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INVARIANTS OF A MAPPING OF A SET TO THE TWO-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. Let E_2 be the 2-dimensional Euclidean space and T be a set such that it has at least two elements. A mapping $\alpha : T \to E_2$ will be called a T-figure in E_2. Let $\mathbb{R}$ be the field of real numbers and $O(2,\mathbb{R})$ be the group of all orthogonal transformations of E_2. Put $SO(2,\mathbb{R}) = \{g \in O(2,\mathbb{R}) | detg = 1\}$, $MO(2,\mathbb{R}) = \{F : E_2 \to E_2 \mid Fx = gx + b, g \in O(2,\mathbb{R}), b \in E_2\}$, $MSO(2,\mathbb{R}) = \{F \in MO(2,\mathbb{R}) | detg = 1\}$. The present paper is devoted to solutions of problems of G-equivalence of T-figures in E_2 for groups $G = O(2,\mathbb{R}), SO(2,\mathbb{R})$, $MO(2,\mathbb{R})$, $MSO(2,\mathbb{R})$. Complete systems of G-invariants of T-figures in E_2 for these groups are obtained. Complete systems of relations between elements of the obtained complete systems of G-invariants are given for these groups.

1. Introduction

Let $\mathbb{R}$ be the field of real numbers, and let E_2 be the 2-dimensional Euclidean space.

The present paper is devoted to solution of problems of G-equivalence of T-figures in E_2 for groups $G = O(2,\mathbb{R}), SO(2,\mathbb{R}), MO(2,\mathbb{R}), MSO(2,\mathbb{R})$ in terms of G-invariants of a T-figure. We have obtain complete systems of G-invariants of T-figures for these groups and describe complete systems of relations between elements of the obtained complete systems of G-invariants.

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Let V be a finite dimensional vector space over a field K and β be a non-degenerate bilinear form on V. Denote by $O(\beta, K)$ the group of all β-orthogonal (that is the form β preserving) transformations of V. Let $MO(\beta, K)$ be the group generated by the group $O(\beta, K)$ and all translations of V. In the paper [6], for the orthogonal group $O(\beta, K)$ in the Euclidean, spherical, hyperbolic and de-Sitter geometries, the orbit of m vectors is characterized by their Gram matrix and an additional subspace. In the book [2, Proposition 9.7.1], for the group $MO(\beta, K)$ in the Euclidean geometry, the orbit of m-vectors is characterized by distances between m-vectors. A complete system of relations between elements of this complete system is also given in [2, Theorem 9.7.3.4]. In the paper [13], a complete system of invariants of m-tuples in the two-dimensional pseudo-Euclidean geometry of index 1 and a complete system relations between the obtained complete system of invariants are given. In the paper [15], a complete system of invariants of m-tuples in the one-dimensional projective space and a complete system relations between the obtained complete system of invariants are given. Invariants of m-points in Lorentzian geometry investigated in the paper [23]. Invariants of m-points appear also in the theory of invariants of Bezier curves ([5,22]), in Computer vision theory ([19,27]), in Computational Geometry ([21]). General theory of m-point invariants considered in the invariant theory (see [3, 8, 20, 30, 31]).

Complete systems of global invariants of paths and curves are investigated in papers [1, 7–9, 12, 14, 24–26]. Complete systems of global invariants of surfaces and vector fields are investigated in papers [10, 11, 28]. Complete systems of global invariants of T-figures in the affine geometry are investigated in the paper [17, 18].

This paper is organized as follows. In Section 1, some known results (Propositions 1-4) on the linear representation of the field of complex numbers in two-dimensional real space are given. Definitions of T-figures in the field $\mathbb{C}$ of complex numbers and in the two-dimensional linear space $\mathbb{R}^2$ are given. Put $S(\mathbb{C}^*) = \{z \in \mathbb{C} | |z| = 1\}$. A definition of $S(\mathbb{C}^*)$-equivalence of T-figures in $\mathbb{C}$ with respect to the group $S(\mathbb{C}^*)$ is given. A definition of $\Lambda(S(\mathbb{C}^*))$-equivalence of T-figures in $\mathbb{R}^2$ with respect to the group $\Lambda(S(\mathbb{C}^*))$ of linear transformation of $\mathbb{R}^2$ is given. It is proved Theorem 1 on a relation between the $S(\mathbb{C}^*)$-equivalence of T-figures in $\mathbb{C}$ and $\Lambda(S(\mathbb{C}^*))$-equivalence of T-figures in $\mathbb{R}^2$. In Section 2, evident forms of elements of groups $SO(2, \mathbb{R})$ and $O(2, \mathbb{R})$ are given. In Section 3, a complete system of G-invariants of a T-figure in the two-dimensional linear space $\mathbb{R}^2$ over the field $\mathbb{R}$ of real numbers for the group $G = SO(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of invariants are given. In Section 4, a complete system of G-invariants of a T-figure in $\mathbb{R}^2$ for the group $G = O(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of G-invariants is given. In Section 5, a complete system of G-invariants of a T-figure in $\mathbb{R}^2$ for the group $G = MSO(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of G-invariants is given. In Section 6, a complete system of G-invariants of a T-figure in $\mathbb{R}^2$ for the group $G = MO(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of G-invariants is given.

2. Some properties of a linear representation of the field of complex numbers in two-dimensional real space

A part of results of this section is known (see [16]).

Denote the field of complex numbers by $\mathbb{C}$. Let $c = c_1 + ic_2 \in \mathbb{C}$. Denote by Λ_c the matrix of the form $\begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix}$. Denote by $\Lambda(\mathbb{C})$ the set $\{\Lambda_c | c \in \mathbb{C}\}$. We consider on the set $\Lambda(\mathbb{C})$ following matrix operations: the component-wise addition and the multiplication of matrices. Then $\Lambda(\mathbb{C})$ is a field with respect to these operations. In it the unit element is the unit matrix.

Proposition 1. *The mapping $\Lambda : \mathbb{C} \to \Lambda(\mathbb{C})$, where $\Lambda : c \to \Lambda_c, \forall c \in \mathbb{C}$, is an isomorphism of the fields $\mathbb{C}$ and $\Lambda(\mathbb{C})$.*

Proof. It is obvious. □

Let $a = a_1 + ia_2 \in \mathbb{C}, b = b_1 + ib_2 \in \mathbb{C}$. Put $\langle a, b\rangle = a_1b_1 + a_2b_2$. Then $\langle a, b\rangle$ is a bilinear form on $\mathbb{R}^2$ and $\langle a, a\rangle = a_1^2 + a_2^2$ is a quadratic form on $\mathbb{R}^2$. For convenience, we denote by $Q(a)$ the quadratic form $\langle a, a\rangle$.

The following propositions 2, 3 and 4 are known.

Proposition 2. *The following equalities $Q(x) = det(\Lambda_x)$ and $Q(xy) = Q(x)Q(y)$ hold for all $x = x_1 + ix_2, y = y_1 + iy_2 \in \mathbb{C}$.*

For $x = x_1 + ix_2 \in \mathbb{C}$, we set $\overline{x} = x_1 - ix_2$.

Proposition 3. *The mapping $x \to \overline{x}$ is an involution of the field $\mathbb{C}$ and the following equalities $x + \overline{x} = 2x_1, \langle x, x\rangle = x\overline{x} = \overline{x}x = x_1^2 + x_2^2$, $Q(x) = Q(\overline{x})$ hold for all $x = x_1 + ix_2 \in \mathbb{C}$.*

Proposition 4. *Let $x \in \mathbb{C}$. Then the element x^{-1} exists if and only if $Q(x) \neq 0$. In the case $Q(x) \neq 0$, the equalities $x^{-1} = \frac{\overline{x}}{Q(x)}$ and $Q(x^{-1}) = \frac{1}{Q(x)}$ hold.*

Put $\mathbb{C}^* = \{x \in \mathbb{C} \mid Q(x) \neq 0\}$. $\mathbb{C}^*$ is a group with respect to the multiplication operation in the field $\mathbb{C}$. Denote by $\Lambda(\mathbb{C}^*)$ the set of all matrices Λ_a, where $a \in \mathbb{C}^*$. For $a \in \mathbb{C}^*$, we have $Q(a) = a_1^2 + a_2^2 \neq 0$ and $Q(a) = det(\Lambda_a) \neq 0$.

Below everywhere we will consider every element $x \in \mathbb{R}^2$ and $x \in E_2$ as a column vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Denote by Γ the following mapping $\Gamma : \mathbb{C} \to \mathbb{R}^2$, where $\Gamma(x_1 + ix_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. It is obvious that the mapping Γ is an isomorphism of linear spaces $\mathbb{C}$ and $\mathbb{R}^2$. Hence there exists the converse isomorphism $\Gamma^{-1}$of Γ and $\Gamma^{-1}(x) = x_1 + ix_2, \forall x \in \mathbb{R}^2$.

Denote by W the following matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Denote by L_a the following linear operator on $\mathbb{C}$: $L_a(x) = a \cdot x, \forall x \in \mathbb{C}$, $a \in \mathbb{C}^*$. Then the following equalities are obvious:

$\Gamma(a_1 + ia_2) = W\Gamma(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix} = \Gamma(\overline{a}), \forall a = a_1 + ia_2 \in \mathbb{C}^*$.

$$\Gamma(L_a(x)) = \Gamma(a \cdot x) = \begin{pmatrix} a_1x_1 - a_2x_2 \\ a_1x_2 + a_2x_1 \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Lambda_a \cdot \Gamma(x), \tag{1}$$

$\forall a \in \mathbb{C}^*, \forall x \in \mathbb{C}$, where $\Lambda_a \cdot \Gamma(x)$ is the multiplication of matrices Λ_a and $\Gamma(x)$.

Hence $\Lambda_a \in \Lambda(\mathbb{C}^*)$ and the mapping $\Lambda : \mathbb{C}^* \to \Lambda(\mathbb{C}^*)$, where $\Lambda(a) = \Lambda_a$, is a linear representation of the groups.

Put $S(\mathbb{C}^*) = \{x \in \mathbb{C} \mid Q(x) = 1\}$. It is a subgroup of the group $\mathbb{C}^*$. $\Lambda(S(\mathbb{C}^*))$ is a subgroup of the group $\Lambda(\mathbb{C}^*)$ and the mapping $\Lambda : S(\mathbb{C}^*) \to \Lambda(\mathbb{C}^*)$, where $\Lambda(a) = \Lambda_a$, is a linear representation of the group $S(\mathbb{C}^*)$ in $\mathbb{R}^2$. $\Lambda(\mathbb{C}^*)$ is a group with respect to the multiplication of matrices. Let T be a set such that it has at least two elements. Denote by $\mathbb{C}^T$ the set of all mappings of the set T to the field $\mathbb{C}$. An element of $\alpha \in \mathbb{C}^T$ will be called a T-figure in the field $\mathbb{C}$. For the figure α, we also use the notation $\alpha(t)$, considering α as a function on T with values in $\mathbb{C}$. Denote by E_2^T the set of all mappings of the set T to E_2. An element $\gamma \in E_2^T$ will be called a T-figure in the space E_2. For the figure γ, we also use the notation $\gamma(t)$, considering γ as a function on T with values in E_2.

Let G be a subgroup of the group $\mathbb{C}^*$.

Definition 1. *Two T-figures $\alpha \in \mathbb{C}^T$ and $\beta \in \mathbb{C}^T$ is called G-equivalent if there exists $g \in G$ such that $\beta(t) = g \cdot \alpha(t), \forall t \in T$. In this case, we also write as follows: $\alpha \overset{G}{\sim} \beta$ or $\alpha(t) \overset{G}{\sim} \beta(t), \forall t \in T$.*

Let G be a subgroup of the group $\mathbb{C}^*$.

Definition 2. *Two T-figures $\gamma \in E_2^T$ and $\eta \in E_2^T$ is called $\Lambda(G)$-equivalent if there exists $a \in G$ such that $\eta(t) = \Lambda_a\gamma(t), \forall t \in T$. In this case, we also write as follows: $\gamma \overset{\Lambda(G)}{\sim} \eta$ or $\gamma(t) \overset{\Lambda(G)}{\sim} \eta(t), \forall t \in T$.*

Theorem 1. *Let $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ be two T-figures in $\mathbb{C}$. Then T-figures $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent if and only if T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ in E_2 are $\Lambda(S(\mathbb{C}^*))$-equivalent.*

Proof. Assume that T-figures $\alpha(t) = \alpha_1 + i\alpha_2(t)$ and $\beta(t) = \alpha_1 + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent. Then there exists $a = a_1 + ia_2 \in S(\mathbb{C}^*)$ such that $\beta(t) = a \cdot \alpha(t), \forall t \in T$.

Using this equality and the equality (1), we obtain following equality:

$$\Gamma(\beta(t)) = \Gamma(a\cdot\alpha(t)) = \begin{pmatrix} a_1\alpha_1(t) - a_2\alpha_2(t) \\ a_1\alpha_2(t) + a_2\alpha_1(t) \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \Lambda_a \Gamma(\alpha(t)), \forall t \in T.$$

This equality means that T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ are $\Lambda(S(\mathbb{C}^*))$-equivalent .

Conversely, assume that T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ are $\Lambda(S(\mathbb{C}^*))$-equivalent. Since Γ is an isomorphism, Γ^{-1} exists. Then the above equality implies that $\beta(t) = \Gamma^{-1}(\Gamma(\beta(t))) = \Gamma^{-1}(\Gamma(a\cdot\alpha(t))) = a\cdot\alpha(t), \forall t \in T$. Hence T-figures $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent. □

3. Fundamental groups of transformations of the 2-dimensional Euclidean space

Let E_2 be the 2-dimensional Euclidean space with the scalar product $\langle a, b\rangle = a_1b_1 + a_2b_2$, where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in E_2$.

Definition 3. *A mapping $F : E_2 \to E_2$ is called orthogonal if $\langle Fx, Fy\rangle = \langle x, y\rangle$ for all $x, y \in E_2$.*

Denote the set of all orthogonal transformations of E_2 by $O(2, \mathbb{R})$.

The following propositions 5-7 are well known.

Proposition 5. *([4], p.221) Every orthogonal transformation of E_2 is linear.*

Proposition 6. *$O(2, \mathbb{R})$ is a group with respect to the multiplication operation of matrices.*

Let $a = a_1 + ia_2, b = b_1 + ib_2 \in \mathbb{C}$. Denote the identity matrix of the bilinear form $\langle a, b\rangle = a_1b_1 + a_2b_2$ by $I = \|\delta_{ij}\|_{i,j=1,2}$, where $\delta_{11} = \delta_{22} = 1, \delta_{12} = \delta_{21} = 0$. By Proposition 5, we can consider every element of $O(2, \mathbb{R})$ as a 2×2-matrix. Let $H \in O(2, \mathbb{R})$, where $H = \|h_{ij}\|_{i,j=1,2}$. Let H^T be the transpose matrix of H. It is known that the equality $\langle Hx, Hy\rangle = \langle x, y\rangle$ for all $x, y \in E_2$ is equivalent to the equality

$$H^T H = I. \tag{2}$$

This equality implies the following

Proposition 7. *Let $H \in O(2, \mathbb{R})$. Then $det(H) = 1$ or $det(H) = -1$.*

We denote by $SO(2, \mathbb{R})$ the set $\{H \in O(2, \mathbb{R}) : det(H) = 1\}$. $SO(2, \mathbb{R})$ is a subgroup of $O(2, \mathbb{R})$. $O(2, \mathbb{R}) = SO(2, \mathbb{R}) \cup \{HW \mid H \in SO(2, \mathbb{R})\}$, where HW is the multiplication of matrices H and W, where $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Theorem 2. *The equality $SO(2, \mathbb{R}) = \Lambda(S(\mathbb{C}^*))$ holds.*

Proof. $\Leftarrow$. We assume that $H \in \Lambda(S(\mathbb{C}^*))$. Then it has the following form $H = \|h_{ij}\|_{i,j=1,2}$, where $h_{11} = h_{22} = c, h_{21} = d, h_{12} = -d$, $c, d \in \mathbb{R}$ and $det(H) = c^2 + d^2 = 1$. We prove that $H \in SO(2,\mathbb{R})$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in E_2$. We have

$$H(x) = \begin{pmatrix} cx_1 - dx_2 \\ dx_1 + cx_2 \end{pmatrix}, H(y) = \begin{pmatrix} cy_1 - dy_2 \\ dy_1 + cy_2 \end{pmatrix}.$$

Using the equality $c^2 + d^2 = 1$, we obtain

$$\langle H(x), H(y)\rangle = (cx_1 - dx_2)(cy_1 - dy_2) + (dx_1 + cx_2)(dy_1 + cy_2) = \\ (c^2 + d^2)(x_1y_1 + x_2y_2) = \langle x, y\rangle .$$

Hence $H \in SO(2,\mathbb{R})$.

$\Rightarrow$. We assume that $H \in SO(2,\mathbb{R})$, where $H = \|h_{ij}\|_{i,j=1,2}$. Then $det(H) = h_{11}h_{22} - h_{12}h_{21} = 1$ and the equality (2) holds. These equalities imply the following system of equalities

$$h_{11}^2 + h_{21}^2 = 1 \qquad (3)$$
$$h_{11}h_{12} + h_{21}h_{22} = 0 \qquad (4)$$
$$h_{12}^2 + h_{22}^2 = 1 \qquad (5)$$
$$h_{11}h_{22} - h_{12}h_{21} = 1 \qquad (6)$$

We consider two cases $h_{12} = 0$ and $h_{12} \neq 0$.

Let $h_{12} = 0$. Then (5) implies $h_{22}^2 = 1$. Hence $h_{22} = 1$ or $h_{22} = -1$. Let $h_{22} = 1$. Then the equalities $h_{22} = 1, h_{12} = 0$ and (4) imply $h_{21} = 0$. Using equalities $h_{21} = 0$ and (3), we obtain $h_{11}^2 = 1$. Hence $h_{11} = 1$ or $h_{11} = -1$. Thus, in the case $h_{12} = 0$ and $h_{22} = 1$, we obtain $h_{21} = 0$ and $h_{11} = 1$ or $h_{11} = -1$. Hence, in this case, we obtain only the following two matrices:

$$A_1 = \{h_{11} = h_{22} = 1, h_{12} = h_{21} = 0\}, A_2 = \{h_{11} = -1, h_{12} = h_{21} = 0, h_{22} = 1\}.$$

It is obviously that $A_1 \in \Lambda(S(\mathbb{C}^*))$ and $A_2 \notin SO(2,\mathbb{R})$.

Let $h_{22} = -1$. Then the equalities $h_{22} = -1, h_{12} = 0$ and (4) imply $h_{21} = 0$. Using equalities $h_{21} = 0$ and (3), we obtain $h_{11}^2 = 1$. Hence $h_{11} = 1$ or $h_{11} = -1$. Thus, in the case $h_{12} = 0$ and $h_{22} = -1$, we obtain $h_{21} = 0$ and $h_{11} = 1$ or $h_{11} = -1$. Hence, in this case, we obtain only the following two matrices:

$$A_3 = \{h_{11} = 1, h_{12} = h_{21} = 0, h_{22} = -1\}, A_4 = \{h_{11} = h_{22} = -1, h_{12} = h_{21} = 0\}.$$

It is obviously that $A_4 \in \Lambda(S(\mathbb{C}^*))$ and $A_3 \notin SO(2,\mathbb{R})$.

Let $h_{12} \neq 0$. Using (4), we obtain

$$h_{11} = -\frac{h_{21}h_{22}}{h_{12}}.$$

Using this equality and equalities (3), (5), we obtain:

$$(-\frac{h_{21}h_{22}}{h_{12}})^2 + h_{21}^2 = 1 \Rightarrow h_{21}^2h_{22}^2 + h_{12}^2h_{21}^2 = h_{12}^2 \Rightarrow h_{21}^2(h_{22}^2 + h_{12}^2) = h_{12}^2 \Rightarrow h_{21}^2 = h_{12}^2 \Rightarrow h_{12}^2 - h_{21}^2 = 0.$$

Hence we obtain $h_{12} - h_{21} = 0$ or $h_{12} + h_{21} = 0$. We consider two cases $h_{12} - h_{21} = 0$ and $h_{12} + h_{21} = 0$.

Let $h_{12} - h_{21} = 0$. Then $h_{12} = h_{21}$. Since $h_{12} \neq 0$, we obtain $h_{21} \neq 0$. Using the equality $h_{12} = h_{21}$ and (4), we obtain $h_{11}h_{21} - h_{21}h_{22} = 0$. Hence $h_{21}(h_{11} + h_{22}) = 0$. Since $h_{21} \neq 0$, this equality implies $h_{11} = -h_{22}$. Thus we have obtained the following equalities: $h_{12} = h_{21}$ and $h_{11} = -h_{22}$. Using (6), we obtain $-h_{11}^2 - h_{12}^2 = 1$. Since $h_{12} \neq 0$ and $-(h_{11}^2 + h_{12}^2) = 1$, we have a contradiction. Hence this case is not possible.

Consider the case $h_{12} + h_{21} = 0$. This equality implies the equality $h_{12} = -h_{21}$. Using this equality and the equality (4) : $h_{11}h_{12} + h_{21}h_{22} = 0$, we obtain $h_{11}h_{12} - h_{12}h_{22} = 0$. Hence $h_{12}(h_{11} - h_{22}) = 0$. Since $h_{12} \neq 0$, this equality implies $h_{11} = h_{22}$. Hence the equalities $h_{12} = -h_{21}$, $h_{11} = h_{22}$ hold. These equalities and (3) imply that the matrix H has the form $\begin{pmatrix} h_{11} & -h_{21} \\ h_{21} & h_{11} \end{pmatrix}$, where $det(H) = 1$. Hence $H \in \Lambda(S(\mathbb{C}^*))$. □

Corollary 1. *Let $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ be T-figures in $\mathbb{C}$. Then T-figures $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent if and only if T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ in E_2 are $SO(2,\mathbb{R})$-equivalent.*

Proof. It follows from Theorems 1 and 2. □

Denote by $MO(2,\mathbb{R})$ the group of all transformations of E_2 generated by the group $O(2,\mathbb{R})$ and all translations of E_2. Elements of the group $MO(2,\mathbb{R})$ has the following form $F : E_2 \to E_2$, where $F(x) = g(x) + a$, $g \in O(2,\mathbb{R}), a \in \mathbb{R}^2$. Denote by $MSO(2,\mathbb{R})$ the group of all transformations of E_2 generated by the group $SO(2,\mathbb{R})$ and all translations of E_2. Elements of the group $MSO(2,\mathbb{R})$ has the following form $F : E_2 \to E_2$, where $F(x) = g(x) + a$, $g \in SO(2,\mathbb{R}), a \in \mathbb{R}^2$.

4. Complete systems of G-invariants of a T-figure in E_2 for the group $G = SO(2,\mathbb{R})$

Let G be a subgroup of the group $MO(2,\mathbb{R})$.

Definition 4. *Two T-figures α and β in E_2 are called G-equivalent if there exists $g \in G$ such that $\alpha = g\beta$. In this case, we also write as follows: $\alpha \overset{G}{\sim} \beta$ or $\alpha(t) \overset{G}{\sim} \beta(t), \forall t \in T$.*

Definition 5. *A function $f(\alpha(t), \beta(t), \ldots, \gamma(t))$ of a finite number of T-figures $\alpha(t), \beta(t), \ldots, \gamma(t)$ is called G-invariant function if*

$f(F\alpha(t), F\beta(t), \dots, F\gamma(t)) = f(\alpha(t), \beta(t), \dots, \gamma(t))$ for all $F \in G$, all T-figures $\alpha(t), \beta(t), \dots, \gamma(t)$ and all $t \in T$.

Example 1. *By the definitions of the groups $O(2,\mathbb{R})$ and $SO(2,\mathbb{R})$, we obtain that the quadratic form $Q : E_2 \to \mathbb{R}$,$Q(x) = \langle x, x\rangle$ is $O(2,\mathbb{R})$-invariant function on E_2 and the bilinear form $f : E_2 \times E_2 \to \mathbb{R}$, $f(x, y) = \langle x, y\rangle$ are $O(2,\mathbb{R})$-invariant functions on the set $E_2 \times E_2$.*

Example 2. *Denote by $[x\, y]$ the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in E_2$. Consider the function $h : E_2 \times E_2 \to \mathbb{R}$, $h(x, y) = [x\, y]$. Using the equality $det(g) = 1, \forall g \in SO(2,\mathbb{R})$, we obtain $[(gx)\,(gy)] = det(g)[x\, y] = [x\, y], \forall g \in SO(2,\mathbb{R}), \forall x, y \in E_2$. This means that $[x\, y]$ is an $SO(2,\mathbb{R})$-invariant function on the set $E_2 \times E_2$. Clearly, $h(x, y)$ is not an $O(2,\mathbb{R})$-invariant function on the set $E_2 \times E_2$.*

Example 3. *By definitions of the groups $G = MO(2,\mathbb{R}), MSO(2,\mathbb{R})$ we obtain that function $f : E_2 \times E_2 \to \mathbb{R}$, $f(x, y) = \langle x - y, x - y\rangle$ is an G-invariant function on the set $E_2 \times E_2$.*

Definition 6. *A system $\{f_1, f_2, \dots, f_m\}$ of G-invariant functions $f_1, f_2, \dots, f_m$ of a T-figure α in E_2^T will be called a complete system of G-invariant functions of T-figure if equalities $f_j(\alpha) = f_j(\beta), \forall j \in 1, 2, \dots, m$ imply $\alpha \overset{G}{\sim} \beta$.*

Denote by θ the vector $\theta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in E_2$. Let α be a T-figure in E_2. Denote by $Z(\alpha)$ the set $\{t \in T | \alpha(t) = \theta\}$. Denote by $\theta_T(t)$ the T-figure such that $\theta_T(t) = \theta, \forall t \in T$.

Denote by 2^T the set of all subsets of the set T.

Proposition 8. (1) *Let G be a subgroup of $\mathbb{C}^*$. Assume that $\alpha, \beta \in \mathbb{C}^T$ such that $\alpha \overset{G}{\sim} \beta$. Then $Z(\alpha) = Z(\beta)$. This means that the function $Z : \mathbb{C}^T \to 2^T$ is a G-invariant function on $\mathbb{C}^T$.*

(2) *Let G be a subgroup of $O(2,\mathbb{R})$. Assume that $\alpha, \beta \in E_2^T$ such that $\alpha \overset{G}{\sim} \beta$. Then $Z(\alpha) = Z(\beta)$ that is the function $Z : E_2^T \to 2^T$ is a G-invariant function on E_2^T.*

Proof. It is obvious. □

Proposition 9. *Let $\mathbb{C}$ be the field of complex numbers and $x = x_1 + ix_2, y = y_1 + iy_2 \in \mathbb{C}$ such that $x \neq 0$. Then,*

(1) *the element* yx^{-1} *exists, the equality* $yx^{-1} = \frac{\langle x,y\rangle}{Q(x)} + i\frac{[x\,y]}{Q(x)}$ *and the following equality hold*

$$\Lambda_{yx^{-1}} = \begin{pmatrix} \frac{\langle x,y\rangle}{Q(x)} & -\frac{[x\,y]}{Q(x)} \\ \frac{[x\,y]}{Q(x)} & \frac{\langle x,y\rangle}{Q(x)} \end{pmatrix} \tag{7}$$

where $\langle x,y\rangle = x_1y_1 + x_2y_2$ *and* $[x\,y] = x_1y_2 - x_2y_1$.

(2) $\det(\Lambda_{yx^{-1}}) \neq 0$ *if and only if* $Q(y) \neq 0$.

Proof. It is given in [16, Proposition 4. 9]. □

Now we consider the G-equivalence problem of T-figures in the field $\mathbb{C}$ for the group $S(\mathbb{C}^*)$.

Let α and β be T-figures in $\mathbb{C}$ such that $\alpha(t) = \beta(t) = 0, \forall t \in T$, that is $Z(\alpha) = Z(\beta) = T$. In this case, it is obvious that $\alpha \overset{S(\mathbb{C}^*)}{\sim} \beta$.

Theorem 3. *Let* α *be a* T*-figure in the field* $\mathbb{C}$ *such that* $Z(\alpha) \neq T$, *and* $t_0 \in T \setminus Z(\alpha)$.

(i) *Suppose that a* T*-figure* β *in* $\mathbb{C}$ *such that* $\alpha \overset{S(\mathbb{C}^*)}{\sim} \beta$. *Then the following equalities hold:*

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [\alpha(t_0)\alpha(t)] = [\beta(t_0)\,\beta(t)]\,, \forall t \in T \setminus Z(\alpha). \end{cases} \tag{8}$$

(ii) *Conversely, assume that a* T*-figure* β *in* $\mathbb{C}$ *such that the equalities* (8) *hold. Then there exists a single element* $g \in S(\mathbb{C}^*)$ *such that* $\beta = g \cdot \alpha$. *In this case, it has the following form* $g = \beta(t_0)(\alpha(t_0))^{-1}$.

Proof. Assume that $\alpha \overset{S(\mathbb{C}^*)}{\sim} \beta$. Then there exists $a \in S(\mathbb{C}^*)$ such that $\beta(t) = a \cdot \alpha(t), \forall t \in T$. By Proposition 8-(1), we obtain the equality $Z(\alpha) = Z(\beta)$. Hence the equality $Z(\alpha) = Z(\beta)$ in (8) is proved.

The equality $Z(\alpha) = Z(\beta)$ and the inequality $Z(\alpha) \neq T$ imply inequality $Z(\beta) \neq T$. Since $t_0 \in T \setminus Z(\alpha) = T \setminus Z(\beta)$, we obtain that $\alpha(t_0) \neq 0$ and $\beta(t_0) \neq 0$. The inequality $\alpha(t_0) \neq 0$ implies an existence of $(\alpha(t_0))^{-1}$. Consider following functions $\alpha(t) \cdot (\alpha(t_0))^{-1}$ and $\beta(t) \cdot (\beta(t_0))^{-1}$ on T. The above equality $\beta(t) = a \cdot \alpha(t), \forall t \in T$, implies following equality: $\beta(t) \cdot (\beta(t_0))^{-1} = a \cdot \alpha(t) \cdot (a \cdot \alpha(t_0))^{-1} = (a \cdot a^{-1}) \cdot \alpha(t) \cdot (\alpha(t_0))^{-1} = \alpha(t) \cdot (\alpha(t_0))^{-1}, \forall t \in T$. Hence following equality holds:$\beta(t) \cdot (\beta(t_0))^{-1} = \alpha(t) \cdot (\alpha(t_0))^{-1}, \forall t \in T$. Using Proposition 9, we obtain following equalities:
$\alpha(t) \cdot (\alpha(t_0))^{-1} = \frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))}, \beta(t) \cdot (\beta(t_0))^{-1} = \frac{\langle \beta(t_0), \beta(t)\rangle}{Q(\beta(t_0))} + i\frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}$. These equalities and the equality $\beta(t) \cdot (\beta(t_0))^{-1} = \alpha(t) \cdot (\alpha(t_0))^{-1}, \forall t \in T$, imply following equality: $\frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0), \beta(t)\rangle}{Q(\beta(t_0))} + i\frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}, \forall t \in T$. This

equality imply following equalities:

$$\begin{cases} \frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0), \beta(t) \rangle}{Q(\beta(t_0))}, \forall t \in T \\ \frac{[\alpha(t_0)\, \alpha(t)]}{Q(\alpha(t_0))} = \frac{[\beta(t_0)\, \beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \end{cases} \tag{9}$$

The equality $\beta(t) = a \cdot \alpha(t), \forall t \in T$, implies following equality $Q(\beta(t_0)) = Q(a \cdot \alpha(t_0))$. Using Proposition 2, we obtain following equality $Q(\beta(t_0)) = Q(a) \cdot Q(\alpha(t_0))$. Since $a \in S(\mathbb{C}^*)$, we have $Q(a) = 1$. This equality and previous equality $Q(\beta(t_0)) = Q(a) \cdot Q(\alpha(t_0))$ imply following equality $Q(\beta(t_0)) = Q(\alpha(t_0))$. This equality and (9) imply following equalities:

$$\begin{cases} \frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0), \beta(t) \rangle}{Q(\alpha(t_0))}, \forall t \in T \\ \frac{[\alpha(t_0)\, \alpha(t)]}{Q(\alpha(t_0))} = \frac{[\beta(t_0)\, \beta(t)]}{Q(\alpha(t_0))}, \forall t \in T. \end{cases}$$

These equalities imply following equalities in (8):

$$\begin{cases} \langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle, \forall t \in T \\ [\alpha(t_0)\, \alpha(t)] = [\beta(t_0)\, \beta(t)], \forall t \in T. \end{cases}$$

Hence equalities (8) is proved.

Conversely, assume that T-figures α and β in $\mathbb{C}$ such that the equalities (8) hold. By the supposition in the present theorem $t_0 \in T \setminus Z(\alpha(t))$. This implies $\alpha(t_0) \neq 0$. This inequality and the equality $Z(\alpha(t)) = Z(\beta(t))$ in (8) imply the inequality $\beta(t_0) \neq 0$. In the equality $\langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle, \forall t \in T$, in (8) we put $t = t_0$. Then we obtain following equality $\langle \alpha(t_0), \alpha(t_0) \rangle = \langle \beta(t_0), \beta(t_0) \rangle$. This equality and the following equalities $Q(\alpha(t_0)) = \langle \alpha(t_0), \alpha(t_0) \rangle$, $Q(\beta(t_0)) = \langle \beta(t_0), \beta(t_0) \rangle$ imply following equality $Q(\alpha(t_0)) = Q(\beta(t_0))$. The inequality $\alpha(t_0) \neq 0$ implies following inequality $Q(\alpha(t_0)) \neq 0$. This inequality, the equality $Q(\alpha(t_0)) = Q(\beta(t_0))$ and the equalities in (8) imply following equality:

$$\begin{cases} \frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0), \beta(t) \rangle}{Q(\beta(t_0))}, \forall t \in T \\ \frac{[\alpha(t_0)\, \alpha(t)]}{Q(\alpha(t_0))} = \frac{[\beta(t_0)\, \beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \end{cases}$$

These equalities imply following equalities:

$$\frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} + i \frac{[\alpha(t_0)\, \alpha(t)]}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0), \beta(t) \rangle}{Q(\beta(t_0))} + i \frac{[\beta(t_0)\, \beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \tag{10}$$

By Proposition 9, we obtain following equalities:

$$\alpha(t) \cdot (\alpha(t_0))^{-1} = \frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} + i \frac{[\alpha(t_0)\, \alpha(t)]}{Q(\alpha(t_0))}, \tag{11}$$

$$\beta(t) \cdot (\beta(t_0))^{-1} = \frac{\langle \beta(t_0), \beta(t) \rangle}{Q(\beta(t_0))} + i \frac{[\beta(t_0)\, \beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \tag{12}$$

Equalities (10), (11) and (12) imply following equality:

$$\beta(t) \cdot (\beta(t_0))^{-1} = \alpha(t) \cdot (\alpha(t_0))^{-1}, \forall t \in T. \quad (13)$$

This equality implies following equality:

$$\beta(t) = \beta(t_0) \cdot (\alpha(t_0))^{-1} \cdot \alpha(t), \forall t \in T. \quad (14)$$

Since $Q(\alpha(t_0)) = Q(\beta(t_0))$, using this equality and Propositions 2, 4, we obtain following equality: $Q(\beta(t_0) \cdot (\alpha(t_0))^{-1}) = Q(\beta(t_0)) \cdot (Q(\alpha(t_0)))^{-1} = Q(\beta(t_0)) \cdot (Q(\beta(t_0)))^{-1} = 1$. This means that $\beta(t_0)(\alpha(t_0))^{-1} \in S(\mathbb{C}^*)$. Hence (14) implies that $\alpha(t) \overset{S(\mathbb{C}^*)}{\sim} \beta(t), \forall t \in T$.

Prove the uniqueness of $h \in S(\mathbb{C}^*)$ satisfying the conditions $\beta(t) = h\alpha(t), \forall t \in T$. Assume that $h \in S(\mathbb{C}^*)$ such that $\beta(t) = h\alpha(t), \forall t \in T$. In particularly, for $t = t_0$, the equality $\beta(t) = h\alpha(t)$ implies following equality: $\beta(t_0) = h\alpha(t_0)$. This equality and the inequality $\alpha(t_0) \neq 0$ imply following equality $\beta(t_0)(\alpha(t_0))^{-1} = h$. Hence the uniqueness of h is proved. □

Theorem 4. *Let α be a T-figure in E_2 such that $Z(\alpha) \neq T$, and $t_0 \in T \setminus Z(\alpha)$.*

- (i) *Suppose that a T-figure β in E_2 such that $\alpha \overset{SO(2,\mathbb{R})}{\sim} \beta$. Then the following equalities hold:*

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [\alpha(t_0)\alpha(t)] = [\beta(t_0)\,\beta(t)]\,, \forall t \in T \setminus Z(\alpha). \end{cases} \quad (15)$$

- (ii) *Conversely, assume that a T-figure β in E_2 such that the equalities* (15) *hold. Then there exists a single matrix $H \in SO(2,\mathbb{R})$ such that $\beta = H\alpha$. In this case, H has the following form*

$$H = \begin{pmatrix} \frac{\langle \alpha(t_0), \beta(t_0)\rangle}{\langle \alpha(t_0), \alpha(t_0), \rangle} & -\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0)\rangle} \\ \frac{[\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0)\rangle} & \frac{\langle \alpha(t_0), \beta(t_0)\rangle}{\langle \alpha(t_0), \alpha(t_0)\rangle} \end{pmatrix}, \quad (16)$$

where $\det(H) = (\frac{\langle \alpha(t_0), \beta(t_0)\rangle}{\langle \alpha(t_0), \alpha(t_0)\rangle})^2 + (\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0)\rangle})^2 = 1.$

Proof. We consider T-figures α and β in E_2 as column vector functions: $\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}$, $\beta(t) = \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}$. Assume that $\alpha \overset{SO(2,\mathbb{R})}{\sim} \beta$. Then, by Proposition 8-(2), $Z(\alpha) = Z(\beta)$. This equality and the inequality $Z(\alpha) \neq T$ imply inequality $Z(\beta) \neq T$. Since functions $\langle \alpha(t_0), \alpha(t)\rangle$ and $[\alpha(t_0)\alpha(t)]$ are $SO(2,\mathbb{R})$-invariant, the $SO(2,\mathbb{R})$-equivalence $\alpha \overset{SO(2,\mathbb{R})}{\sim} \beta$, and the equality $Z(\alpha) = Z(\beta)$ imply equalities (15).

Conversely, assume that a T-figures α and β in E_2 such that the equalities (15) hold. Consider following T-figures in the field $\mathbb{C}$: $\Gamma^{-1}(\alpha(t)) = \alpha_1(t) + i\alpha_2(t), \forall t \in T$, $\Gamma^{-1}(\beta(t)) = \beta_1(t) + i\beta_2(t), \forall t \in T$. For these T-figures in $\mathbb{C}$ the equalities (15) also hold. Then, by Theorem 3, these T-figures are $S(\mathbb{C}^*)$-equivalent and there exists a single element $g \in S(\mathbb{C}^*)$ such that $\beta_1(t) + i\beta_2(t) = g \cdot (\alpha_1(t) + i\alpha_2(t)), \forall t \in T$. In

this case, by Theorem 3, g has the following form:
$g = \frac{\beta_1(t_0)+i\beta_2(t_0)}{\alpha_1(t_0)+i\alpha_2(t_0)} = \frac{(\beta_1(t_0)+i\beta_2(t_0))\cdot(\alpha_1(t_0)-i\alpha_2(t_0))}{(\alpha_1(t_0)+i\alpha_2(t_0))\cdot(\alpha_1(t_0)-i\alpha_2(t_0))}$
$= \frac{(\alpha_1(t_0)\beta_1(t_0)+\alpha_2(t_0)\beta_2(t_0))+i(\alpha_1(t_0)\beta_2(t_0)-\alpha_2(t_0)\beta_1(t_0))}{(\alpha_1(t_0))^2+(\alpha_2(t_0))^2} = \frac{\langle\alpha(t_0),\beta(t)\rangle+i[\alpha(t_0)\,\beta(t_0)]}{Q(\alpha(t_0))}$.

The $S(\mathbb{C}^*)$-equivalence of the T-figures $\Gamma^{-1}(\alpha)$, and $\Gamma^{-1}(\beta(t)) = \beta_1(t)+i\beta_2(t), \forall t \in T$ in $\mathbb{C}$, by Theorem 3, implies $SO(2,\mathbb{R})$-equivalence of T-figures α and β in E_2. In this case there exists a single element $H \in SO(2,\mathbb{R})$ such that $H = \Lambda_g$ and $\beta(t) = H \cdot \alpha(t), \forall t \in T$. By Proposition 9, the above form of $g = \frac{\langle\alpha(t_0),\beta(t)\rangle+i[\alpha(t_0)\,\beta(t_0)]}{Q(\alpha(t_0))}$ implies that H has the form (16), where $det(H) = 1$. □

Remark 1. *Assume that T be a set such that it has at least two elements. By Theorem 4, the system*

$$\{Z(\alpha), \langle\alpha(t_0), \alpha(t)\rangle, [\alpha(t_0)\,\alpha(t)]\} \tag{17}$$

is a complete system of $SO(2,\mathbb{R})$-invariant functions on the set of all T-figures α in E_2 such that $Z(\alpha) \neq T$, and $t_0 \in T \setminus Z(\alpha)$.

Now let us find a complete system of relations between elements of this complete system.

Theorem 5. *Let (17) be the complete system of $SO(2,\mathbb{R})$-invariants of a T-figure α in E_2. Assume that:*
(1.1) *U is a subset of T such that $U \neq T$*
(1.2) *$t_0 \in T \setminus U$*
(1.3) *r be a real number such that $r > 0$*
(1.4) *$a(t) = (a_1(t), a_2(t))$ be a mapping $a : T \to E_2$ such that following two properties hold:*
(1.4.1) *$a_1(t) = 0, \forall t \in U$, and $a_1(t_0) = r$*
(1.4.2) *$a_2(t) = 0, \forall t \in U$, and $a_2(t_0) = 0$.*
Then there exists a T-figure α in E_2 such that following equalities hold:
(2.1) *$Z(\alpha) = U$*
(2.2) *$\langle\alpha(t_0), \alpha(t)\rangle = a_1(t), \forall t \in T$*
(2.3) *$[\alpha(t_0)\,\alpha(t)] = a_2(t), \forall t \in T$.*

Proof. Assume that α is a T-figure in E_2 such that $Z(\alpha) \neq T$ and $t_0 \in T \setminus Z(\alpha)$.

(2.1) – (2.3) We choose a T-figure α as follows. Put $\alpha(t_0) = (\sqrt{r}, 0)$. Then we obtain $\langle\alpha(t_0), \alpha(t_0)\rangle = r$. This equality implies $Q(\alpha(t_0)) = \langle\alpha(t_0), \alpha(t_0)\rangle = r$. Hence $\langle\alpha(t_0), \alpha(t_0)\rangle = a_1(t_0) = r$. We choose α on the set U as follows. We put $\alpha(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \forall t \in U$. This equality implies $\langle\alpha(t), \alpha(t)\rangle = a(t) = 0, \forall t \in U$.

For fixed $t \in T$, we consider $a(t)$ and $\alpha(t)$ as elements of the field $\mathbb{C}$ of complex numbers: $a(t) = a_1(t) + ia_2(t)$, $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$. We put $\alpha(t) = \frac{a(t)\alpha(t_0)}{r}, \forall t \in T \setminus (U \cup \{t_0\})$. Since $\alpha(t_0) = \sqrt{r} \neq 0$, $(\alpha(t_0))^{-1}$ exists. Then the equalities $\alpha(t) = \frac{a(t)\alpha(t_0)}{r}, \forall t \in T \setminus (U \cup \{t_0\})$, imply equalities $(\alpha(t_0))^{-1}\alpha(t) = \frac{a(t)}{r}, \forall t \in$

$T \setminus (U \cup \{t_0\})$. By Proposition 9, $(\alpha(t_0))^{-1}\alpha(t) = \frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))}, \forall t \in T$. The equality $Q(\alpha(t_0)) = \langle \alpha(t_0), \alpha(t_0)\rangle = r$, the last two equalities $(\alpha(t_0))^{-1}\alpha(t) = \frac{a(t)}{r}, \forall t \in T \setminus (U \cup \{t_0\})$, $(\alpha(t_0))^{-1}\alpha(t) = \frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))}, \forall t \in T$, and equalities $\langle \alpha(t), \alpha(t)\rangle = a(t) = 0, \forall t \in U$, imply equalities $\frac{\langle \alpha(t_0), \alpha(t)\rangle}{r} + i\frac{[\alpha(t_0)\,\alpha(t)]}{r} = \frac{a(t)}{r}, \forall t \in T$. These equalities imply $Z(\alpha) = U$, $\langle \alpha(t_0), \alpha(t)\rangle = a_1(t), \forall t \in T$, and $[\alpha(t_0)\,\alpha(t)] = a_2(t)$, $\forall t \in T$. The statements (2.1)-(2.3) are proved. □

5. Complete systems of G-invariants of a T-figure in E_2 for the group $G = O(2, \mathbb{R})$

By Proposition 7, the following equality holds:
$O(2, \mathbb{R}) = SO(2, \mathbb{R}) \cup \{HW \mid H \in SO(2, \mathbb{R})\}$, where HW is the multiplication of matrices H and W, where $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For shortness, denote the set $\{HW \mid H \in SO(2, \mathbb{R})\}$ by $SO(2, \mathbb{R}) \cdot W$. We note that $SO(2, \mathbb{R}) \cap SO(2, \mathbb{R}) \cdot W = \emptyset$.

Let α and β be T-figures in E_2. Assume that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$. Then there exists $F \in O(2, \mathbb{R})$ such that $\beta(t) = F\alpha(t), \forall t \in T$. Denote by $Equ(\alpha, \beta)$ the set of all $F \in O(2, \mathbb{R})$ such that $\beta(t) = F\alpha(t), \forall t \in T$.

Proposition 10. *Let α and β be T-figures in E_2 such that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$. Then there exist only following three possibilities for the set $Equ(\alpha, \beta)$:*
(I) $Equ(\alpha, \beta) = \{F\}$, where $F \in SO(2, \mathbb{R})$.
(II) $Equ(\alpha, \beta) = \{F\}$, where $F \in SO(2, \mathbb{R}) \cdot W$.
(III) $Equ(\alpha, \beta) = \{F_1, F_2\}$, where $F_1 \in SO(2, \mathbb{R})$, $F_2 \in SO(2, \mathbb{R}) \cdot W$.

Proof. Assume that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$. Then there exists $F \in O(2, \mathbb{R})$ such that $F \in Equ(\alpha, \beta)$. Since $F \in O(2, \mathbb{R})$ and $F \in O(2, \mathbb{R}) = SO(2, \mathbb{R}) \cup \{HW \mid H \in SO(2, \mathbb{R})\}$, then $F \in SO(2, \mathbb{R})$ or $F \in \{HW \mid H \in SO(2, \mathbb{R})\}$.

(I) Let $F \in Equ(\alpha, \beta)$, where $F \in SO(2, \mathbb{R})$. By Theorem 4, in this case there exists only one $F \in SO(2, \mathbb{R})$ such that following equalities $\beta(t) = F\alpha(t), \forall t \in T$, hold. Hence, in this case, the set $Equ(\alpha, \beta)$ has a only one element of $SO(2, \mathbb{R})$. Assume that the set $Equ(\alpha, \beta)$ has not elements of $SO(2, \mathbb{R}) \cdot W$. Then, in this case, the set $Equ(\alpha, \beta)$ has only a single element $F \in O(2, \mathbb{R})$ and it is such that $F \in SO(2, \mathbb{R})$.

(II) Let $F \in Equ(\alpha, \beta)$, where $F \in \{HW \mid H \in SO(2, \mathbb{R})\}$. Then following equality $\beta(t) = F\alpha(t), \forall t \in T$, holds. Since $F \in \{HW \mid H \in SO(2, \mathbb{R})\}$, there exists $H \in SO(2, \mathbb{R})$ such that $F = HW$. Then we have following equality $\beta(t) = HW\alpha(t), \forall t \in T$. By Theorem 4, in this case there exists only one $H \in SO(2, \mathbb{R})$ such that following equalities $\beta(t) = HW\alpha(t), \forall t \in T$, hold. Hence, in this case, the set $Equ(\alpha, \beta)$ has only one element of $\{HW \mid H \in SO(2, \mathbb{R})\}$. Assume that the set $Equ(\alpha, \beta)$ has not elements of $SO(2, \mathbb{R})$. Then, in this case, the set $Equ(\alpha, \beta)$ has only one element of $\{HW \mid H \in SO(2,\mathbb{R})\}$ such that $Equ(\alpha,\beta) = \{F\}$, where $F \in \{HW \mid H \in SO(2,\mathbb{R})\}$.

(III) Let $Equ(\alpha,\beta)$ be such that $F_1 \in Equ(\alpha,\beta)$ and $F_2 \in Equ(\alpha,\beta)$, where $F_1 \in SO(2,\mathbb{R})$ and $F_2 \in \{HW \mid H \in SO(2,\mathbb{R})\}$. Then following equalities hold: $\beta(t) = F_1\alpha(t), \forall t \in T$, and $\beta(t) = F_2\alpha(t) = HW\alpha(t), \forall t \in T$, where $H \in SO(2,\mathbb{R})$. By Theorem 4, in the case $\beta(t) = F_1\alpha(t), \forall t \in T$, there exists only one $F_1 \in SO(2,\mathbb{R})$ such that following equalities $\beta(t) = F_1\alpha(t), \forall t \in T$, hold. Hence, in this case, the set $Equ(\alpha,\beta)$ has only one element of $SO(2,\mathbb{R})$. By Theorem 4, in the case $\beta(t) = F_2\alpha(t) = HW\alpha(t), \forall t \in T$, where $H \in SO(2,\mathbb{R})$, there exists only one element $F_2 \in \{HW \mid H \in SO(2,\mathbb{R})\}$ such that following equalities $\beta(t) = F_2\alpha(t) = HW\alpha(t), \forall t \in T$ hold, where $H \in SO(2,\mathbb{R})$. Then, in this case, the set $Equ(\alpha,\beta)$ have only two elements: only one element of $SO(2,\mathbb{R})$ and only one element of $SO(2,\mathbb{R}) \cdot W$.

□

Theorem 6. *Let α be a T-figure in E_2 such that $Z(\alpha) \neq T$ and $t_0 \in T \setminus Z(\alpha)$.*

(i) *Suppose that a T-figure β in E_2 such that the following equalities $\beta(t) = HW\alpha(t), \forall t \in T$, hold for some $H \in SO(2,\mathbb{R})$. Then following equalities hold:*

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ -[\alpha(t_0)\alpha(t)] = [\beta(t_0)\,\beta(t)], \forall T \setminus Z(\alpha). \end{cases} \tag{18}$$

(ii) *Conversely, assume that a T-figure β in E_2 such that the equalities* (18) *hold. Then there exists only one matrix $U \in SO(2,\mathbb{R})$ such that $\beta(t) = UW\alpha(t), \forall t \in T$. In this case, U has the following form*

$$U = \begin{pmatrix} \frac{\langle W\alpha(t_0),\beta(t_0)\rangle}{\langle \alpha(t_0),\alpha(t_0)\rangle} & -\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0),\alpha(t_0)\rangle} \\ \frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0),\alpha(t_0)\rangle} & \frac{\langle W\alpha(t_0),\beta(t_0)\rangle}{\langle \alpha(t_0),\alpha(t_0)\rangle} \end{pmatrix}, \tag{19}$$

where $\det(U) = (\frac{\langle W\alpha(t_0),\beta(t_0)\rangle}{\langle \alpha(t_0),\alpha(t_0)\rangle})^2 + (\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0),\alpha(t_0)\rangle})^2 = 1.$

Proof. Suppose that a T-figure β in E_2 such that the following equalities $\beta(t) = HW\alpha(t), \forall t \in T$, hold for some $H \in SO(2,\mathbb{R})$. This means T-figures $W\alpha$ and β are $SO(2,\mathbb{R})$-equivalent. Then, by Theorem 4, we obtain following equalities:

$$\begin{cases} Z(W\alpha) = Z(\beta) \\ \langle W\alpha(t_0), W\alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [W\alpha(t_0)W\alpha(t)] = [\beta(t_0)\,\beta(t)], \forall t \in T \setminus Z(\alpha). \end{cases} \tag{20}$$

These equalities and equalities $Z(W\alpha) = Z(\alpha)$, $\langle W\alpha(t_0), W\alpha(t)\rangle = \langle \alpha(t_0), \alpha(t)\rangle$, $[W\alpha(t_0)W\alpha(t)] = -[\alpha(t_0)\alpha(t)]$ imply equalities (18).

Conversely, assume that a T-figure β in E_2 such that the equalities (18) hold. Then equalities (18) and equalities $Z(W\alpha) = Z(\alpha)$, $\langle W\alpha(t_0), W\alpha(t)\rangle = \langle \alpha(t_0), \alpha(t)\rangle$,

$[W\alpha(t_0)W\alpha(t)] = -[\alpha(t_0)\alpha(t)]$ imply equalities (20). By Theorem 4, equalities (20) and Proposition 10 imply an existence of only one $U \in SO(2,\mathbb{R})$ such that following equalities $\beta(t) = UW\alpha(t), \forall t \in T$, hold. By Theorem 4, the matrix U has the form (19). □

Remark 2. *Assume that T be a set such that it has at least two elements. By Theorem 6, the system $\{Z(\alpha), \langle\alpha(t_0),\alpha(t)\rangle, [W\alpha(t_0)\,W\alpha(t)]\}$ is a complete system of $SO(2,\mathbb{R})$-invariant functions on the set of all T-figures $W\alpha$ such that $Z(\alpha) \neq T$, and $t_0 \in T \setminus Z(\alpha)$. Complete system of relations between elements of this system follows easy from Theorem 5.*

Theorem 7. *Let α and β be T-figures in E_2. Assume that $Z(\alpha) \neq T$ and $t_0 \in T \setminus Z(\alpha)$.*

(*i*) *Suppose that matrices $H_1, H_2 \in SO(2,\mathbb{R})$ exist such that $\beta(t) = H_1\alpha(t), \forall t \in T$, and $\beta(t) = H_2 W\alpha(t), \forall t \in T$. Then following equalities hold:*

$$\left\{ \begin{array}{c} Z(\alpha) = Z(\beta) \\ \langle\alpha(t_0),\alpha(t)\rangle = \langle\beta(t_0),\beta(t)\rangle \\ rank(\alpha) = rank(\beta) = 1 \end{array} \right. \tag{21}$$

for all $t \in T \setminus Z(\alpha(t))$.

(*ii*) *Conversely, assume that the equalities* (21) *hold. Then only two matrices $H_1 \in SO(2,\mathbb{R})$ and $H_2 \in SO(2,\mathbb{R})$ exist such that following equalities $\beta(t) = H_1\alpha(t), \forall t \in T$, $\beta(t) = H_2 W\alpha(t), \forall t \in T$, hold. Here the matrix H_1 has the following form:*

$$H_1 = \begin{pmatrix} \frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} & -\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} \\ \frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} & \frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} \end{pmatrix}, \tag{22}$$

where $\det(H_1) = (\frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 + (\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 = 1.$

Here the matrix $H_2 \in SO(2,\mathbb{R})$ has the following form

$$H_2 = \begin{pmatrix} \frac{\langle W\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} & -\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} \\ \frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} & \frac{\langle W\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} \end{pmatrix}, \tag{23}$$

where $\det(H_2) = (\frac{W\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 + (\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 = 1.$

Proof. (*i*) Suppose that there exist $H_1 \in SO(2,\mathbb{R})$ such that $\beta(t) = H_1\alpha(t), \forall t \in T$. Then, by Theorem 4 the following equalities hold:

$$\left\{ \begin{array}{c} Z(\alpha) = Z(\beta) \\ \langle\alpha(t_0),\alpha(t)\rangle = \langle\beta(t_0),\beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [\alpha(t_0)\alpha(t)] = [\beta(t_0)\,\beta(t)]\,, \forall T \setminus Z(\alpha). \end{array} \right. \tag{24}$$

Suppose that there exist $H_2 \in SO(2,\mathbb{R})$ such that $\beta(t) = H_2 W\alpha(t), \forall t \in T$. Then, by Theorem 6, the following equalities hold:

$$\left\{ \begin{array}{c} Z(\alpha) = Z(\beta) \\ \langle\alpha(t_0), \alpha(t)\rangle = \langle\beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [\alpha(t_0)\alpha(t)] = -\left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \end{array} \right. \tag{25}$$

Equalities (24) and (25)imply the following equalities:

$$\left\{ \begin{array}{c} Z(\alpha) = Z(\beta) \\ \langle\alpha(t_0), \alpha(t)\rangle = \langle\beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha). \end{array} \right. \tag{26}$$

Equalities (24) implies the following equalities:

$$[\alpha(t_0)\alpha(t)] = \left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \tag{27}$$

Equalities (25) implies the following equalities:

$$[\alpha(t_0)\alpha(t)] = -\left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \tag{28}$$

Equalities (27)and (28) imply following equalities:

$$[\beta(t_0)\,\beta(t)] = -\left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \tag{29}$$

These equalities imply following equalities:

$$[\beta(t_0)\,\beta(t)] = 0, \forall T \setminus Z(\alpha). \tag{30}$$

These equalities and the equalities (27) imply following equalities

$$[\alpha(t_0)\alpha(t)] = 0, \forall T \setminus Z(\alpha). \tag{31}$$

The equalities (31) imply that there exists a real function $a(t)$ on T such that $a(t) = 0, \forall t \in Z(\alpha)$, $a(t) \neq 0, \forall T \setminus Z(\alpha)$ and equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$ hold.

Similarly, equalities (30) imply that there exists a real function $b(t)$ on T such that $b(t) = 0, \forall t \in Z(\alpha)$, $b(t) \neq 0, \forall T \setminus Z(\alpha)$ and equalities $\beta(t) = b(t)\beta(t_0), \forall t \in T$ hold.

The above equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$ and $\beta(t) = b(t)\beta(t_0), \forall t \in T$ imply the equality $rank(\alpha) = rank(\beta) = 1$ in the equalities (21). This equality and the equalities (24)imply the equalities (21).

Conversely, assume that the equalities (21) hold. Then the equality $rank(\alpha) = 1$ in (21) implies an existence of a real function $a(t)$ on T such that $a(t) = 0, \forall t \in Z(\alpha)$, $a(t) \neq 0, \forall T \setminus Z(\alpha)$ and $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$.

Similarly, the equality $rank(\beta) = 1$ in (21) implies an existence of a real function $b(t)$ on T such that $b(t) = 0, \forall t \in Z(\alpha)$, $b(t) \neq 0, \forall T \setminus Z(\alpha)$, and $\beta(t) = b(t)\beta(t_0), \forall t \in T$. The equalities $Z(\alpha) = Z(\beta)$, and $\langle\alpha(t_0), \alpha(t)\rangle = \langle\beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha)$, imply following equality $a(t) = b(t), \forall t \in T$. Hence we obtain following equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$, and $\beta(t) = a(t)\beta(t_0), \forall t \in T$.

Since $t_0 \in T \setminus Z(\alpha)$, we have $a(t_0) \neq 0$. By the equality $Z(\alpha) = Z(\beta)$, we obtain $\beta(t_0) \neq 0$. By [16, Theorem 5.1], only two matrices $H_1 \in SO(2,\mathbb{R})$ and

$H_2 \in SO(2,\mathbb{R})$ exist such that $\beta(t_0) = H_1\alpha(t_0)$ and $\beta(t_0) = H_2W\alpha(t_0)$. By [16, Theorem 5.1.], H_1 has the form (23) and H_2 has the form (24).

The above equalities $\beta(t) = a(t)\beta(t_0), \forall t \in T$, $\beta(t_0) = H_1\alpha(t_0)$, $\beta(t_0) = H_2W\alpha(t_0)$ imply following equalities: $\beta(t) = H_1\alpha(t), \forall t \in T$, and $\beta(t) = H_2W\alpha(t), \forall t \in T$. □

Remark 3. *Assume that T be a set such that it has at least two elements. By Theorem 7, the system $\{Z(\alpha), \langle\alpha(t_0), \alpha(t)\rangle, rank(\alpha)\}$ is a complete system of $SO(2,\mathbb{R})$-invariant functions on the set of all T-figures α such that $Z(\alpha) \neq T$, $rank(\alpha) = 1$ and $t_0 \in T \setminus Z(\alpha)$. Complete system of relations between elements of this system follows easy from Theorem 5.*

Corollary 2. *Let α and β be a T-figures in E_2 such that $Z(\alpha) \neq T$ and $Z(\beta) \neq T$. Assume that there exists a single matrix $F \in O(2,\mathbb{R})$ such that $\beta(t) = F\alpha(t), \forall t \in T$. Then $rank(\alpha) = rank(\beta) = 2$.*

Conversely, assume that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$, and $rank(\alpha) = rank(\beta) = 2$. Then there exists a single matrix $F \in O(2,\mathbb{R})$such that $\beta(t) = F\alpha(t), \forall t \in T$.

Proof. It follows from Theorems 4,6 and 7. □

6. Complete systems of invariants of a T-figure in E_2 for the group $MSO(2,\mathbb{R})$

Let $G = O(2,\mathbb{R})$ or $G = SO(2,\mathbb{R})$. Denote by $G \ltimes Tr(2,\mathbb{R})$ the group of all transformations of E_2 generated by elements of G and all translations of E_2. In particularly, $MO(2,\mathbb{R}) = O(2,\mathbb{R}) \ltimes Tr(2,\mathbb{R})$ and $MSO(2,\mathbb{R}) = SO(2,\mathbb{R}) \ltimes Tr(2,\mathbb{R})$.

Assume that the set T has only one element. Let α and β be T-figures. Then they are $Tr(2,\mathbb{R})$-equivalent. Hence they are $G \ltimes Tr(2,\mathbb{R})$-equivalent. Below we assume that T has at last two elements.

Proposition 11. *Let $G = O(2,\mathbb{R})$ or $G = SO(2,\mathbb{R})$ and T be a set such that it has at last two elements.*

(1) *Assume that $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$, and t_0 is a fixed element of T. Then $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$.*

(2) *Assume that $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$, for some element $t_0 \in T$. Then $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$.*

Proof. $\Rightarrow$ Assume that $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$. Then there exists $F \in G$ and $a \in E_2$ such that $\beta(t) = F\alpha(t) + a, \forall t \in T$. In particularly, for $t = t_0$, we have $\beta(t_0) = F\alpha(t_0) + a$. This equality implies $a = \beta(t_0) - F\alpha(t_0)$. This equality and equalities $\beta(t) = F\alpha(t) + a, \forall t \in T$, imply equalities $\beta(t) = F\alpha(t) + \beta(t_0) - F\alpha(t_0), \forall t \in T$. These equalities imply equalities $\beta(t) - \beta(t_0) = F(\alpha(t) - \alpha(t_0)), \forall t \in T$, that is $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$.

$\Leftarrow$ Assume that $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$. Then there exists $F \in G$ such that $\beta(t) - \beta(t_0) = F(\alpha(t) - \alpha(t_0)), \forall t \in T$. Put $a = \beta(t_0) - F\alpha(t_0)$.

This equality implies $\beta(t_0) = F\alpha(t_0) + a$. The equality $a = \beta(t_0) - F\alpha(t_0)$ and equalities $\beta(t)-\beta(t_0) = F(\alpha(t)-\alpha(t_0)), \forall t \in T$, $\beta(t_0) = F\alpha(t_0)+a$ imply equalities $\beta(t) = F\alpha(t) + a, \forall t \in T$. Hence $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$. □

Proposition 12. *Let $G = SO(2,\mathbb{R})$ or $G = O(2,\mathbb{R})$. Assume that α and β are T-figures such that $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$ and $t_0 \in T$. Then $Z(\alpha(t)-\alpha(t_0)) = Z(\beta(t)-\beta(t_0))$.*

Proof. This statement follows from Propositions 8 and 11. □

This proposition means that the function $Z(\alpha(t) - \alpha(t_0))$ is a $G \ltimes Tr(2,\mathbb{R})$-invariant function of a T-figure $\alpha(t)$ for any $t_0 \in T$.

Proposition 13. *Let $G = SO(2,\mathbb{R})$ or $G = O(2,\mathbb{R})$. Assume that $t_0 \in T$ and $Z(\alpha(t) - \alpha(t_0)) = Z(\beta(t) - \beta(t_0) = T$. Then $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$.*

Proof. In this case, we have $\alpha(t) = \alpha(t_0), \forall t \in T$, and $\beta(t) = \beta(t_0), \forall t \in T$. These equalities imply $\beta(t) = \alpha(t) + (\beta(t_0) - \alpha(t_0)), \forall t \in T$. Hence T-figures α and β are $G \ltimes Tr(2,\mathbb{R})$-equivalent. □

Theorem 8. *Let $t_0 \in T$, α be a T-figure in E_2 such that $Z(\alpha(t)-\alpha(t_0)) \neq T$, and $t_1 \in T \setminus Z(\alpha(t) - \alpha(t_0))$ be fixed.*

(*i*) *Suppose that a T-figure β in E_2 such that $\alpha \overset{MSO(2,\mathbb{R})}{\sim} \beta$. Then following equalities hold:*

$$\begin{cases} Z(\alpha(t) - \alpha(t_0)) = Z(\beta(t) - \beta(t_0) \\ \langle\alpha(t_1) - \alpha(t_0), \alpha(t) - \alpha(t_0)\rangle = \langle\beta(t_1) - \beta(t_0), \beta(t) - \beta(t_0)\rangle \\ [(\alpha(t_1) - \alpha(t_0))\,(\alpha(t) - \alpha(t_0))] = [(\beta(t_1) - \beta(t_0))\,(\beta(t) - \beta(t_0)] \end{cases} \tag{32}$$

for all $t \in T \setminus Z(\alpha(t) - \alpha(t_0))$.

(*ii*) *Conversely, assume that a T-figure β in E_2 such that the equalities* (32) *hold. Then there exists only one element $F \in MSO(2,\mathbb{R})$ such that $\beta = F\alpha$. The evident form of F as follows:$F\alpha(t) = H\alpha(t) + a, \forall t \in T$, where $H \in SO(2,\mathbb{R})$, $a \in E_2$. Here evident form of H as follows*

$$H = \begin{pmatrix} \frac{\langle\alpha(t_1)-\alpha(t_0),\, \beta(t_1)-\beta(t_0)\rangle}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} & -\frac{[(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} \\ \frac{[(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} & \frac{\langle\alpha(t_1)-\alpha(t_0),\, \beta(t_1)-\beta(t_0)\rangle}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} \end{pmatrix}, \tag{33}$$

where $\det(H) = (\frac{\langle\alpha(t_1)-\alpha(t_0),\, \beta(t_1)-\beta(t_0)\rangle}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle})^2 + (\frac{[(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle\alpha(t_1)-\alpha(t_0),\, \alpha(t_1)-\alpha(t_0)\rangle})^2 = 1$. *The element a has the following form: $a = \beta(t_0) - H\alpha(t_0)$.*

Proof. It follows from Proposition 11 and Theorem 4 □

Corollary 3. *Let α and β be T-figures in E_2. Assume that α and $t_0 \in T$ are such that $Z(\alpha(t) - \alpha(t_0)) \neq T$. Assume that $F_1 \in SO(2,\mathbb{R})$, $a_1 \in E_2$, $F_2 \in SO(2,\mathbb{R})$, $a_2 \in E_2$ such that:*
1) $\beta(t) = F_1\alpha(t) + a_1, \forall t \in T$,
2) $\beta(t) = F_2\alpha(t) + a_2, \forall t \in T$.

Then $F_1 = F_2, a_1 = a_2$.

Proof. It follows easy from Proposition 11and Theorem 8. □

Remark 4. *Let* $t_0 \in T$. *By Theorem* 8, *the system* $\{Z(\alpha(t)-\alpha(t_0)), \langle\alpha(t_1)-\alpha(t_0), \alpha(t)-\alpha(t_0)\rangle, [(\alpha(t_1)-\alpha(t_0))\,(\alpha(t)-\alpha(t_0))]\}$ *is a complete system of* $MSO(2,\mathbb{R})$*-invariant functions on the set of all* T*-figures* α *in* E_2 *such that* $Z(\alpha(t)-\alpha(t_0)) \neq T$, *where* $t_1 \in T \setminus Z(\alpha(t)-\alpha(t_0))$ *be fixed. A complete system of relations between elements of this complete system is obtained as in Theorem* 5.

7. Complete systems of invariants of a T-figure in E_2 for the group $MO(2,\mathbb{R})$

Let α and β be T-figures in E_2. Assume that α and $t_0 \in T$ such that $Z(\alpha(t)-\alpha(t_0)) \neq T$. Then, by Proposition 11 $\alpha \overset{MO(2,\mathbb{R})}{\sim} \beta$ if and only if $(\alpha(t)-\alpha(t_0)) \overset{O(2,\mathbb{R})}{\sim} (\beta(t)-\beta(t_0), \forall t \in T$. In this case, by Proposition 10, there exist only three following possibilities for the set $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$:
(I) $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ has only one element F, where $F \in SO(2,\mathbb{R})$.
(II) $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ has only one element F, where $F \in SO(2,\mathbb{R})\cdot W$.
(III) $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ has only two elements F_1 and F_2, where $F_1 \in SO(2,\mathbb{R})$ and $F_2 \in SO(2,\mathbb{R})\cdot W$.

A description of the set $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ and a complete system of invariants of a T-figure in E_2 in the case (I) are given in Section 5.

Consider the case (II).

Theorem 9. *Let* α *be a* T*-figure in* E_2 *such that* $Z(\alpha(t)-\alpha(t_0)) \neq T$ *for some* $t_0 \in T$ *and* $t_1 \in T \setminus Z(\alpha(t)-\alpha(t_0))$ *be fixed.*

(i) *Suppose that a* T*-figure* β *such that the following equalities* $\beta(t) = HW\alpha(t) + d, \forall t \in T$, *hold for some* $H \in SO(2,\mathbb{R})$ *and some* $d \in E_2$. *Then following equalities hold:*

$$\begin{cases} Z(\alpha(t)-\alpha(t_0)) = Z(\beta(t)-\beta(t_0)) \\ \langle\alpha(t_1)-\alpha(t_0), \alpha(t)-\alpha(t_0)\rangle = \langle\beta(t_1)-\beta(t_0), \beta(t)-\beta(t_0)\rangle \\ -[\alpha(t_1)-\alpha(t_0)\;\alpha(t)-\alpha(t_0)] = [\beta(t_1)-\beta(t_0)\;\beta(t)-\beta(t_0)]. \end{cases} \tag{34}$$

for all $t \in T \setminus Z(\alpha(t)-\alpha(t_0))$.

(ii) *Conversely, assume that a* T*-figure* β *in* E_2 *such that the equalities* (34)*hold. Then a single matrix* $U \in SO(2,\mathbb{R})$ *and a single* $d \in E_2$ *exist such that* $\beta(t) = UW\alpha(t) + d, \forall t \in T$. *In this case,* U *has following form*

$$U = \begin{pmatrix} \frac{\langle W(\alpha(t_1)-\alpha(t_0)),(\beta(t_1)-\beta(t_0))\rangle}{\langle(\alpha(t_1)-\alpha(t_0)),(\alpha(t_1)-\alpha(t_0))\rangle} & -\frac{[W(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle(\alpha(t_1)-\alpha(t_0)),(\alpha(t_1)-\alpha(t_0))\rangle} \\ \frac{[W(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle(\alpha(t_1)-\alpha(t_0)),(\alpha(t_1)-\alpha(t_0))\rangle} & \frac{\langle W(\alpha(t_1)-\alpha(t_0)),(\beta(t_1)-\beta(t_0))\rangle}{\langle(\alpha(t_1)-\alpha(t_0)),(\alpha(t_1)-\alpha(t_0))\rangle} \end{pmatrix}, \tag{35}$$

where
$\det(U) = (\frac{\langle W(\alpha(t_1)-\alpha(t_0)),(\beta(t_1)-\beta(t_0))\rangle}{\langle(\alpha(t_1)-\alpha(t_0)),(\alpha(t_1)-\alpha(t_0))\rangle})^2 + (\frac{[W(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle(\alpha(t_1)-\alpha(t_0)),(\alpha(t_1)-\alpha(t_0))\rangle})^2 = 1$. *The element* d *has following form:* $d = \beta(t_0) - UW\alpha(t_0)$.

Proof. It follows easy from Proposition 11 and Theorem 6 □

Consider the case (III).

Theorem 10. *Let α be a T-figure in E_2 such that $Z(\alpha(t)-\alpha(t_0)) \neq T$ for some $t_0 \in T$ and $t_1 \in T \setminus Z(\alpha(t)-\alpha(t_0))$ be fixed.*

(i) *Suppose that matrices $F_1 \in SO(2,\mathbb{R})$, $F_2 \in SO(2,\mathbb{R})$ and vectors $d_1 \in E_2, d_2 \in E_2$ exist such that $\beta(t) = F_1\alpha(t) + d_1, \forall t \in T$, and $\beta(t) = F_2 W\alpha(t) + d_2, \forall t \in T$. Then following equalities hold:*

$$\begin{cases} Z(\alpha(t)-\alpha(t_0)) = Z(\beta(t)-\beta(t_0)) \\ \langle \alpha(t_1)-\alpha(t_0), \alpha(t)-\alpha(t_0)\rangle = \langle \beta(t_1)-\beta(t_0), \beta(t)-\beta(t_0)\rangle \\ rank(\alpha(t)-\alpha(t_0)) = rank(\beta(t)-\beta(t_0)) = 1, \end{cases} \quad (36)$$

for all $t \in T \setminus Z(\alpha(t)-\alpha(t_0))$.

(ii) *Conversely, assume that the equalities* (36) *hold. Then only two matrices $H_1 \in SO(2,\mathbb{R})$, $H_2 \in SO(2,\mathbb{R})$ and only two vectors $d_1 \in E_2, d_2 \in E_2$ exist such that following equalities $\beta(t) = H_1\alpha(t) + d_1, \forall t \in T$, $\beta(t) = H_2 W\alpha(t) + d_2, \forall t \in T$, hold. Here the matrix H_1 has following form:*

$$H_1 = \begin{pmatrix} \frac{\langle \alpha(t_1)-\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & -\frac{[\alpha(t_1)-\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \\ \frac{[\alpha(t_1)-\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & \frac{\langle \alpha(t_1)-\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \end{pmatrix}, \quad (37)$$

where $\det(H_1) = (\frac{\langle \alpha(t_1)-\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 + (\frac{[\alpha(t_1)-\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 = 1$. *Vector d_1 has following form $d_1 = \beta(t_0) - H_1\alpha(t_0)$.*

Here the matrix $H_2 \in SO(2,\mathbb{R})$ has following form

$$H_2 = \begin{pmatrix} \frac{\langle W\alpha(t_1)-W\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & -\frac{[W\alpha(t_1)-W\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \\ \frac{[W\alpha(t_1)-W\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & \frac{\langle W\alpha(t_1)-W\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \end{pmatrix}, \quad (38)$$

where
$\det(H_2) = (\frac{W\alpha(t_1)-W\langle \alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 + (\frac{[W\alpha(t_1)-W\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 = 1$. *Vector d_2 has following form $d_2 = \beta(t_0) - H_2W\alpha(t_0)$.*

Proof. It follows easy from Proposition 11and Theorem 7 □

8. Conclusion

Results and methods of the present paper are useful in the theory of G-invariants of systems of points, curves, vector fields, topological figures and polynomial figures in the two-dimensional Euclidean space E_2 for groups $G = SO(2,\mathbb{R})$, $O(2,\mathbb{R})$, $MSO(2,\mathbb{R})$ and $MO(2,\mathbb{R})$. Results and methods of the present paper are also useful in the theory of G-invariants of mechanical figures in the two-dimensional Euclidean space E_2 for Galilei groups.

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