Sigma J Eng & Nat Sci 38 (1), 2020, 475-480

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

APPLICATION OF CRANK-NICOLSON METHOD TO A RANDOM COMPONENT HEAT EQUATION

Halil ANAÇ*[1], Mehmet MERDAN[2], Tülay KESEMEN[3]

[1]*Gümüşhane University, Dept. of Mathematical Engineering, GÜMÜŞHANE;* ORCID: 0000-0002-1316-3947
[2]*Gümüşhane University, Dept. of Mathematical Engineering, GÜMÜŞHANE;* ORCID: 0000-0002-8509-3044
[3]*Karadeniz Teknik University, Department of Mathematics, TRABZON;* ORCID: 0000-0002-8807-5677

Received: 30.10.2019 Revised: 26.11.2019 Accepted: 22.01.2020

ABSTRACT

In this study, the solution of a random component heat equation is obtained by using Crank-Nicolson Method. The initial condition of this equation is examined by Normal distribution. The expected value and variance of solution of this equation are obtained. Crank-Nicolson method is applied to analyze the solution of this equation. Also, the solution and the graphs of the expected value and variance are obtained by using MATLAB software. The results of the heat equation are compared with random characteristics of this equation. Firstly, a random component heat equation is solved by this method.
Keywords: Random component heat equation, expected value, Crank-Nicolson method, variance.
2010 Mathematics Classification: 35R11, 35R60.

1. INTRODUCTION

The random partial differential equations (RPDEs) are defined as random partial differential equations with random inputs that can be a random variable or a stochastic process. The subject of the random component partial differential equations is one of much current interests due to the great importance of many applications in engineering, mathematics, biology, physics and a lot of applied sciences.

In the literature, there are very few studies on random partial differential equations (RPDEs). Babuska et al. proposed and analysed a stochastic collocation method to solve elliptic partial differential equations by random coefficients and forcing terms [2]. Nobile et al. proposed and analysed an anisotropic sparse grid stochastic collocation method for solving partial differential equations with random coefficients and forcing terms [21]. Charrier examined the problem of the numerical solution of an elliptic partial differential equation by random coefficients and homogeneous Dirichlet boundary conditions [3]. Kuo applied quasi-Monte Carlo methods to a class of elliptic partial differential equations by random coefficients which was parametrized by a countably infinite number of terms in a Karhunen–Loeve expansion [17]. Gunzburger used the stochastic finite element methods for solving partial differential equations by random input data [10]. It is generally impossible to solve random nonlinear partial differential equations

[*] Corresponding Author: e-mail: halilanac0638@gmail.com, tel: (456) 233 10 00 / 1955

analytically. Thus, va rious numerical methods and approximation schemes for RPDEs, ordinary differential equations and partial differential equations have been developed. There are a lot of numerical methods and approximation schemes as adomian decomposition method (ADM) [16], homotopy perturbation method [12-14,22,27], collocation method [24,25,28], Galerkin-type method [29], differential transformation method (DTM) [1,15,19,20,26,30], variational iteration method (VIM) [11], finite difference method [7,8,23], Crank-Nicolson method [4,6,18,23], random finite difference scheme [5,8] and many other methods. The main motivation of this paper is to analyze a random component heat equation by the Crank-Nicolson method which is the reliable computational method.

This work studies a random component heat equation solve numerically by the Crank-Nicolson method. Crank and Nicolson established a practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type [4]. The aim of this study is to present the application of Crank-Nicolson method for obtaining the approximate analytical solution of the random component heat equation and for calculating the expected value and variance of this solution. It is observed that the numerical solution obtained by Crank-Nicolson method for this equation is almost similar to exact solution for this equation.

2. CRANK-NICOLSON METHOD

In this section, we present Crank-Nicolson method [18].

Consider the one-dimensional heat equation, for $0 \leq x < a,$ and $0 < t < b$,

$$u_t(x,t) = c^2 u_{xx}(x,t) \tag{1}$$

with the initial condition

$$u(x,0) = f(x) \qquad \text{for } t = 0 \text{ and } 0 \leq x \leq a \tag{2}$$

with the boundary conditions

$$u(0,t) = c_1 \text{ for } x = 0 \text{ and } 0 \leq t \leq b,$$
$$u(1,t) = c_2 \text{ for } x = a \text{ and } 0 \leq t \leq b.$$

If the Crank- Nicolson discretization is applied to the problem (1) and (2), then it is obtained as [18]

$$\frac{u_{i,j+1}-u_{i,j}}{k} = c^2 \frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}+u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{2h^2} \tag{3}$$

Moreover, $r = \frac{c^2 k}{h^2}$ is substituted in Eq. (3). If Eq. (3) is rearranged, then the following implicit difference formula is obtained. For $i = 2,3,\ldots,n-1$ [18]

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = (2-2r)u_{i,j} + r\left(u_{i-1,j} + u_{i+1,j}\right) \tag{4}$$

where the terms on the right hand side of this equation are all known. Formula (4) is sometimes implemented by using ratio r=1. In this case, Eq. (4) become [18]

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \tag{5}$$

for $i = 2,3,\ldots,n-1$. Also, The boundary conditions gives the first and last equations (i.e. $u_{1,j} = u_{1,j+1} = c_1$ and $u_{n,j} = u_{n,j+1} = c_2$, respectively). Equations (5) are usually solved in their tridiagonal matrix form $Ax = B$ [18]

$$\begin{bmatrix} 4 & -1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 4 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{p,j+1} \\ \vdots \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 2c_1 + u_{3,j} \\ u_{2,j} + u_{4,j} \\ \vdots \\ u_{p-1,j} + u_{p+1,j} \\ \vdots \\ u_{n-3,j} + u_{n-1,j} \\ u_{n-2,j} + 2c_2 \end{bmatrix}.$$

If the Crank-Nicolson m ethod is applied in Matlab software, then this linear system $Ax = B$ is solved by either direct means or by iteration.

3. NORMAL DISTRIBUTION

Definition (Normal random variable): If the probability density function of a random variable X has the following form, this random variable has the Normal distribution and is called a Normal random variable.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma^2 > 0 \tag{6}$$

If the random variable X gets a normal distribution with parameters μ and σ^2, the expected value and variance of the random variable X are given as [9]

$$E(X) = \mu, Var(X) = \sigma^2 \tag{7}$$

4. NUMERICAL EXPERIMENT

Consider the following random component heat equation. For $0 < x < 1$ and $0 < t < 0.1$,

$$u_t(x,t) = u_{xx}(x,t) \tag{8}$$

with the initial condition

$$u(x,0) = Asin(\pi x) + Bsin(3\pi x) \text{ for } t = 0 \text{ and } 0 \le x \le 1 \tag{9}$$

with the boundary conditions

$$u(0,t) = 0 \text{ for } x = 0 \text{ and } 0 \le t \le 0.1$$
$$u(1,t) = 0 \text{ for } x = 1 \text{ and } 0 \le t \le 0.1$$

where A and B are normal distributed random variable with parameters $\mu = 1, \sigma = 1$, i.e. $A, B \sim N(\mu = 1, \sigma^2 = 1)$.

If we use the step sizes $\Delta x = h = 0.1$ and $\Delta t = k = 0.01$, the ratio $r = 1$. The grid is obtained $n = 11$ columns wide with $m = 11$ rows high. If we apply the algorithm, then it is generated the expected values of the approximate solutions of Eq. (8) for $0 < x_i < 1$ and $0 \le t_j \le 0.1$. Also, Monte Carlo simulation is performed 1000 times for the expected values of these solutions of the Eq. (8) using the Matlab software. Then the expected values of approximate solutions of Eq. (8) in Table 1 is obtained for $0 < x_i < 1$ and $0 \le t_j \le 0.1$.

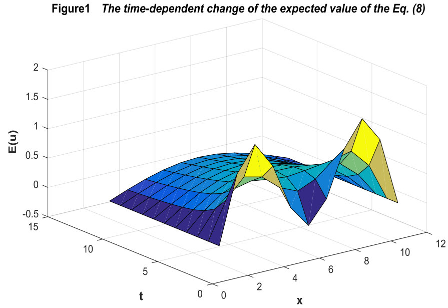
If we compare these values obtained from Crank-Nicolson method, it is seen that these are almost similar with the analytical solution $u(x,t) = sin(\pi x)e^{-\pi^2 t} + sin(3\pi x)e^{-9\pi^2 t}$ for $A = B = 1$.

Table 1. The expected values of approximate solutions of Eq. (8)

	$x_2 = 0.1$	$x_3 = 0.2$	$x_4 = 0.3$	$x_5 = 0.4$	$x_6 = 0.5$	$x_7 = 0.6$	$x_8 = 0.7$	$x_9 = 0.8$	$x_{10} = 0.9$
t_1	1,14413	1,55603	1,08649	0,27368	-0,11527	0,27368	1,08649	1,55603	1,14413
t_2	0,61866	0,91861	0,82517	0,55235	0,41301	0,55235	0,82517	0,91861	0,61866
t_3	0,38665	0,62802	0,68160	0,62741	0,58988	0,62741	0,68160	0,62802	0,38665
t_4	0,27804	0,48414	0,59028	0,62156	0,62449	0,62156	0,59028	0,48414	0,27804
t_5	0,22190	0,40347	0,52367	0,58549	0,60353	0,58549	0,52367	0,40347	0,22190
t_6	0,18863	0,35105	0,47000	0,53998	0,56274	0,53998	0,47000	0,35105	0,18863
t_7	0,16580	0,31214	0,42414	0,49339	0,51669	0,49339	0,42414	0,31214	0,16580
t_8	0,14815	0,28045	0,38373	0,44893	0,47116	0,44893	0,38373	0,28045	0,14815
t_9	0,13341	0,25322	0,34757	0,40769	0,42831	0,40769	0,34757	0,25322	0,13341
t_{10}	0,12059	0,22914	0,31499	0,36992	0,38881	0,36992	0,31499	0,22914	0,12059
t_{11}	0,10918	0,20757	0,28554	0,33551	0,35272	0,33551	0,28554	0,20757	0,10918

Also, the graph of the expected values of approximate solutions of the Eq. (8) is given in figure 1.

Figure1 *The time-dependent change of the expected value of the Eq. (8)*



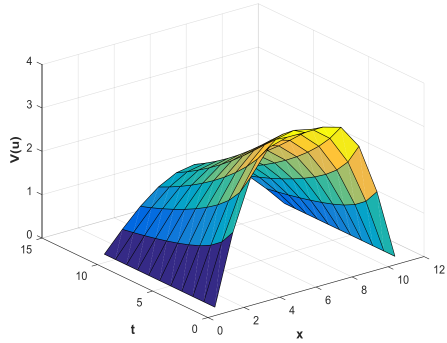
Moreover, if we apply the algorithm, then it is generated the variances of the approximate solutions of Eq. (8) for $0 < x_i < 1$ and $0 \leq t_j \leq 0.1$. Also, Monte Carlo simulation is performed 1000 times for the variances of these solutions of the Eq. (8) using the Matlab software. Then then the variances of approximate solutions of Eq. (8) in Table 2 is obtained for $0 < x_i < 1$ and $0 \leq t_j \leq 0.1$.

Table 2. The variances of approximate solutions of Eq. (8)

	$x_2 = 0.1$	$x_3 = 0.2$	$x_4 = 0.3$	$x_5 = 0.4$	$x_6 = 0.5$	$x_7 = 0.6$	$x_8 = 0.7$	$x_9 = 0.8$	$x_{10} = 0.9$
t_1	1,60787	2,77220	3,32875	3,45015	3,44174	3,45015	3,32875	2,77220	1,60787
t_2	1,26553	2,28933	2,95050	3,27783	3,36993	3,27783	2,95050	2,28933	1,26553
t_3	1,06739	1,98239	2,64701	3,03423	3,15924	3,03423	2,64701	1,98239	1,06739
t_4	0,93447	1,75856	2,38828	2,77700	2,90762	2,77700	2,38828	1,75856	0,93447
t_5	0,83340	1,57829	2,16053	2,52864	2,65426	2,52864	2,16053	1,57829	0,83340
t_6	0,74986	1,42428	1,95688	2,29716	2,41405	2,29716	1,95688	1,42428	0,74986
t_7	0,67748	1,28856	1,77342	2,08466	2,19189	2,08466	1,77342	1,28856	0,67748
t_8	0,61326	1,16715	1,60758	1,89090	1,98864	1,89090	1,60758	1,16715	0,61326
t_9	0,55561	1,05775	1,45741	1,71476	1,80360	1,71476	1,45741	1,05775	0,55561
t_{10}	0,50359	0,95884	1,32135	1,55487	1,63551	1,55487	1,32135	0,95884	0,50359
t_{11}	0,45652	0,86927	1,19801	1,40983	1,48298	1,40983	1,19801	0,86927	0,45652

Also, the graph of the variances of approximate solutions of Eq. (8) is given in figure 2.

Figure2 ***The time-dependent change of variance of the Eq. (8)***



5. CONCLUSION

In this study, we successfully applied Crank-Nicolson method to solve a random component heat equation. Also, the expected value and variance of this solution are obtained. Graphs of the expected value and variance are plotted with MAPLE software. Numerical results show that this method is very effective and practical.

REFERENCES

[1] Ayaz, F. (2004). Solutions of the system of differential equations by differential transform method. Applied Mathematics and Computation, 147(2), 547-567.

[2] Babuška, I., Nobile, F., & Tempone, R. (2007). A stochastic collocation method for elliptic partial differential equations with random input data. SIAM Journal on Numerical Analysis, 45(3), 1005-1034.

[3] Charrier, J. (2012). Strong and weak error estimates for elliptic partial differential equations with random coefficients. SIAM Journal on numerical analysis, 50(1), 216-246.

[4] Crank, J., & Nicolson, P. (1947). A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type. Proc. Camb. Phil. Soc., 43(1), 50-67, doi:10.1007/BF02127704.

[5] Cortés, J.C., Jódar, L., Villafuerte, L., & Villanueva, R.J. (2007). Computing mean square approximations of random diffusion models with source term. Mathematics and Computers in Simulation, 76(1-3), 44-48.

[6] Çelik, C., & Duman, M. (2012). Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative, Journal of computational physics, 231(4), 1743-1750.

[7] Ekolin, G. (1991). Finite difference methods for a nonlocal boundary value problem for the heat equation. BIT 31, 245-261.

[8] El-Tawil, & Sohaly, M.A. (2012). Mean square convergent three points finite difference scheme for random partial differential equations. J. of the Egyptian Math. Soc., 20(3), 188-204.

[9] Feller, W. (2008). An introduction to probability theory and its applications (Vol. 2). John Wiley & Sons.

[10] Gunzburger, M. D., Webster, C. G., & Zhang, G. (2014). Stochastic finite element methods for partial differential equations with random input data. Acta Numerica, 23, 521-650.

[11] He, J.H. (1999). Variational iteration method-a kind of non-linear analytical technique:some examples. Int. J. of Non-Linear Mech., 34(4), 699-708.

[12] He, J.H. (2003). Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and computation, 135(1), 73-79.

[13] He, J.H. (2006). Homotopy perturbation method for solving boundary value problems. Physics letters A, 350(1-2), 87-88.

[14] He, J.H. (2006). Addendum: new interpretation of homotopy perturbation method. International Journal of Modern Physics B, 20(18), 2561-2568.

[15] Kangalgil, F., & Ayaz, F. (2009). Solitary wave solutions for the KdV and mKdV equations by differential transform method. Chaos, Solitons & Fractals, 41(1), 464-472.

[16] Khudair, A.R., Ameen, A.A., & Khalaf, S.L. (2011). Mean square solutions of second-order random differential equations by using adomian decomposition method. Applied Mathematical Sciences, 5, 2521-2535.

[17] Kuo, F. Y., Schwab, C., & Sloan, I. H. (2012). Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. SIAM Journal on Numerical Analysis, 50(6), 3351-3374.

[18] Mathews, J.H., & Fink, K.D. (2004). Numerical methods using MATLAB (Vol. 4). Upper Saddle River, NJ: Pearson Prentice Hall.

[19] Merdan, M. (2010). A new application of modified differential transformation method for modeling the pollution of a system of lakes. Selçuk Journal of Applied Mathematics, 11(2), 27-40.

[20] Merdan, M., Anac, H., Bekiryazici, Z., & Kesemen, T. (2019). Solving of Some Random Partial Differential Equations by Using Differential Transformation Method and Laplace-Pade Method. Gümüşhane Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 9(1), 108-118.

[21] Nobile, F., Tempone, R., & Webster, C. G. (2008). An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. SIAM Journal on Numerical Analysis, 46(5), 2411-2442.

[22] Odibat, Z., & Momani, S. (2008). Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order. Chaos, Solitons & Fractals, 36(1), 167-174.

[23] Smith, G.D. (1965). Numerical Solution of Partial Differential Equations. Oxford University Press.

[24] Thirumalai, S., Seshadri, R., & Yuzbasi, S. (2019). Population dynamics between a prey and a predator using spectral collocation method. International Journal of Biomathematics, 1950049.

[25] Yüzbaşı, Ş. (2017). A numerical method for solving second-order linear partial differential equations under Dirichlet, Neumann and Robin boundary conditions. International Journal of Computational Methods, 14(02), 1750015.

[26] Yüzbaşi, Ş., & Ismailov, N. (2017). Differential Transform Method to Solve Two-Dimensional Volterra Integral Equations with Proportional Delays. New Trends in Mathematical Sciences, 5(4), 65-71.

[27] Yüzbaşi, Ş., & Karaçayir, M. (2017). Application of homotopy perturbation method to solve two models of delay differential equation systems. International Journal of Biomathematics, 10(06), 1750080.

[28] Yüzbaşı, Ş. (2018). A collocation approach for solving two-dimensional second-order linear hyperbolic equations. Applied Mathematics and Computation, 338, 101-114.

[29] Yüzbaşı, Ş., & Karaçayır, M. (2018). A Galerkin-Type Method to Solve One-Dimensional Telegraph Equation Using Collocation Points in Initial and Boundary Conditions. International Journal of Computational Methods, 15(05), 1850031.

[30] Zhou, J.K. (1986). Differential Transform and Its Applications for Electrical Circuits. Huazhong University Press, Wuhan, China.