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An Extension of the Adams-type Theorem to the Vanishing Generalized Weighted Morrey Spaces

Abdulhamit Kucukaslan

Department of Aerospace Engineering, Faculty of Aeronautics and Astronautics Sciences, Ankara Yildirim Beyazit University, Ankara, Turkey.

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Abstract. In this paper, we generalize Adams-type theorems given in [1, 13] (which are the following Theorem A and Theorem B, respectively) to the vanishing generalized weighted Morrey spaces. We prove the Adams-type boundedness of the generalized fractional maximal operator M_ρ from the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n, w)$ to another one $\mathcal{VM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ with $w \in A_{p,q}$ for $1 < p < \infty, q > p$; and from the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{1,\varphi}(\mathbb{R}^n, w)$ to the vanishing generalized weighted weak Morrey spaces $\mathcal{VWM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ with $w \in A_{1,q}$ for $p = 1, 1 < q < \infty$. The all weight functions belong to Muckenhoupt-Weeden classes $A_{p,q}$.

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1. Introduction

The classical Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ defined by Morrey in [20] to study the local behavior of solutions to second order elliptic PDEs. Morrey spaces have important applications to potential theory, function spaces and applied mathematics, for instance see the papers [1, 18, 27].

The boundedness of some classical operators of harmonic analysis in the weighted Lebesgue spaces $L_p(\mathbb{R}^n, w)$ were obtained by Muckenhoupt [22], Muckenhoupt and Wheeden [21], and Coifman and Fefferman [2]. In [12], Komori and Shirai defined the weighted Morrey spaces $M_{p,\kappa}(\mathbb{R}^n, w)$ as follows: For $1 \le p \le \infty, 0 < \kappa < 1$ and w be a weight, $f \in M_{p,\kappa}(\mathbb{R}^n, w)$ if $f \in L_p^{loc}(\mathbb{R}^n, w)$ and

$$\|f\|_{M_{p,\kappa}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{L_p(B(x,r),w)} < \infty.$$

Weighted inequalities for fractional operators have good applications to potential theory and quantum mechanics. For more detail we refer the book [6].

Firstly, Vitanza in [30] (see also [26]), introduced the vanishing Morrey space $\mathcal{VM}_{p,\lambda}(\mathbb{R}^n)$ and applied there to obtain a regularity result for elliptic PDEs. Later in [31], Vitanza proved an existence theorem for a Dirichlet problem, under weaker assumptions then those introduced by Miranda in [19], and a Sobolev space $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces

Email address: a.kucukaslan@aybu.edu.tr (A. Kucukaslan)

depending on the dimension. Persson et al. [25] showed the boundedness of commutators of Hardy operators on vanishing Morrey spaces. Also Ragusa [26] obtained a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $\mathcal{VM}_{p,\lambda}(\mathbb{R}^n)$.

The vanishing generalized Morrey space $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n)$ and vanishing generalized local Morrey space $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ was introduced by Samko (see, [28, 29]). The boundedness of the multi-dimensional Hardy type operators, maximal, potential and singular operators in these spaces were proved in [28, 29]. Kucukaslan et al. [13] proved the Spanne-type and Adams-type boundedness of generalized fractional integral operators on vanishing generalized local Morrey spaces. Guliyev et al. [11] proved the commutators of Riesz potential operator in the vanishing generalized weighted Morrey spaces with variable exponent.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. Then, the generalized fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ are defined by the following equalities:

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)|dy, \quad I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)dy.$$

If $\rho(t) \equiv t^\alpha$, then $M_\alpha \equiv M_{t^\alpha}$ is the fractional maximal operator and $I_\alpha \equiv I_{t^\alpha}$ is the Riesz potential operator.

The generalized fractional maximal and integral operators M_ρ and I_ρ were initially investigated in [5, 24]. Nakai [24] introduced the the generalized Morrey spaces $M_{p,\varphi}$ and proved the boundedness of the generalized fractional integral operator I_ρ in these spaces. Nowadays many authors have been culminating important observations about these two-operators M_ρ and I_ρ especially in connection with Morrey-type spaces (see [3, 10, 14–16]).

During the last decades, the theory of boundedness of classical operators of the harmonic analysis in the generalized Morrey spaces have been well studied so far [3, 4, 10, 11, 13–17, 23, 24, 28, 29]. But, Adams-type boundedness of the generalized fractional maximal operator M_ρ in the vanishing generalized weighted Morrey spaces has not been studied, yet.

Guliyev [8] proved the Adams-type boundedness of Riesz potential operator I_α from the spaces $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$ without any assumption on monotonicity of φ_1, φ_2.

In this present paper, by using the method given by Guliyev in [7] (see also, [8]) we prove the Adams-type boundedness of the generalized fractional maximal operator M_ρ from the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n, w)$ to another one $\mathcal{VM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ with $w \in A_{p,q}$ for $1 < p < q < \infty$, and from the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{1,\varphi}(\mathbb{R}^n, w)$ to the vanishing generalized weighted weak Morrey spaces $\mathcal{VWM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ with $w \in A_{1,q}$ for $p = 1, 1 < q < \infty$. The all weight functions belong to Muckenhoupt-Weeden classes $A_{p,q}$.

Throughout the paper, we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C.

2. Preliminaries

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r) \subset \mathbb{R}^n$ the open ball centered at x of radius r. Let $|B(x, r)|$ be the Lebesgue measure of ball $B(x, r)$ and $\mathbb{R}^n$ is the Euclidean space. A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$, in the special case of $w \equiv 1$ we get that $w(E) = |E|$. The characteristic function of E by χ_E. If w is a weight function, for all $f \in L_1^{loc}(\mathbb{R}^n)$ and $1 \leq p < \infty$ we denote by $L_p^{loc}(\mathbb{R}^n, w)$ the weighted Lebesgue space defined by the norm

$$\|f\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n,w)} = \left(\int_{B(x,r)} |f(x)|^p w(x)dx \right)^{\frac{1}{p}} < \infty.$$

We recall that a weight function w belongs to the Muckenhoupt-Wheeden class $A_{p,q}$ (see [21]) for $1 < p < q < \infty$, if

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{\frac{1}{q}} \left(\frac{1}{|B|} \int_B w(x)^{-p'} dx \right)^{\frac{1}{p'}} \leq C,$$

if $p = 1$, w is in the $A_{1,q}$ with $1 < q < \infty$, then

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{\frac{1}{q}} \left(ess\sup_{x\in B} \frac{1}{w(x)} \right) \leq C,$$

where $C > 0$ and the supremum is taken with respect to all balls B.

We find it convenient to define the generalized weighted Morrey spaces as the following.

Definition 2.1. ([9]). Let $1 \le p < \infty$, w be a weight function on $\mathbb{R}^n$ and $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We denote by $M_{p,\varphi}(\mathbb{R}^n,w)$ the generalized weighted Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n,w)$ with finite norm

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)}.$$

Also, by $WM_{p,\varphi}(\mathbb{R}^n,w)$ we denote the generalized weighted weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n,w)$ for which

$$\|f\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)},$$

where $WL_p(B(x,r),w)$ denotes the weighted weak L_p space of measurable functions f for which

$$\|f\|_{WL_p(B(x,r),w)} = \sup_{t>0} \left(\int_{\{y\in B(x,r): |f(y)|>t\}} w(y)dy \right)^{\frac{1}{p}}.$$

If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Remark 2.2. (i) If $w \equiv 1$, then $M_{p,\varphi}(\mathbb{R}^n,w) = M_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space.
(ii) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(\mathbb{R}^n,w) = M_{p,\kappa}(\mathbb{R}^n,w)$ is the weighted Morrey space.
(iii) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$ then $M_{p,\varphi}(\mathbb{R}^n,w) = M_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(\mathbb{R}^n,w) = WM_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.
(iv) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(\mathbb{R}^n,w) = L_p(\mathbb{R}^n,w)$ is the weighted Lebesgue space.

Inspired by Samko in [28] and extending the definition of vanishing generalized Morrey spaces to the case of weighted Morrey-type spaces, we introduce the following definition.

Definition 2.3. ([11]). The vanishing generalized weighted Morrey space $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)$ is defined as the space of functions $f \in M_{p,\varphi}(\mathbb{R}^n,w)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{w(B(x,r))^{-\frac{1}{p}}}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0.$$

The vanishing generalized weighted weak Morrey space $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n,w)$ is defined as the space of functions $f \in WM_{p,\varphi}(\mathbb{R}^n,w)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{w(B(x,r))^{-\frac{1}{p}}}{\varphi(x,r)} \|f\|_{WL_p(B(x,r),w)} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0 \quad \text{and} \quad \sup_{0<r<\infty} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty, \tag{2.1}$$

which makes the spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)$ and $VWM_{p,\varphi}(\mathbb{R}^n,w)$ non-trivial, because bounded functions with compact support belong then to this space, see [29]. That is, these conditions are sufficient conditions for the non-triviality of the spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)$ and $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n,w)$.

Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We say that φ belongs to the class $\mathfrak{M}_{\rm glob}$, if it satisfies the assumptions in (2.1).

The spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)$ and $VWM_{p,\varphi}(\mathbb{R}^n,w)$ are closed subspaces of the Banach spaces $M_{p,\varphi}(\mathbb{R}^n,w)$ and $WM_{p,\varphi}(\mathbb{R}^n,w)$, respectively, which may be shown by standard means, such that the spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)$ and $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n,w)$ are Banach spaces with respect to the norm

$$\|f\|_{\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)} \equiv \|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)},$$

$$\|f\|_{\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n,w)} \equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)},$$

respectively.

We will also use the following notations

$$\mathfrak{A}_{p,\varphi,w}(f;x,r) := \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)}$$

and

$$\mathfrak{A}^W_{p,\varphi,w}(f;x,r) := \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)}$$

for brevity, so that

$$\mathcal{VM}_{p,\varphi,w}(\mathbb{R}^n) = \left\{ f \in M_{p,\varphi}(\mathbb{R}^n, w) : \lim_{r\to 0} \mathfrak{A}_{p,\varphi,w}(f;x,r) = 0 \right\}$$

and similarly for $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n, w)$.

3. Adams-type Estimate for the Operator M_ρ in the Spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$

In this section, we obtain the Adams-type boundedness of the generalized fractional maximal operator M_ρ in the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$.

In the following theorem, Adams [1] studied boundedness of the Riesz potential in the Morrey spaces.

Theorem A (Adams, [1])**.** Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then, for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.

In the following theorem, we give Adams-type results for the boundedness of the operator M_ρ on the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$.

Theorem B (Adams-type result, [16])**.** Let $1 \le p < q < \infty$, $w \in A_{p,q}$, $\frac{\rho(t)}{t^n}$ be almost decreasing, and let $\rho(t)$ satisfy the condition (3.2) and the inequality

$$\int_0^{k_2 r} \frac{\rho(s)}{s} ds \le C\rho(r),$$

where k_2 is given by the condition (3.2) and C does not depend on $r > 0$. Let also $\varphi(x,t)$ satisfy the conditions

$$\sup_{r<t<\infty} w(B(x,t))^{-1} \left(\operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s)\, w(B(x,s)) \right) \le C\, \varphi(x,r),$$

$$\rho(r)\varphi(x,r) + \left(\sup_{t>r} \frac{\varphi(x,t)^{\frac{1}{p}} w(B(x,t))^{\frac{1}{p}} \rho(t)}{t^{\frac{n}{p}}} \right) \le C\varphi(x,r)^{\frac{p}{q}},$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then, the operator M_ρ is bounded from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n, w)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$ and for $p > 1$ from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n, w)$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n, w)$. Moreover, for $1 \le p < q < \infty$

$$\|M_\rho f\|_{WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n,w)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n,w)},$$

and for $1 < p < q < \infty$

$$\|M_\rho f\|_{M_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n,w)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n,w)}$$

In order to achieve our purpose, we assume that

$$\sup_{1\le t<\infty} \frac{\rho(t)}{t^n} < \infty, \tag{3.1}$$

so that the fractional maximal function $M_\rho f$ is well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n \setminus B(0,1)}(x)}{|x|^{2n}}.$$

In addition, we shall also assume that ρ satisfies the growth condition: there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r<s\le 2r} \frac{\rho(s)}{s^n} \le C \sup_{k_1 r<t<k_2 r} \frac{\rho(t)}{t^n}, \quad r > 0. \tag{3.2}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C > 0$ such that

$$\frac{1}{C}\frac{\rho(t)}{t^n} \le \frac{\rho(r)}{r^n} \le C\frac{\rho(t)}{t^n},$$

whenever r and t satisfy $r, t > 0$ and $\frac{1}{2} \le \frac{r}{t} \le 2$. In the sequel for the generalized fractional maximal operator M_ρ we always assume that ρ satisfies the condition (3.2).

The boundedness of the operator I_ρ in the spaces $L_p(\mathbb{R}^n)$ can be found in [4]. Let $\frac{\rho(t)}{t^n}$ be almost decreasing, that is, there exists a constant C such that $\frac{\rho(t)}{t^n} \le C\ \frac{\rho(s)}{s^n}$ for $s < t$. In this case, there is a close and strong relation between the operators M_ρ and I_ρ (see, [14]) such that

$$M_\rho f(x) \lesssim I_\rho(|f|)(x).$$

Hence, the following lemma is valid for the operator M_ρ, which was used in the proof of our main result.

Lemma 3.1. *([16]). Let $w \in A_{p,q}, 1 \le p < \infty, q > p$, the function ρ satisfies the conditions(3.1)-(3.2), and $f \in L_1^{loc}(\mathbb{R}^n, w)$.*

(i) If $1 < p < \infty, q > p$, then there exist $C > 0$ for all $r > 0$ such that the inequality

$$\rho(r) \le C r^{\frac{n}{p}-\frac{n}{q}} \tag{3.3}$$

is sufficient condition for the boundedness of generalized fractional maximal operator M_ρ from $L_p(\mathbb{R}^n, w)$ to $L_q(\mathbb{R}^n, w)$.

(ii) If $p = 1, 1 < q < \infty$, then there exist $C > 0$ for all $r > 0$ such that the inequality

$$\rho(r) \le C r^{n-\frac{n}{q}} \tag{3.4}$$

is sufficient condition for the boundedness of generalized fractional maximal operator M_ρ from $L_1(\mathbb{R}^n, w)$ to $WL_q(\mathbb{R}^n, w)$, where the constant C does not depend on f.

The following lemma is weighted local strong and weak L_p-estimates for the operator M_ρ which is our main tool to prove our main results.

Lemma 3.2. *([16]). Let $1 \le p < \infty, q > p$, $w \in A_{p,q}$, and $\rho(t)$ satisfy the conditions (3.1)-(3.2).*

(i) If $1 < p < q < \infty$ and the condition (3.3) is fulfill, then the inequality

$$\|M_\rho f\chi_{B(x,r)}\|_{L_q(\mathbb{R}^n,w)} \lesssim \|f\chi_{B(x,2r)}\|_{L_p(\mathbb{R}^n,w)} + w(B(x,r))^{\frac{1}{q}}\left(\sup_{t>r}\frac{\rho(t)}{t^{\frac{n}{p}}}\|f\chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,w)}\right) \tag{3.5}$$

and,

(ii) if $p = 1, 1 < q < \infty$ and the condition (3.4) is fulfill, then the inequality

$$\|M_\rho f\chi_{B(x,r)}\|_{WL_q(\mathbb{R}^n,w)} \lesssim \|f\chi_{B(x,2r)}\|_{L_1(\mathbb{R}^n,w)} + w(B(x,r))^{\frac{1}{q}}\left(\sup_{t>r}\frac{\rho(t)}{t^n}\|f\chi_{B(x,t)}\|_{L_1(\mathbb{R}^n,w)}\right)$$

hold for any ball $B(x, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n, w)$.

An extension theorem of Theorem B also containing Theorem A, which is the following theorem is our main result in which we generalize the Adams-type boundedness of the generalized fractional maximal operator M_ρ in the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$.

Theorem 3.3. *Let $1 \le p < q < \infty, w \in A_{p,q}$, $\varphi, \in \mathfrak{M}_{\rm glob}$ and the function ρ satisfy the conditions (3.1)-(3.2) and (3.3)-(3.4). Let also φ satisfy the conditions*

$$\sup_{t<r<\infty} \varphi(x, r) \le C\,\varphi(x, t), \tag{3.6}$$

$$m_\delta = \sup_{\delta<r<\infty}\sup_{x\in\mathbb{R}^n} \varphi(x, r) < \infty, \tag{3.7}$$

and

$$\sup_{r<t<\infty} \rho(t)\varphi(x, t)^{\frac{1}{p}} \le C\,\rho(r)^{-\frac{p}{q-p}}, \tag{3.8}$$

where C does not depend on x and r. Then, the generalized fractional maximal operator M_ρ is bounded from vanishing generalized weighted Morrey spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$ to $\mathcal{VM}_{q,\varphi}(\mathbb{R}^n, w)$ for $p > 1$ and from the vanishing space $\mathcal{VM}_{1,\varphi_1}(\mathbb{R}^n, w)$ to the vanishing weak space $\mathcal{VWM}_{q,\varphi}(\mathbb{R}^n, w)$ for $p = 1$.

Proof. Since the norm inequality is already provided by Theorem B, so we only have to prove that

$$\text{If } \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{p,\varphi^{1/p},w}(f;x,r)=0, \text{ then } \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{q,\varphi^{1/q},w}(M_\rho f;x,r)=0, \tag{3.9}$$

and

$$\text{if } \lim_{r\to 0}\mathfrak{A}^W_{1,\varphi,w}(f;x,r)=0, \text{ then } \lim_{r\to 0}\mathfrak{A}^W_{q,\varphi^{1/q},w}(M_\rho f;x,r)=0. \tag{3.10}$$

Under the conditions (3.2), (3.6) and (3.8) we know that (see, proved in [16], p. 64) for all $x\in\mathbb{R}^n$

$$M_\rho f(x)\le C(Mf(x))^{\frac{p}{q}}\|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n,w)}. \tag{3.11}$$

To check (3.9), i.e., to show that

$$\sup_{x\in\mathbb{R}^n}\frac{w(B(x,r))^{-\frac{1}{q}}\|M_\rho f\|_{L^q(B(x,r),w)}}{\varphi(x,r)^{1/q}}<\varepsilon \quad\text{for small } r,$$

we use the estimates (3.5) and (3.11) where we split the right-hand side:

$$\frac{w(B(x,r))^{-\frac{1}{q}}\|M_\rho f\|_{L^q(B(x,r),w)}}{\varphi(x,r)^{1/q}}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{3.12}$$

with $\delta_0>0$ and $r<\delta_0$, where

$$I_{\delta_0}(x,r):=\frac{1}{\varphi(x,r)^{1/q}}\sup_{r<t<\delta_0}t^{-\frac{n}{q}}\|f\|^{p/q}_{L_p(B(x,t),w)}$$

and

$$J_{\delta_0}(x,r):=\frac{1}{\varphi(x,r)^{1/q}}\sup_{t>\delta_0}w(B(x,t))^{-\frac{1}{q}}\|f\|^{p/q}_{L_p(B(x,t),w)}.$$

We use the fact that $f\in VM_{p,\varphi^{1/p}}(\mathbb{R}^n,w)$ and choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\frac{w(B(x,t))^{-\frac{1}{q}}\|f\|_{L^p(B(x,t),w)}}{\varphi(x,t)^{1/p}}<[\frac{\varepsilon}{2C^{p/q^2}}]^{q/p},\quad t\le\delta_0,$$

where C is constants from (3.6) and (3.12), which yields the estimate of the second term uniform in $r\in(0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\quad 0<r<\delta_0.$$

For the second term, we have

$$J_{\delta_0}(x,r)\le\frac{m^{1/q}_{\delta_0}\|f\|^{p/q}_{M_{p,\varphi^{1/p}}(\mathbb{R}^n,w)}}{\varphi(x,r)^{1/q}},$$

where m_{δ_0} is the constant from (3.7) with $\delta=\delta_0$. Then, by (2.1) we choose small r such that

$$\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi(x,r)}\le\left[\frac{\varepsilon}{2m^{1/q}_{\delta_0}\|f\|^{p/q}_{M_{p,\varphi^{1/p}}(\mathbb{R}^n,w)}}\right]^q,$$

which completes the estimation of the second term and the proof. The proof of (3.10) is, line by line, similar to the proof of (3.9). □

Conclusion

We generalize Adams-type theorems to the vanishing generalized weighted Morrey spaces. We show that the Adams-type boundedness of the generalized fractional maximal operator M_ρ from the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{p,\varphi^{\frac{1}{p}}}(\mathbb{R}^n,w)$ to another one $\mathcal{VM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n,w)$ with $w\in A_{p,q}$ for some suitable parameter of p which are in the interval $1<p<\infty, q>p$; and from the vanishing generalized weighted Morrey spaces $\mathcal{VM}_{1,\varphi}(\mathbb{R}^n,w)$ to the vanishing generalized weighted weak Morrey spaces $\mathcal{VWM}_{q,\varphi^{\frac{1}{q}}}(\mathbb{R}^n,w)$ with $w\in A_{1,q}$ for the special case $p=1$, and $1<q<\infty$. The all weight functions belong to Muckenhoupt-Weeden classes $A_{p,q}$.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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