(s)} + J_{n+1}^{(s)}\mathbb{i} + J_{n+2}^{(s)}\mathbb{j} + J_{n+3}^{(s)}\mathbb{k}\right)\left(J_n^{(s)} + J_{n+1}^{(s)}\mathbb{i} + J_{n+2}^{(s)}\mathbb{j} + J_{n+3}^{(s)}\mathbb{k}\right)$$

$$= -\left(\left(J_n^{(s)}\right)^2 + \left(J_{n+1}^{(s)}\right)^2 + \left(J_{n+2}^{(s)}\right)^2 + \left(J_{n+3}^{(s)}\right)^2\right) + 2J_n^{(s)}\left(J_n^{(s)} + J_{n+1}^{(s)}\mathbb{i} + J_{n+2}^{(s)}\mathbb{j} + J_{n+2}^{(s)}\mathbb{k}\right).$$

From (8), we get

$$\left(OJ_n^{(s)}\right)^2 = -OJ_n^{(s)}OJ_n^{(s)^*} + 2J_n^{(s)}OJ_n^{(s)}.$$

Theorem 2.2.4. The Binet formula of the higher order Jacobsthal quaternions as follow

$$OJ_n^{(s)} = \frac{(a^s)^n\hat{a} - (b^s)^n\hat{b}}{a^s - b^s} \quad (6)$$

where $\hat{a} = (1 + a^s\mathbb{i} + a^{2s}\mathbb{j} + a^{3s}\mathbb{k})$ and $\hat{b} = (1 + b^s\mathbb{i} + b^{2s}\mathbb{j} + b^{3s}\mathbb{k})$.

Proof. Using (2) and (3), we have

$$OJ_n^{(s)} = J_n^{(s)} + J_{n+1}^{(s)}\mathbb{i} + J_{n+2}^{(s)}\mathbb{j} + J_{n+3}^{(s)}\mathbb{k}$$

$$= \frac{(a^s)^n}{a^s - b^s}[1 + a^s\mathbb{i} + a^{2s}\mathbb{j} + a^{3s}\mathbb{k}] - \frac{(b^s)^n}{a^s - b^s}[1 + b^s\mathbb{i} + b^{2s}\mathbb{j} + b^{3s}\mathbb{k}]$$

$$= \frac{(a^s)^n\hat{a}}{a^s-b^s} - \frac{(b^s)^n\hat{b}}{a^s-b^s} = \frac{(a^s)^n\hat{a}-(b^s)^n\hat{b}}{a^s-b^s}.$$

Thus, the proof ends.

Theorem 2.2.5. The higher order Jacobsthal quaternions is given by

$$OJ_{n+1}^{(s)} = j_s OJ_n^{(s)} - (-2)^s OJ_{n-1}^{(s)}.$$

Proof. From (1) and (6), we find that

$$OJ_{n+1}^{(s)} = \frac{(a^s)^{n+1}\hat{a} - (b^s)^{n+1}\hat{b}}{a^s - b^s} = \frac{1}{a^s - b^s}\left(a^s(a^s)^n\hat{a} - b^s(b^s)^n\hat{b}\right)$$

$$= \frac{1}{a^s - b^s}\left(a^s(a^s)^n\hat{a} - a^s(b^s)^n\hat{b} + a^s(b^s)^n\hat{b} - b^s(b^s)^n\hat{b}\right)$$

$$= \frac{1}{a^s - b^s}a^s\left((a^s)^n\hat{a} - (b^s)^n\hat{b}\right) + \frac{1}{a^s - b^s}a^s\left((b^s)^n\hat{b} - b^s(b^s)^n\hat{b}\right)$$

$$= (a^s + b^s)OJ_n^{(s)} - b^s OJ_n^{(s)} + \frac{1}{a^s - b^s}\left(a^s(b^s)^n\hat{b} - b^s(b^s)^n\hat{b}\right)$$

$$= j_s OJ_n^{(s)} - \frac{a^s b^s}{a^s-b^s}\left((a^s)^{n-1}\hat{a} - (b^s)^{n-1}\hat{b}\right) = j_s OJ_n^{(s)} - (-2)^s OJ_{n-1}^{(s)}.$$

Thus, the proof is obtained.

Theorem 2.2.6. If we take n and s negative integer numbers for $OJ_n^{(s)}$ then we get the following properties:

i) $OJ_{-n}^{(s)} = (-2)^{-sn}\frac{(b^s)^n\hat{a}-(a^s)^n\hat{b}}{a^s-b^s}$,

ii) $OJ_{-n}^{(-s)} = -(-2)^s OJ_n^{(s)}$,

iii) $OJ_n^{(-s)} = -(-2)^s OJ_{-n}^{(s)}$.

Proof. By using (6), we have

i) $$OJ_{-n}^{(s)} = \frac{(a^s)^{-n}\hat{a} - (b^s)^{-n}\hat{b}}{a^s - b^s} = \frac{\frac{\hat{a}}{(a^s)^n} - \frac{\hat{b}}{(b^s)^n}}{a^s - b^s} = \frac{(b^s)^n\hat{a} - (a^s)^n\hat{b}}{(a^s)^n(b^s)^n}\frac{1}{a^s - b^s}$$

$$= (-2)^{-sn}\frac{(b^s)^n\hat{a}-(a^s)^n\hat{b}}{a^s-b^s},$$

ii) $$OJ_{-n}^{(-s)} = \frac{(a^{-s})^{-n}\hat{a} - (b^{-s})^{-n}\hat{b}}{a^{-s} - b^{-s}} = \frac{(a^s)^n\hat{a} - (b^s)^n\hat{b}}{\frac{b^s-a^s}{a^s b^s}} = (-2)^s\frac{(a^s)^n\hat{a} - (b^s)^n\hat{b}}{b^s - a^s}$$

$$= -(-2)^s OJ_n^{(s)},$$

iii) $$OJ_n^{(-s)} = \frac{(a^{-s})^n\hat{a} - (b^{-s})^n\hat{b}}{a^{-s} - b^{-s}} = \frac{(a^s)^{-n}\hat{a} - (b^s)^{-n}\hat{b}}{\frac{b^s-a^s}{a^s b^s}} = -(-2)^s\frac{(a^s)^{-n}\hat{a} - (b^s)^{-n}\hat{b}}{a^s - b^s}$$

$$= -(-2)^s OJ_{-n}^{(s)}.$$

So, the desired is achieved.

Now let's give the following lemma which we need to obtain the generating function and the sum formula.

Lemma 2.2.7. The following equations are provided

i) $\hat{a}-\hat{b}=(a^s-b^s)(\mathbb{i}+j_s\mathbb{j}+(j_{2s}+(-2)^s)\mathbb{k})$,

ii) $\hat{a}b^s-\hat{b}a^s=(a^s-b^s)(-1+(-2)^s\mathbb{j}+(-2)^s j_s\mathbb{k})$,

iii) $\hat{a}a^s-\hat{b}b^s=(a^s-b^s)\big(1+j_s\mathbb{i}+(j_s+(-2)^s)\mathbb{j}+\big({j_s}^3+(-2)^{s+1}j_s\big)\mathbb{k}\big)$.

Proof.

i) $\hat{a}-\hat{b}=(1+a^s\mathbb{i}+a^{2s}\mathbb{j}+a^{3s}\mathbb{k})-(1+b^s\mathbb{i}+b^{2s}\mathbb{j}+b^{3s}\mathbb{k})$

$=(a^s-b^s)\mathbb{i}+(a^{2s}-b^{2s})\mathbb{j}+(a^{3s}-b^{3s})\mathbb{k}$

$=(a^s-b^s)(\mathbb{i}+j_s\mathbb{j}+(j_{2s}+(-2)^s)\mathbb{k})$.

ii) $\hat{a}b^s-\hat{b}a^s=(1+a^s\mathbb{i}+a^{2s}\mathbb{j}+a^{3s}\mathbb{k})b^s-(1+b^s\mathbb{i}+b^{2s}\mathbb{j}+b^{3s}\mathbb{k})a^s$

$=-(a^s-b^s)+(a^{2s}b^s-b^{2s}a^s)\mathbb{j}+(a^{3s}b^s-b^{3s}a^s)\mathbb{k}$

$=(a^s-b^s)(-1+(-2)^s\mathbb{j}+(-2)^s(a^s+b^s)\mathbb{k})$

$=(a^s-b^s)(-1+(-2)^s\mathbb{j}+(-2)^s j_s\mathbb{k})$.

iii) $\hat{a}a^s-\hat{b}b^s=(1+a^s\mathbb{i}+a^{2s}\mathbb{j}+a^{3s}\mathbb{k})a^s-(1+b^s\mathbb{i}+b^{2s}\mathbb{j}+b^{3s}\mathbb{k})b^s$

$=(a^s-b^s)+(a^{2s}-b^{2s})\mathbb{i}+(a^{3s}-b^{3s})\mathbb{j}+(a^{4s}-b^{4s})\mathbb{k}$

$=(a^s-b^s)\big(1+j_s\mathbb{i}+(j_s+(-2)^s)\mathbb{j}+\big({j_s}^3+(-2)^{s+1}j_s\big)\mathbb{k}\big)$.

Theorem 2.2.8. The generating function of $OJ_n^{(s)}$ is given as follow

$$G^{(s)}(x)=\sum_{n=0}^{\infty}OJ_n^{(s)}x^n=\frac{(\mathbb{i}+j_s\mathbb{j}+(j_{2s}+(-2)^s)\mathbb{k})-(1+(-2)^s\mathbb{j}+(-2)^s j_s\mathbb{k})x}{(1-j_s x+(-2)^s x^2)}.$$

Proof.

$$G^{(s)}(x)=\sum_{n=0}^{\infty}OJ_n^{(s)}x^n=\sum_{n=0}^{\infty}\left(J_n^{(s)}+J_{n+1}^{(s)}\mathbb{i}+J_{n+2}^{(s)}\mathbb{j}+J_{n+3}^{(s)}\mathbb{k}\right)x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{(a^n)^s-(b^n)^s}{a^s-b^s}+\frac{(a^{n+1})^s-(b^{n+1})^s}{a^s-b^s}\mathbb{i}+\frac{(a^{n+2})^s-(b^{n+2})^s}{a^s-b^s}\mathbb{j}+\frac{(a^{n+3})^s-(b^{n+3})^s}{a^s-b^s}\mathbb{k}\right]x^n$$

$$=\frac{1}{a^s-b^s}\sum_{n=0}^{\infty}(a^n)^s x^n(1+a^s\mathbb{i}+a^{2s}\mathbb{j}+a^{3s}\mathbb{k})-\frac{1}{a^s-b^s}\sum_{n=0}^{\infty}(b^n)^s x^n(1+b^s\mathbb{i}+b^{2s}\mathbb{j}+b^{3s}\mathbb{k})$$

$$= \frac{1}{a^s - b^s}\sum_{n=0}^{\infty}(a^n)^s x^n \hat{a} - \frac{1}{a^s - b^s}\sum_{n=0}^{\infty}(b^n)^s x^n \hat{b}$$

$$= \left(\frac{\hat{a}}{a^s - b^s}\right)\left(\frac{1}{1 - a^s x}\right) - \left(\frac{\hat{b}}{a^s - b^s}\right)\left(\frac{1}{1 - b^s x}\right)$$

$$= \frac{\hat{a} - \hat{a}b^s x - \hat{b} + \hat{b}a^s x}{(a^s - b^s)(1 - a^s x)(1 - b^s x)}$$

$$= \frac{\hat{a} - \hat{b} - (\hat{a}b^s - \hat{b}a^s)x}{(a^s - b^s)(1 - (a^s + b^s)x + (-2)^s x^2)}$$

$$= \frac{(a^s - b^s)(\mathbb{i} + j_s\mathbb{j} + (j_{2s} + (-2)^s)\mathbb{k}) - ((a^s - b^s)(-1 + (-2)^s\mathbb{j} + (-2)^s j_s\mathbb{k}))x}{(a^s - b^s)(1 - (a^s + b^s)x + (-2)^s x^2)}$$

$$= \frac{(\mathbb{i} + j_s\mathbb{j} + (j_{2s} + (-2)^s)\mathbb{k}) - (1 + (-2)^s\mathbb{j} + (-2)^s j_s\mathbb{k})x}{(1 - j_s x + (-2)^s x^2)}.$$

Thus, the desired is obtained.

Theorem 2.2.9. Sum of the higher order Jacobsthal quaternions $OJ_n^{(s)}$ is

$$SOJ_n^{(s)} = \sum_{n=0}^{\infty} OJ_n^{(s)} = \frac{(\mathbb{i} + j_s\mathbb{j} + (j_{2s} + (-2)^s)\mathbb{k}) - (1 + (-2)^s\mathbb{j} + (-2)^s j_s\mathbb{k})}{(1 - j_s + (-2)^s)}.$$

Proof. If we write $x = 1$ in Theorem 2.2.8, the proof is clear.

Theorem 2.2.10. For $n, m \in \mathbb{Z}$, we have

$$\sum_{n=0}^{\infty} OJ_{n+m}^{(s)} x^n = \frac{OJ_n^{(s)} - (-2)^s OJ_{m-1}^{(s)} x}{1 + j_s x + (-2)^s x^2}.$$

Proof.

$$\sum_{n=0}^{\infty} OJ_{n+m}^{(s)} x^n = \sum_{n=0}^{\infty} \frac{(a^s)^{n+m}\hat{a} - (b^s)^{n+m}\hat{b}}{a^s - b^s} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(a^s)^{n+m}\hat{a}}{a^s - b^s} x^n - \sum_{n=0}^{\infty} \frac{(b^s)^{n+m}\hat{b}}{a^s - b^s} x^n = \frac{\hat{a}a^{sm}}{a^s - b^s}\sum_{n=0}^{\infty} a^{sn}x^n - \frac{\hat{b}b^{sm}}{a^s - b^s}\sum_{n=0}^{\infty} b^{sn}x^n$$

$$= \left(\frac{\hat{a}a^{sm}}{a^s - b^s}\right)\left(\frac{1}{1 - a^s x}\right) - \left(\frac{\hat{b}b^{sm}}{a^s - b^s}\right)\left(\frac{1}{1 - b^s x}\right)$$

$$= \left(\frac{1}{a^s - b^s}\right)\left[\frac{\hat{a}a^{sm} - \hat{a}a^{sm}b^s x - \hat{b}b^{sm} + \hat{b}b^{sm}a^s x}{1 - b^s x - a^s x + (ab)^s x^2}\right]$$

$$= \left(\frac{1}{a^s - b^s}\right)\left[\frac{\hat{a}(a^s)^m - \hat{b}(b^s)^m}{1 - j_s x + (-2)^s x^2} - \frac{a^s b^s(\hat{a}(a^s)^{m-1} - \hat{b}(b^s)^{m-1})x}{1 - j_s x + (-2)^s x^2}\right]$$

$$= \left[\frac{OJ_m{}^{(s)}}{1 - j_s x + (-2)^s x^2} - \frac{(-2)^s OJ_{m-1}{}^{(s)} x}{1 - j_s x + (-2)^s x^2}\right]$$

$= \frac{OJ_n^{(s)} - (-2)^s OJ_{m-1}^{(s)} x}{1 + j_s x + (-2)^s x^2}.$

So, the proof is done.

Theorem 2.2.11. Exponential generating function of $OJ_n^{(s)}$ is given by

$\sum_{n=0}^{\infty} OJ_{n+m}^{(s)} \frac{x^n}{n!} = \frac{\hat{a}e^{a^s x} - \hat{b}e^{b^s x}}{a^s - b^s}.$

Proof. Let $U_x{}^{(s)} = \sum_{n=0}^{\infty} OJ_n^{(s)} \frac{x^n}{n!}$ be the exponential generating function of $OJ_n^{(s)}$. From (6), the following is obtained.

$$U_x{}^{(s)} = \sum_{n=0}^{\infty} OJ_n^{(s)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{(a^s)^n \hat{a} - (b^s)^n \hat{b}}{a^s - b^s}\right) \frac{x^n}{n!}$$

$$= \frac{\hat{a}}{a^s - b^s} \sum_{n=0}^{\infty} \frac{(a^s x)^n}{n!} - \frac{\hat{b}}{a^s - b^s} \sum_{n=0}^{\infty} \frac{(b^s x)^n}{n!}$$

$= \frac{\hat{a}e^{a^s x}}{a^s - b^s} - \frac{\hat{b}e^{b^s x}}{a^s - b^s} = \frac{\hat{a}e^{a^s x} - \hat{b}e^{b^s x}}{a^s - b^s}.$

Thus, the proof is obtained.

2.3. Some Identities for Higher Order Jacobsthal Quaternions

Lemma 2.3.1. There are the following equations

$$\hat{a}\hat{b} = \alpha - \nabla\beta \tag{7}$$

and

$$\hat{b}\hat{a} = \alpha + \nabla\beta \tag{8}$$

where $\alpha = (1 - (-2)^s - (-2)^{2s} - (-2)^{3s} + j_s \mathbb{i} + j_{2s} \mathbb{j} + j_{3s} \mathbb{k}),$

$\beta = (-2)^{2s} J_1^{(s)} \mathbb{i} - (-2)^s J_2^{(s)} \mathbb{j} + (-2)^s J_1^{(s)} \mathbb{k}$ and $\nabla = (a^s - b^s).$

Proof.
$\hat{a}\hat{b} = (1 + a^s \mathbb{i} + a^{2s} \mathbb{j} + a^{3s} \mathbb{k})(1 + b^s \mathbb{i} + b^{2s} \mathbb{j} + b^{3s} \mathbb{k})$

$= 1 + b^s \mathbb{i} + b^{2s} \mathbb{j} + b^{3s} \mathbb{k} + a^s \mathbb{i} - a^s b^s + a^s b^{2s} \mathbb{k} - a^s b^{3s} \mathbb{j} + a^{2s} \mathbb{j} - a^{2s} b^s \mathbb{k} - a^{2s} b^{2s}$

$+ a^{2s} b^{3s} \mathbb{i} + a^{3s} \mathbb{k} + a^{3s} b^s \mathbb{j} - a^{3s} b^{2s} \mathbb{i} - a^{3s} b^{3s}$

$= 1 + b^s \mathbb{i} + b^{2s} \mathbb{j} + b^{3s} \mathbb{k} + a^s \mathbb{i} - (-2)^s + a^s b^{2s} \mathbb{k} - a^s b^{3s} \mathbb{j} + a^{2s} \mathbb{j} - a^{2s} b^s \mathbb{k} - (-2)^{2s}$

$+ a^{2s} b^{3s} \mathbb{i} + a^{3s} \mathbb{k} + a^{3s} b^s \mathbb{j} - a^{3s} b^{2s} \mathbb{i} - (-2)^{3s}$

$$= (1-(-2)^s-(-2)^{2s}-(-2)^{3s})+(a^s+b^s+a^{2s}b^{3s}-a^{3s}b^{2s})\mathbb{i}$$

$$+(a^{2s}+b^{2s}+a^{3s}b^s-a^sb^{3s})\mathbb{j}+(a^{3s}+b^{3s}+a^sb^{2s}-a^{2s}b^s)\mathbb{k}$$

$$= (1-(-2)^s-(-2)^{2s}-(-2)^{3s}+j_s\mathbb{i}+j_{2s}\mathbb{j}+j_{3s}\mathbb{k})-(-2)^{2s}(a^s-b^s)\mathbb{i}$$

$$+(-2)^s(a^{2s}-b^{2s})\mathbb{j}-(-2)^s(a^s-b^s)\mathbb{k}$$

$$= (1-(-2)^s-(-2)^{2s}-(-2)^{3s}+j_s\mathbb{i}+j_{2s}\mathbb{j}+j_{3s}\mathbb{k})-(-2)^{2s}\nabla J_1^{(s)}\mathbb{i}$$

$$+(-2)^s\nabla J_2^{(s)}\mathbb{j}-(-2)^s\nabla J_1^{(s)}\mathbb{k}$$

$$= (1-(-2)^s-(-2)^{2s}-(-2)^{3s}+j_s\mathbb{i}+j_{2s}\mathbb{j}+j_{3s}\mathbb{k})-\nabla\left((-2)^{2s}J_1^{(s)}\mathbb{i}-(-2)^sJ_2^{(s)}\mathbb{j}+(-2)^sJ_1^{(s)}\mathbb{k}\right)$$

$$= \alpha-\nabla\beta.$$

Equation (8) can be similarly proved.

Theorem 2.3.2. (Vajda Identity) For $n, m, r \in \mathbb{Z}$, we get

$$OJ_{n+m}^{(s)}OJ_{n+r}^{(s)}-OJ_n^{(s)}OJ_{n+m+r}^{(s)} = (-2)^{sn}J_m^{(s)}\left[\alpha J_r^{(s)}+\beta j_{sr}\right].$$

Proof. By using (6), we find that

$$OJ_{n+m}^{(s)}OJ_{n+r}^{(s)}-OJ_n^{(s)}OJ_{n+m+r}^{(s)} = \left(\frac{(a^s)^{n+m}\hat{a}-(b^s)^{n+m}\hat{b}}{a^s-b^s}\right)\left(\frac{(a^s)^{n+r}\hat{a}-(b^s)^{n+r}\hat{b}}{a^s-b^s}\right)$$
$$-\left(\frac{(a^s)^n\hat{a}-(b^s)^n\hat{b}}{a^s-b^s}\right)\left(\frac{(a^s)^{n+m+r}\hat{a}-(b^s)^{n+m+r}\hat{b}}{a^s-b^s}\right)$$

$$= \left(\frac{1}{(a^s-b^s)^2}\right)\big(-(a^s)^{n+m}\hat{a}(b^s)^{n+r}\hat{b}-(b^s)^{n+m}\hat{b}(a^s)^{n+r}\hat{a}+(a^s)^n\hat{a}(b^s)^{n+m+r}\hat{b}$$
$$+(b^s)^n\hat{b}(a^s)^{n+m+r}\hat{a}\big)$$

$$= \frac{1}{\nabla^2}\left(-\hat{a}\hat{b}(a^s)^n(b^s)^{n+r}((a^s)^m-(b^s)^m)+\hat{b}\hat{a}(b^s)^n(a^s)^{n+r}((a^s)^m-(b^s)^m)\right)$$

$$= \frac{1}{\nabla^2}(a^s)^n(b^s)^n((-\hat{a}\hat{b}(b^s)^r+\hat{b}\hat{a}(a^s)^r))((a^s)^m-(b^s)^m))$$

$$= \frac{(-2)^{sn}}{\nabla^2}(-\hat{a}\hat{b}(b^s)^r+\hat{b}\hat{a}(a^s)^r)((a^s)^m-(b^s)^m)$$

$$= \frac{(-2)^{sn}}{\nabla}(-(\alpha-\nabla\beta\,)(b^s)^r+(\alpha+\nabla\beta\,)(a^s)^r)J_m^{(s)}$$

$$= \frac{(-2)^{sn}}{\nabla}J_m^{(s)}(\alpha((a^s)^r-(b^s)^r)+\nabla\beta((a^s)^r+(b^s)^r))$$

$$= (-2)^{sn}J_m^{(s)}\left(\alpha J_r^{(s)}+\beta j_{sr}\right).$$

Thus, the equality is proved.

Theorem 2.3.3. (Catalan Identity) For $n, m, r \in \mathbb{Z}$, we get

$$OJ_{n-r}^{(s)} OJ_{n+r}^{(s)} - \left(OJ_n^{(s)}\right)^2 = (-2)^{sn} J_{-r}^{(s)} \left[\alpha J_r^{(s)} + \beta j_{sr}\right].$$

Proof. The proof is derived from the special case of Vajda identity.

$$OJ_{n+m}^{(s)} OJ_{n+r}^{(s)} - OJ_n^{(s)} OJ_{n+m+r}^{(s)} = (-2)^{sn} J_m^{(s)} \left[\alpha J_r^{(s)} + \beta j_{sr}\right].$$

For $m = -r$, we get

$$OJ_{n-r}^{(s)} OJ_{n+r}^{(s)} - \left(OJ_n^{(s)}\right)^2 = (-2)^{sn} J_{-r}^{(s)} \left[\alpha J_r^{(s)} + \beta j_{sr}\right].$$

Theorem 2.3.4. (Cassini Identity) We have

$$OJ_{n-1}^{(s)} OJ_{n+1}^{(s)} - \left(OJ_n^{(s)}\right)^2 = -(-2)^{s(n-1)} [\alpha + \beta j_s].$$

Proof. If we take $m = -1$ and $r = 1$ in Catalan identity, the following is obtained.

$$OJ_{n-1}^{(s)} OJ_{n+1}^{(s)} - \left(OJ_n^{(s)}\right)^2 = (-2)^{sn} J_{-1}^{(s)} \left[\alpha J_1^{(s)} + \beta j_s\right].$$

$$= (-2)^{sn} - (-2)^{-s} [\alpha + \beta j_s]$$

$$= -(-2)^{s(n-1)} [\alpha + \beta j_s].$$

Theorem 2.3.5. (d' Ocagne Identity) We have

$$OJ_k^{(s)} OJ_{n+1}^{(s)} - OJ_n^{(s)} OJ_{k+1}^{(s)} = (-2)^{sn} J_{k-n}^{(s)} [\alpha + \beta j_s].$$

Proof. If we take for $m + n = k$ and $r = 1$ in Vajda identity, the following is obtained.

$$OJ_{n+m}^{(s)} OJ_{n+r}^{(s)} - OJ_n^{(s)} OJ_{n+m+r}^{(s)} = (-2)^{sn} J_m^{(s)} \left[\alpha J_r^{(s)} + \beta j_{sr}\right]$$

$$OJ_k^{(s)} OJ_{n+1}^{(s)} - OJ_n^{(s)} OJ_{k+1}^{(s)} = (-2)^{sn} J_{k-n}^{(s)} [\alpha + \beta j_s].$$

3. CONCLUSION

In this work, we introduced higher order Jacobsthal numbers with recurrence relations. We defined higher order Jacobsthal quaternions by using these numbers. We gave the concept of the norm and conjugate for higher order Jacobsthal quaternions. We proved some propositions for these quaternions. Also, we obtained the recurrence relation, the Binet formula and the generating function which are basic concepts in number sequences for these quaternions. Finally, we gave Cassini, Catalan, Vajda and d'Ocagne identities which are the main identities in the literature for higher order Jacobsthal quaternions.
This work can be extended to higher order Jacobsthal-Lucas quaternions.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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