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Upper and Lower θp−continuous Multifunctions

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Abstract. We introduce two new types of multifunctions, namely upper (lower) θp−continuous multifunctions, between topological spaces. Besides characterising these multifunctions, we study some properties of upper θp−continuous multifunctions.

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1. Introduction and Preliminaries

Throughout this paper, (Δ, τ), (Λ, σ), and (Υ, ϕ) (or simply Δ, Λ, and Υ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. By a multifunction $M : \Delta \longrightarrow \Lambda$, we mean a point-to-set correspondence from Δ into Λ, and we always assume that $M(u) \neq \emptyset$ for all $u \in \Delta$. For a multifunction $M : \Delta \longrightarrow \Lambda$, following [1, 2] we shall denote the upper and lower inverse of a set W of Λ by $M^+(W)$ and $M-(W)$, respectively, that is, $M^+(W) = \{u \in \Delta : M(u) \subset W\}$ and $M-(W) = \{u \in \Delta : M(u) \cap W \neq \emptyset\}$. For each $U \subset \Delta, M(U) = \bigcup_{u \in U} M(u)$. Then, M is said to be a surjection if $M(\Delta) = \Lambda$, or equivalently if for each $v \in \Lambda$ there exists an $u \in \Delta$ such that $v \in M(u)$. Moreover,$M : \Delta \longrightarrow \Lambda$ is called upper semi continuous (resp. lower semi continuous) if $M^+(V)$ (resp. $M-(V)$) is open in Δ for every open set W of Λ. Let U be a subset of a space Δ. We denote the interior and the closure of a set U by $i(U)$ and $c(U)$, respectively. A subset U is said to be preopen [6] (resp. α-open [9]) if $U \subset i(c(U))$ (resp. $U \subset i(c(i(U)))$). The complement of a preopen set is called preclosed [6]. The intersection of all preclosed sets containing U is called the preclosure [5] of U and is denoted by $pc(U)$. The preinterior of U is defined by the union of all preopen sets contained in U and is denoted by $pi(U)$. The family of all preopen sets of Δ is denoted by $po(\Delta)$. We set $po(\Delta, u) = \{V : u \in V \text{ and } V \in po(\Delta)\}$. A point u of U is called a θ-cluster [13] point of A if $c(V) \cap U \neq \emptyset$ for every open set V of Δ containing u. The set of all θ-cluster points of U is called the θ-closure [13] of U and is denoted by $c_\theta(U)$. A subset U is said to be

θ-closed [13] if $U = c_\theta(U)$. The complement of a θ-closed set is said to be θ-open. A point u of Δ is called a pre-θ-cluster [10] point of A if $pcl(V) \cap U \neq \emptyset$ for every preopen set V of Δ containing u. The set of all pre-θ-cluster points of U is called the pre-θ-closure [[10] of U and is denoted by $pc_\theta(U)$. A subset U is said to be pre-θ-closed [10] if $U = pc_\theta(U)$. The complement of a pre-θ-closed set is said to be pre-θ-open. Alternatively, a set U of (Δ, τ) is called pre-θ-open [3] iff for each $u \in U$, there exists a preopen set W with $u \in W$ such that $pc(W) \subset U$. The family of all pre-θ-open sets of Δ is denoted by $p_\theta o(X)$. We set $p_\theta o(\Delta, u) = \{V : u \in V \text{ and } V \in p_\theta o(\Delta)\}$.

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Definition 1.1. A multifunction $M : \Delta \longrightarrow \Lambda$ is said to be lower (upper) θ^*-continuous [8] at a point $u_0 \in \Delta$ if for each open set W in Λ with $W \cap M(u_0) \neq \emptyset$ (resp. $M(u_0) \subseteq W$), i.e. $u_0 \in M^-(W)$ (resp. $u_0 \in M^+(W)$), there exists an open neighbourhood U of u_0 such that $u \in cl(U) \Longrightarrow M(u) \cap W \neq \emptyset$ (resp. $u \in cl(U) \Longrightarrow M(u) \subseteq W$). The multifunction M is said to be lower or upper θ^*-continuous on Δ if M is respectively so at each point of Δ .

Definition 1.2. A multifunction $M : \Delta \longrightarrow \Lambda$ is said to be upper (lower) precontinuous [12] if for each $u \in \Delta$ and each open set W of Λ such that $M(u) \subset W$ (resp. $M(u) \cap W \neq \emptyset$), there exists a preopen set U containing u such that $U \subset M^+(W)$ (resp. $U \subset M^-(W)$).

2. Upper and Lower θp−continuous Multifunctions

Definition 2.1. A multifunction $M : \Delta \longrightarrow \Lambda$ is said to be:

(a) Upper θp−continuous at a point $u \in \Delta$ if for each open set W of Λ such that $M(u) \subset W$, there exists $U \in po(\Delta, u)$ such that $M(v) \subset W$ for every $v \in pc(U)$;

(b) Lower θp−continuous at a point $x \in \Delta$ if for each open set W of Λ such that $M(u) \cap W \neq \emptyset$, there exists $U \in po(\Delta, u)$ such that $M(v) \cap W \neq \emptyset$ for every $v \in pc(U)$;

(c) Upper (Lower) θp−continuous if M has this property at each point of Δ.

Theorem 2.2. *For a multifunction $M : \Delta \longrightarrow \Lambda$, the following are equivalent:*

(a) M is upper θp−continuous on Δ;

(b) for each point u in Δ and each open set W in Λ containing $M(u)$, there exists a pre-θ-open set U in Δ containing u such that $M(U) \subset W$;

(c) $M^+(W)$ is pre-θ-open in Δ for any open set W of Λ;

(d) $M^-(K)$ is pre-θ-closed in Δ for any closed set K of Λ;

(e) $pc_\theta(M^-(T)) \subset M^-(c(T))$ for any subset T of Λ.

Proof. (a)⇒(c): Let W be any open set of Λ and $x \in M^+(W)$. There exists $U \in po(\Delta, u)$ such that $M(v) \subset W$ for every $v \in pc(U)$. Therefore, $u \in pc(U) \subset M^+(W)$. This shows that $M^+(W)$ is pre-θ-open in Δ.

(c)⇒(d): It is clear from the fact that $M^+(\Lambda - W) = \Delta$-$M^-(W)$ for any subset W of Λ.

(d)⇒(e): Let W be any subset of Λ. Then $c(U)$ is closed in Λ and hence $M^-(c(U))$ is pre-θ-closed in Δ. Therefore, we have $pc_\theta(M^-(U)) \subset M^-(c(U))$.

(e)⇒(a): Let $u \in \Delta$ and W be any open subset of Λ such that $M(u) \subset W$. Since $\Lambda - W$ is closed in Λ, we have $pc_\theta(M^-(\Lambda\text{-}W)) \subset M^-(\Lambda\text{-}W)$. Then $M^-(\Lambda\text{-}W)$ is pre-θ-closed in Δ. Since $M^-(\Lambda\text{-}W) = \Delta - M^+(W)$, $M^+(W)$ is pre-θ-open in Δ and hence there exists $U \in po(\Delta, u)$ such that $pcl(U) \subset M^+(W)$. This shows that M is upper θp−continuous.

(b)⇒(c): Let W be any open set of Λ and $u \in M^+(V)$ (i.e, $M(u) \subset W$). Then there exists $U \in p_\theta o(\Delta, u)$ such that $u \in U \subset M^+(W)$. Since U is pre-θ-open in Δ, there exists $G \in po(_\theta o(\Delta, u)$ such that $u \in pcl(G) \subset U \subset M^+(W)$. Hence $M^+(W)$ is pre-θ-open in Δ.

(c)⇒(b): Let $u \in \Delta$ and W be any open set of Λ such that $M(u) \subset W$. Then $M^+(W)$ is pre-θ-open in Δ and $u \in M^+(W)$. Let $U = M^+(W)$. Then $M(U) \subset W$. □

Theorem 2.3. *For a multifunction $M : \Delta \longrightarrow \Lambda$, the following are equivalent:*

(a) M is lower θp−continuous on Δ;

(b) for each point u in Δ and each open set W in Λ with $u \in M^-(W)$, there exists a pre-θ-open set U in Δ containing u such that $U \subset M^-(W)$;

(c) $M^-(W)$ is pre-θ-open in Δ for any open set W of Λ;

(d) $M^+(K)$ is pre-θ-closed in X for any closed set K of Λ;

(e) $pcl_\theta(M^+(T)) \subset M^+(cl(T))$ for any subset T of Λ.

Remark 2.4. For a multifunction $M : \Delta \longrightarrow \Lambda$, the following implications hold:

upper θ^*-continuity $\Longrightarrow$ upper θp−continuity $\Longrightarrow$ upper precontinuity

None of these implications is reversible as shown by the following examples.

Example 2.5. Let τ_c be the cofinite topology on $\mathbb{R}$ and $\sigma = \{\{1, 2\}, \{3\}, \{1, 2, 3\}, \Lambda, \emptyset\}$ be a topology on $\Lambda = \{1, 2, 3, 4\}$. Define a multifunction $M : (\mathbb{R}, \tau_c) \longrightarrow (\Lambda, \sigma)$ as follows:

$$M(u) = \begin{cases} \{1, 2, 3\} & \text{if } u \in \mathbb{Q} \\ \{4\} & \text{if u} \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

We have $M^+(\{1,2\}) = \varnothing,^+ (M\{3\}) = \varnothing$ and $M^+(\{1,2,3\}) = \mathbb{Q}$. Then, M is upper θp−continuous but not upper θ^*-continuous, since $\mathbb{Q}$ is pre-θ-open and is not θ-open in $(\mathbb{R}, \tau_c)$.

Example 2.6. Let $\Delta = \{a,b,c\}, \Lambda = \{1,2,3\}$, $\tau = \{\{a\},\{a,b\},\{a,c\},\Delta,\varnothing\}$ and $\sigma = \{\{1\},\{2\},\{1,2\},\Lambda,\varnothing\}$.Define a multifunction $M : (\Delta,\tau) \longrightarrow (\Lambda,\sigma)$ as follows:

$$M(u) = \begin{cases} \{1,2\} & \text{if u=a,} \\ \{3\} & \text{if u=b,} \\ \{2,3\} & \text{if u=c.} \end{cases}$$

Then $M^+(\{1\}) = \varnothing$, $M^+(\{2\}) = \varnothing$ and $M^+(\{1,2\}) = \{a\}$. M is upper precontinuous but not upper θp−continuous, since $\{a\}$ is preopen in (Δ,τ) while not pre-θ-open in (Δ,τ).

Lemma 2.7 ([7]). *Let U and V be subsets of a space (Δ,τ).*
(1) If $U \in po(\Delta)$ and V is semi-open in X, then $(A \cap V) \in po(V)$.
(2) If $A \in po(V)$ and $V \in po(\Delta)$, then $A \in po(\Delta)$.

Lemma 2.8 ([4]). *Let U and V be subsets of a space Δ such that $U \subset V \subset \Delta$. Let $pc_V(U)$ denote the preclosure of U in the subspace V.*
(1) If is V semi-open in Δ, then $pc_V(U) \subset pc(U)$.
(2) If $A \in po(V)$ and $V \in po(\Delta)$, then $pc(U) \subset pc_V(U)$.

Theorem 2.9. *Let $\{U_\lambda : \lambda \in \Omega\}$ be an α-open cover of a space Δ. Then, a multifunction $M : (\Delta,\tau) \longrightarrow (\Lambda,\sigma)$ is upper (lower) θp−continuous if and only if the restriction $M \mid U_\lambda : U_\lambda \longrightarrow Y$ is upper (lower) θp−continuous for each $\lambda \in \Omega$.*

Proof. We prove only the case for M upper θp−continuous.
(Necessity) Let $\lambda \in \Omega$, $u \in U_\lambda$, W is an open set in Λ such that $(M \mid U_\lambda)(u) \subset W$. Since M is upper θp−continuous and $M(u) = (M \mid U_\lambda)(u)$, there exists a preopen set G in Δ containing x such that $pc(G) \subset M^+(W)$. Set $U = U_\lambda \cap G$. Then, $U \in po(U_\lambda, u)$ and $pc_{U_\lambda}(U) \subset pc(U)$. Therefore, we have $(M \mid U_\lambda)(pc_{U_\lambda}(U)) = M(pc_{U_\lambda}(U)) \subset M(pc(U)) \subset W$. Hence $M \mid U_\lambda : U_\lambda \longrightarrow \Lambda$ is upper θp−continuous for each $\lambda \in \Omega$.
(Sufficiency) Let $u \in \Delta$ and W be any open set in Λ such that $M(u) \subset W$. Then, there exists some $\lambda \in \Omega$ such that $u \in U_\lambda$. Since $M \mid U_\lambda$ is upper θp−continuous, and $(M \mid U_\lambda)(u) = M(u)$, there exists $U \in po(U_\lambda, u)$ such that $pc_{U_\lambda}(U) \subset (M \mid U_\lambda)^+(E)$. By Lemma 2 , we have $U \in po(\Delta)$ and $pc(U) \subset pc_{U_\lambda}(U)$ such that $M(pc(U)) \subset W$. Thus, M is upper θp−continuous. □

Theorem 2.10. *Let (X,τ), (Y,σ) and (Z,φ) be topological spaces. Let $F_1 : X \longrightarrow Y$ and $F_2 : Y \longrightarrow Z$ be multifunctions. If $F_1 : X \longrightarrow Y$ is an upper (lower) θp− continuous multifunction and $F_2 : Y \longrightarrow Z$ is an upper (lower) semi-continuous multifunction, then $F = F_2 \circ F_1 : X \longrightarrow Z$ is an upper (lower) θp−continuous multifunction.*

Proof. We prove only the case for F upper θp−continuous.
Let $G \subset Z$ be any open set. From the definition of $F_2 \circ F_1$, we have $F^+(G) = (F_2 \circ F_1)^+(G) = F_1^+(F_2^+(G))$. Since F_2 is an upper semi-continuous multifunction, $F_2^+(G)$ is open in Y. Since F_1 is an upper θp−continuous multifunction, $F_1^+(F_2^+(G))$ is a pre-θ-open set in X. This shows that, $F = F_2 \circ F_1$ is an upper θp−continuous multifunction. □

Theorem 2.11. *If $M : \Delta \longrightarrow \Lambda$ is an upper θp−continuous multifunction such that $M(x)$ compact for each $x \in \Delta$ and Λ is Hausdorff, then a set $A = \{(x,y) \in \Delta \times \Delta : M(x) \cap M(y) \neq \varnothing\}$ is a pre-θ-closed set in $\Delta \times \Delta$.*

Proof. Let $(x,y) \in (\Delta \times \Delta) - A$. Then, $M(x) \cap M(y) = \varnothing$. Since $M(x)$ and $M(y)$ are compact and Λ is Hausdorff, there exist disjoint open sets V_1 and V_2 of Λ such that $M(x) \subset V_1$ and $M(y) \subset V_2$. Since M is upper θp−continuous, there exist $U_1 \in PO(\Delta,x)$, $U_2 \in PO(\Delta,y)$ such that $x \in pCl(U_1) \subset M^+(V_1)$ and $y \in pCl(U_2) \subset M^+(V_2)$. Since $pCl(U_1 \times U_2) \cap A \subset (pCl(U_1) \times pCl(U_2)) \cap A$ and $(pCl(U_1) \times pCl(U_2)) \cap A = \varnothing$, we have

$$pCl(U_1 \times U_2) \cap A = \varnothing.$$

Since $U_1 \times U_2$ is preopen in $\Delta \times \Delta$ and $(x,y) \in pCl(U_1 \times U_2) \subset \Delta - A$, it follows that A is pre-θ-closed in $\Delta \times \Delta$. □

Let A be a subset of a space (Δ,τ). Then, $M : (\Delta,\tau) \longrightarrow (A,\tau_A)$ is called a retracting multifunction [14] if $x \in M(x)$ for each $x \in A$.

Theorem 2.12. *Let M be an upper θp−continuous multifunction of a Hausdorff space (Δ,τ) into itself. If $M(x)$ is compact for each $x \in \Delta$, then the set $A = \{x : x \in M(x)\}$ is a pre-θ-closed subset.*

Proof. Let $x_0 \in pCl_\theta(A)$. Suppose that $x_0 \notin A$, i.e. $x_0 \notin M(x_0)$. Since (Δ, τ) is Hausdorff and $M(x)$ is compact, there exist disjoint open sets U and V such that $x_0 \in U$ and $M(x_0) \subseteq V$.Since U, V are open, we also have $pCl(U) \cap V = \emptyset$. Let $W \in PO(\Delta, x_0)$ such that $pCl(W) \subset M^+(V)$. We have $pCl(U \cap W) \cap A \neq \emptyset$. Let z$\in pCl(U \cap W) \cap A$. Since $z \in A$, $z \in M(z)$. Also, $z \in pCl(W)$ and $z \in pCl(U)$. This shows that $pCl(W) \nsubseteq M^+(V)$, which is a contradiction. Thus, $x_0 \in A$ and A is pre-θ-closed. □

Corollary 2.13. *Let A be a subset of (Δ, τ) and $M : (\Delta, \tau) \longrightarrow (A, \tau_A)$ an upper θp−continuous retracting multifunction such that $M(x)$ is compact for each $x \in A$. If (Δ, τ) is Hausdorff, then A is pre-θ-closed.*

Definition 2.14. A space X is said to be pre-Urysohn [11] if for each pair of distinct points x and y in X, there exist $U \in PO(X, x)$ and $V \in PO(X, y)$ such that $pCl(U) \cap pCl(V) = \emptyset$.

Theorem 2.15. *Let $M : \Delta \longrightarrow \Lambda$ be an upper θp−continuous multifunction such that $M(x)$ is compact for each $x \in \Delta$ and let $M(x) \cap M(y) = \emptyset$ for each pair of distinct points $x, y \in \Delta$. If Λ is Hausdorff, then Δ is pre-Urysohn.*

Proof. Let x and y be any two distinct points in Δ. Then, $M(x) \cap M(y) = \emptyset$. Since Λ is Hausdorff and $M(x)$ and $M(y)$ are compact, there exist disjoint open sets V_1, V_2 such that $M(x) \subset V_1$ and $M(y) \subset V_2$. Since M is upper θp−continuous, there exist $U_1 \in PO(\Delta, x)$ and $U_2 \in PO(\Delta, y)$ such that $x \in pCl(U_1) \subset M^+(V_1)$, $y \in pCl(U_2) \subset M^+(V_2)$. Then, we have $pCl(U_1) \cap pCl(U_2) = \emptyset$. This shows that, Δ is pre-Urysohn. □

Definition 2.16. A space X is said to be p-closed [4] if every cover of X by preopen sets has a finite subcover whose preclosures cover X.

Theorem 2.17. *Let $M : \Delta \longrightarrow \Lambda$ be an upper θp−continuous surjective multifunction such that $M(x)$ is compact for each $x \in \Delta$. If Δ is p-closed, then Λ is compact.*

Proof. Let $\{V_i : i \in I\}$ be an open cover of Λ. Since $M(x)$ is compact for each $x \in \Delta$, there exist a finite subset $I(x)$ of I such that

$$M(x) \subset \bigcup \{V_i : i \in I(x)\}.$$

Put

$$V(x) = \bigcup \{V_i : i \in I(x)\}.$$

Since M is an upper θp−continuous multifunction, there exists a preopen set $U(x)$ of Δ containing x such that $pCl(U(x)) \subset M^+(V(x))$. Then, the family $\{U(x) : x \in \Delta\}$ is a preopen cover of Δ and since Δ is p-closed, there exist a finite number of points, say, $x_1, ..., x_n$ in Δ such that $\Delta = \bigcup \{pCl(U(x_i)) : i = 1, ..., n\}$. Hence, we have

$$M(X) = \Lambda = M(\bigcup_{i=1}^{n} pCl(U(x_i))) = \bigcup_{i=1}^{n} F(pCl(U(x_i))) \subset \bigcup_{i=1}^{n} V(x_i) = \bigcup_{i=1}^{n} \bigcup_{i \in I(x_i)} V_i.$$

This shows that Λ is compact. □

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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