Turk. J. Math. Comput. Sci.
14(1)(2022) 155–165
© MatDer
DOI : 10.47000/tjmcs.1004423

A New Approach for Smarandache Curves

Semra Kaya Nurkan[1,*], Ilkay Arslan Guven[2]

[1]Department of Mathematics, Faculty of Arts and Sciences, Usak University, 64200, Uşak, Turkey.
[2]Department of Mathematics, Faculty of Arts and Sciences, Gaziantep University, 27310, Gaziantep, Turkey.

Received: 04-10-2021 • Accepted: 18-05-2022

Abstract. In this paper, we introduce new adjoint curves which are associated curves in Euclidean space of three dimension. They are generated with the help of integral curves of special Smarandache curves. We attain some connections between Frenet apparatus of these new adjoint curves and main curve. We characterize these curves in which conditions they are general helix and slant helix. Finally, we exemplify them with figures.

2010 AMS Classification: 53A04,14H45,14J50

Keywords: Adjoint curve, helix, slant helix.

1. Introduction

In differential geometry, one of the primary study area is the curve theory in 3-dimensional Euclidean space $\mathbb{E}^3$ or other special spaces. The geometric properties of curves are described by the help of calculus of differentiation and integral. Curves are used in computer graphics and industry, in roads and railways as methodical bends, in canals to carry through pronged change of direction. They are also rolled in the vertical plane at all changes of stage to prevent the rapid change of stage at the apex. So, determining new types of curves attaches great importance.

Since helices can be seen in many of areas, the most common way of characterizing curves is to figure out whether they are general helix, cylindrical helix or slant helix. A curve which is a helix in $\mathbb{E}^3$ is defined via the feature that its tangent vector field constitutes a constant angle with a fixed direction. Lancret's theorem proves a helix as the proportion of its torsion and curvature is constant [13]. Helix was generalized by Hayden in [9]. Then by the help of Killing vector field through a curve, general helix was defined in real space form of three dimension and in this space form Lancret Theorem was given again for general helices by Barros [3]. Characterizations for a curve which is non-degenerate general helix in terms of its harmonic curvature were given by Camcı et al in [4]. A new helix type which is named as slant helix was defined by Izumiya and Takeuchi in Euclidean 3-space [10]. They described it that it is a curve whose principal normal vector field makes a constant angle with a fixed direction. They also characterized slant helix with constant geodesic curvature function of the principal image of the principal normal indicatrix [11]. Another example to classification of curves is associated curve (direction curve, adjoint curve or conjugate mate) which is defined by taking integral of a vector field created by one of a given curve's Frenet vectors. *A*djoint curve (or conjugate mate) is defined by the help of integral of binormal vector of a curve with any parameter s, in [5–7, 14]. A particular solution to a differential equation or system of equations describes a curve which is parametrized. This curve is called

*Corresponding Author
Email addresses: semra.kaya@usak.edu.tr (S. Kaya Nurkan), iarslan@gantep.edu.tr (I. Arslan Guven)

an integral curve. In case of X is a vector field and $\alpha(t)$ is a parametric curve, the solution of the differential equation $\alpha'(t) = X(\alpha(t))$ imputes an integral curve of X.

Smarandache curves are curves which are included in Smarandache geometry. In [18], Smarandache curves in Minkowski space E_1^4 were described as; Frenet frame vectors on other regular curve constitute the position vector of it. Also they determined a special case of this type and called it Smarandache TB_2 curve. Then, in [2] the author introduced special curves by Frenet-Serret vector fields in Euclidean space which are called Smarandache TN, NB and TNB curves. Smarandache curves in reference to Darboux frame in Minkowski space of 3-dimension were investigated in [1]. They used the known relation between the Darboux and Frenet frame to research special Smarandache curves of timelike curve which is on timelike surface. Additionally the studies of this type of curve can be found out in literature [8, 12, 15–17].

Motivated by these ideas, in this work we study new adjoint curves by combining special Smarandache curves and integral curves. We call these new curves TN-adjoint curve, TB-adjoint curve, NB-adjoint curve and TNB-adjoint curve. We establish some relations between a Frenet curve and these curves. Additionally, we obtain necessary and sufficient conditions for these curves to be general helix and slant helix.

2. Preliminaries

In this part, some basic terms are recollected in relation to differential geometry of space curves in Euclidean space $\mathbb{E}^3$.

Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a regular curve and the Frenet frame of α be $\{T, N, B\}$. $\|\alpha'(s)\| = 1$ if and only if the curve α is a unit speed curve (or has arc-length parametrization s). For a unit speed curve α; $T(s) = \alpha'(s)$ is called the *unit tangent vector* of α. $\kappa(s) = \|\alpha''(s)\|$ denotes the *curvature* of α which measures the amount by which the curve deviates from being a straight line. The *unit principal normal vector* $N(s)$ of α is given by $\alpha''(s) = \kappa(s)N(s)$. The *unit binormal vector* of α is defined by the vector $B(s) = T(s) \times N(s)$. Then, the famous Frenet formula is given as [13]:

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s), \\ B'(s) &= -\tau(s)N(s), \end{aligned}$$

where $\tau(s)$ is the *torsion* of α and it measures the amount by which the curve deviates from a plane.

The Frenet vectors of an arc-length parametrized curve α can be computed as:

$$\begin{aligned} T(s) &= \alpha'(s), \\ N(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|}, \\ B(s) &= T(s) \times N(s). \end{aligned} \tag{2.1}$$

The curvature and torsion of α are calculated respectively by:

$$\begin{aligned} \kappa &= \frac{\|\alpha'(s) \times \alpha''(s)\|}{\|\alpha'(s)\|^3}, \\ \tau(s) &= \frac{\det\left(\alpha'(s), \alpha''(s), \alpha'''(s)\right)}{\|\alpha'(s) \times \alpha''(s)\|^2}. \end{aligned}$$

A curve is given the name *general helix* where the angle between its tangent lines and a fixed direction is constant. This fixed direction is called the axis of the general helix. Lancret expressed the characterization of helix in 1802, that a curve is a general helix if and only if the harmonic curvature or the ratio $\dfrac{\tau}{\kappa}$ is constant, with $\kappa \neq 0$. The general helix is called *circular helix* if both $\kappa \neq 0$ and $\tau \neq 0$ are constants [13].

A *slant helix* which is defined in [11] has constant geodesic curvature function of the principal image of the principal normal indicatrix. This constant function is given by

$$\sigma(s) = \left(\frac{\kappa^2}{\left(\kappa^2 + \tau^2\right)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)'\right)(s). \tag{2.2}$$

The Darboux vector of an arc length parametered curve $\alpha : I \longrightarrow \mathbb{E}^3$ is given by

$$W(s) = \tau(s)T(s) + \kappa(s)B(s), \tag{2.3}$$

where T and B are tangent and binormal vectors, κ is the curvature and τ is the torsion of α. The vector of angular velocity is described with Darboux vector. When an object moves along a curve, a rotation and translation vector depict its motion. This rotation vector is the Darboux vector.

A curve $\alpha : I \longrightarrow \mathbb{E}^n$ with arc length which has the curvature $\kappa \neq 0$ and non zero second derivative is called Frenet curve.

Definition 2.1 ([14]). Let α be s arc length parametrized regular curve with nonvanishing torsion and $\{T_\alpha, N_\alpha, B_\alpha\}$ is the Frenet frame of α. The *adjoint curve* of α is defined as

$$\beta(s) = \int_{s_0}^{s} B_\alpha(s)\, ds.$$

Definition 2.2 ([2]). Let α be s arc length parametrized regular curve with nonvanishing torsion and $\{T_\alpha, N_\alpha, B_\alpha\}$ is the Frenet frame of α. Smarandache TN, NB, TNB curves are defined by:

$$\begin{aligned}
\beta &= \frac{1}{\sqrt{2}}(T_\alpha + N_\alpha), \\
\zeta &= \frac{1}{\sqrt{2}}(N_\alpha + B_\alpha), \\
\psi &= \frac{1}{\sqrt{3}}(T_\alpha + N_\alpha + B_\alpha),
\end{aligned}$$

respectively.

Remark 2.3 ([5]). Let α be s arc length parametrized regular curve and β be the adjoint curve of α. Since β is the integral curve of α, one can take the adjoint curve's (β) arc-length parameter $\overline{s}$ as $\overline{s} = s$.

Therefore, in the following sections, we will make calculations based on this remark.

3. New Adjoint Curves in $\mathbb{E}^3$

In [2], the author introduced that an arc length parameterized curve in E^3 is called Smarandache curve whose position vector is generated by Frenet frame vectors on another regular curve.

In this respect,we adapt this definition to regular curves as the integral of Smarandache curves in the Euclidean 3-space in such ways that:

Definition 3.1. Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve with Frenet-Serret apparatus $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$. TN-adjoint curve, TB-adjoint curve, NB-adjoint curve and TNB-adjoint curve of α are defined by:

$$\beta = \frac{1}{\sqrt{2}}\int (T_\alpha + N_\alpha)\, ds, \tag{3.1}$$

$$\gamma = \frac{1}{\sqrt{2}}\int (T_\alpha + B_\alpha)\, ds, \tag{3.2}$$

$$\zeta = \frac{1}{\sqrt{2}}\int (N_\alpha + B_\alpha)\, ds, \tag{3.3}$$

$$\psi = \frac{1}{\sqrt{3}}\int (T_\alpha + N_\alpha + B_\alpha)\, ds, \tag{3.4}$$

respectively.

3.1. TN-**Adjoint Curve.**

Theorem 3.2. *Let α be arc length parametrized regular curve in $\mathbb{E}^3$ with Frenet-Serret apparatus $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ and β be TN-adjoint curve of α. The Frenet vector fields, curvature and torsion of β are given by*

$$\begin{aligned} T_\beta &= \frac{1}{\sqrt{2}}\left(T_\alpha + N_\alpha\right), \\ N_\beta &= \frac{1}{\sqrt{2+f^2}}\left(-T_\alpha + N_\alpha + fB_\alpha\right), \\ B_\beta &= \frac{1}{\sqrt{4+2f^2}}\left(fT_\alpha - fN_\alpha + 2B_\alpha\right), \\ \kappa_\beta &= \frac{1}{\sqrt{2}}\kappa_\alpha\sqrt{2+f^2}, \\ \tau_\beta &= \frac{1}{\sqrt{2}}\tau_\alpha + \frac{\sqrt{2}f'}{2+f^2}, \end{aligned} \tag{3.5}$$

where $f = \dfrac{\tau_\alpha}{\kappa_\alpha}$.

Proof. By differentiating equation (3.1) and using Frenet formulas, we compute

$$\begin{aligned} \beta'(s) &= \frac{1}{\sqrt{2}}\left(T_\alpha + N_\alpha\right), \\ T_\beta &= \frac{1}{\sqrt{2}}\left(T_\alpha + N_\alpha\right). \end{aligned}$$

In order to obtain the principal normal vector and the curvature of β, we have

$$T_\beta' = \frac{1}{\sqrt{2}}\left(-\kappa_\alpha T_\alpha + \kappa_\alpha N_\alpha + \tau_\alpha B_\alpha\right).$$

Then by using equation (2.1) and definition of curvature, we obtain them as:

$$\kappa_\beta = \left\|T_\beta'\right\| = \frac{1}{\sqrt{2}}\sqrt{2\kappa_\alpha^2 + \tau_\alpha^2}$$

and

$$N_\beta = \frac{1}{\sqrt{2\kappa_\alpha^2 + \tau_\alpha^2}}\left(-\kappa_\alpha T_\alpha + \kappa_\alpha N_\alpha + \tau_\alpha B_\alpha\right).$$

Beside this, we express

$$T_\beta \times N_\beta = \frac{1}{\sqrt{2}}\left(T_\alpha + N_\alpha\right) \times \frac{1}{\sqrt{2\kappa_\alpha^2 + \tau_\alpha^2}}\left(-\kappa_\alpha T_\alpha + \kappa_\alpha N_\alpha + \tau_\alpha B_\alpha\right).$$

Thus, the binormal vector is

$$B_\beta = \frac{1}{\sqrt{4\kappa_\alpha^2 + 2\tau_\alpha^2}}\left(\tau_\alpha T_\alpha - \tau_\alpha N_\alpha + 2\kappa_\alpha B_\alpha\right).$$

In order to find the torsion of β, we differentiate N_β and use the relation $\tau_\beta = \langle N_\beta', B_\beta\rangle$. Then, we have

$$\tau_\beta = \frac{1}{\sqrt{2}}\tau_\alpha + \sqrt{2}\frac{\left(\dfrac{\tau_\alpha}{\kappa_\alpha}\right)'}{2 + \left(\dfrac{\tau_\alpha}{\kappa_\alpha}\right)^2}.$$

Assuming $f = \frac{\tau_\alpha}{\kappa_\alpha}$ and arranging the expressions,we reach the result. □

The following two corollaries are consequences of Theorem 3.2.

Corollary 3.3. *Let a Frenet curve $\alpha : I \to \mathbb{E}^3$ be general helix. Then, the Darboux vectors of both α and TN-adjoint curve of α are equal.*

Proof. Let β be TN-adjoint curve of α. By using equations (2.3) and (3.5), we have

$$\begin{aligned} W_\beta &= \tau_\beta(s)T_\beta(s) + \kappa_\beta(s)B_\beta(s) \\ &= \left(\frac{1}{\sqrt{2}}\tau_\alpha + \frac{\sqrt{2}f'}{2+f^2}\right)\left(\frac{1}{\sqrt{2}}(T_\alpha + N_\alpha)\right) \\ &\quad + \left(\frac{1}{\sqrt{2}}\kappa_\alpha\sqrt{2+f^2}\right)\left(\frac{1}{\sqrt{4+2f^2}}(fT_\alpha - fN_\alpha + 2B_\alpha)\right). \end{aligned}$$

Then, we make some appropriate calculations, and get

$$\begin{aligned} W_\beta &= \tau_\alpha(s)T_\alpha(s) + \kappa_\alpha(s)B_\alpha(s) + \frac{f'}{2+f^2}(T_\alpha + N_\alpha) \\ W_\beta &= W_\alpha + \frac{f'}{2+f^2}(T_\alpha + N_\alpha). \end{aligned}$$

In the last equation since α is general helix, we have $f' = 0$ which means the Darboux vectors of α and β are equal. □

Corollary 3.4. *Let a Frenet curve $\alpha : I \to \mathbb{E}^3$ be general helix. Then, TN-adjoint curve of α is general helix.*

Proof. If we compute the quotient of the torsion and curvature of TN-adjoint curve of α which are in Theorem 3.2, we have

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{\frac{1}{\sqrt{2}}\tau_\alpha + \frac{\sqrt{2}f'}{2+f^2}}{\frac{1}{\sqrt{2}}\kappa_\alpha\sqrt{2+f^2}}.$$

Assuming that α is general helix, then $f' = 0$. So, we get

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{f}{\sqrt{2+f^2}}$$

which means $\frac{\tau_\beta}{\kappa_\beta}$ is constant. □

3.2. TB-Adjoint Curve.

Theorem 3.5. *Let α be arc length parametrized regular curve in $\mathbb{E}^3$ with Frenet-Serret apparatus $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ and γ be TB-adjoint curve of α. The Frenet vector fields, curvature and torsion of γ are given by*

$$\begin{aligned} T_\gamma &= \frac{1}{\sqrt{2}}(T_\alpha + B_\alpha), \\ N_\gamma &= N_\alpha, \\ B_\gamma &= \frac{1}{\sqrt{2}}(-T_\alpha + B_\alpha), \\ \kappa_\gamma &= \frac{1}{\sqrt{2}}\kappa_\alpha(1-f), \\ \tau_\gamma &= \frac{1}{\sqrt{2}}\kappa_\alpha(1+f). \end{aligned} \tag{3.6}$$

Proof. The relation between Ferret-Serret invariants of TB-adjoint curve and α can be easily obtained by using equation (3.3) and Frenet formulas of α. □

Corollary 3.6. *The Darboux vectors of a Frenet curve $\alpha : I \to \mathbb{E}^3$ and TB-adjoint curve of α are equal.*

Proof. Let γ be TB-adjoint curve of α. By using equations (2.3) and (3.6), we have

$$\begin{aligned} W_\gamma &= \tau_\gamma(s)T_\gamma(s) + \kappa_\gamma(s)B_\gamma(s) \\ &= \left(\frac{1}{\sqrt{2}}\kappa_\alpha(1+f)\right)\left(\frac{1}{\sqrt{2}}(T_\alpha + B_\alpha)\right) + \left(\frac{1}{\sqrt{2}}\kappa_\alpha(1-f)\right)\left(\frac{1}{\sqrt{2}}(-T_\alpha + B_\alpha)\right). \end{aligned}$$

Then, we make some calculations and get

$$W_\gamma = \tau_\alpha(s)T_\alpha(s) + \kappa_\alpha(s)B_\alpha(s) = W_\alpha.$$

□

Corollary 3.7. *A Frenet curve $\alpha : I \to \mathbb{E}^3$ is general helix if and only if TB-adjoint curve of α is general helix.*

Proof. If we compute the quotient of the torsion and curvature of TB-adjoint curve of α which are in Theorem 3.5, we have

$$\begin{aligned} \frac{\tau_\gamma}{\kappa_\gamma} &= \frac{\frac{1}{\sqrt{2}}\kappa_\alpha(1+f)}{\frac{1}{\sqrt{2}}\kappa_\alpha(1-f)} \\ &= \frac{(1+f)}{(1-f)}. \end{aligned}$$

Thus, in all cases where the value of the f is not equal to one; if f is constant, the result is apparent. □

Corollary 3.8. *A Frenet curve $\alpha : I \to \mathbb{E}^3$ is a slant helix if and only if TB-adjoint curve of α is a slant helix.*

Proof. In the face of equation (2.2), the geodesic curvature function of the spherical image of the principal normal indicatrix of γ is given by:

$$\delta_\gamma = \frac{\kappa_\gamma^2}{(\kappa_\gamma^2 + \tau_\gamma^2)^{\frac{3}{2}}}.\left(\frac{\tau_\gamma}{\kappa_\gamma}\right)'.$$

We take into account that $\kappa_\gamma = \frac{1}{\sqrt{2}}\kappa_\alpha(1-f)$ and $\tau_\gamma = \frac{1}{\sqrt{2}}\kappa_\alpha(1+f)$, then put these equations in expression of δ_γ. Therefore, we clearly find $\delta_\gamma = \delta_\alpha$. □

Corollary 3.9. *A Frenet curve $\alpha : I \to \mathbb{E}^3$ and TB-adjoint curve of α are Bertrand mate curves.*

Proof. From equation (3.6), we found that $N_\gamma = N_\alpha$, which means the principal normal vector fields of α and TB-adjoint curve of α are linearly dependent. So, they are Bertrand mates. □

3.3. NB-Adjoint Curve.

Theorem 3.10. *Let α be arc length parametrized regular curve in $\mathbb{E}^3$ with Frenet-Serret apparatus $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ and ζ be NB-adjoint curve of α. The Frenet vector fields, curvature and torsion of ζ are given by*

$$\begin{aligned} T_\zeta &= \frac{1}{\sqrt{2}}(N_\alpha + B_\alpha), \\ N_\zeta &= \frac{1}{\sqrt{1+2f^2}}(-T_\alpha - fN_\alpha + fB_\alpha), \\ B_\zeta &= \frac{1}{\sqrt{2+4f^2}}(2fT_\alpha - N_\alpha + B_\alpha), \\ \kappa_\zeta &= \frac{1}{\sqrt{2}}\kappa_\alpha\sqrt{1+2f^2}, \\ \tau_\zeta &= \frac{1}{\sqrt{2}}\kappa_\alpha + \sqrt{2}\frac{f'}{1+2f^2}. \end{aligned} \tag{3.7}$$

Proof. The relation between Frenet-Serret invariants of NB-adjoint curve and α can be easily obtained by using equation (3.3) and Frenet formulas of α. □

Corollary 3.11. *Let a Frenet curve $\alpha : I \to \mathbb{E}^3$ be general helix. Then, the Darboux vectors of both α and NB-adjoint curve of α are equal.*

Proof. By using equations (2.3) and (3.7), we have

$$\begin{aligned} W_\zeta &= \tau_\zeta(s)T_\zeta(s) + \kappa_\zeta(s)B_\zeta(s) \\ &= \left(\frac{1}{\sqrt{2}}\kappa_\alpha + \sqrt{2}\frac{f'}{1+2f^2}\right)\left(\frac{1}{\sqrt{2}}(N_\alpha + B_\alpha)\right) \\ &\quad + \left(\frac{1}{\sqrt{2}}\kappa_\alpha\sqrt{1+2f^2}\right)\left(\frac{1}{\sqrt{2+4f^2}}(2fT_\alpha - N_\alpha + B_\alpha)\right). \end{aligned}$$

Then, we make some appropriate calculations and get

$$\begin{aligned} W_\zeta &= \tau_\alpha(s)T_\alpha(s) + \kappa_\alpha(s)B_\alpha(s) + \frac{f'}{1+2f^2}(N_\alpha + B_\alpha) \\ W_\zeta &= W_\alpha + \frac{f'}{1+2f^2}(N_\alpha + B_\alpha). \end{aligned}$$

In the last equation since α is general helix, we have $f' = 0$ which means the Darboux vectors of α and ζ are equal. □

Corollary 3.12. *Let a Frenet curve $\alpha : I \to \mathbb{E}^3$ be general helix. Then, NB-adjoint curve of α is general helix.*

Proof. If we compute the quotient of the torsion and curvature of NB-adjoint curve of α which are in Theorem 3.10, we have

$$\frac{\tau_\zeta}{\kappa_\zeta} = \frac{\frac{1}{\sqrt{2}}\kappa_\alpha + \sqrt{2}\frac{f'}{1+2f^2}}{\frac{1}{\sqrt{2}}\kappa_\alpha\sqrt{1+2f^2}}.$$

Assuming that α is general helix, then $f' = 0$. So, we get

$$\frac{\tau_\zeta}{\kappa_\zeta} = \frac{1}{\sqrt{1+2f^2}}$$

which means $\frac{\tau_\zeta}{\kappa_\zeta}$ is constant. □

3.4. TNB-Adjoint Curve.

Theorem 3.13. *Let α be arc length parametered regular curve in E^3 with Frenet-Serret apparatus $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ and ψ be TNB-adjoint curve of α. The Frenet vector fields, curvature and torsion of ψ are given by*

$$\begin{aligned} T_\psi &= \frac{1}{\sqrt{3}}(T_\alpha + N_\alpha + B_\alpha), \\ N_\psi &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-f+f^2}}(-T_\alpha + (1-f)N_\alpha + fB_\alpha), \\ B_\psi &= \frac{1}{\sqrt{6}}\frac{1}{\sqrt{1-f+f^2}}\begin{pmatrix} (2f-1)T_\alpha \\ -(1+f)N_\alpha + (2-f)B_\alpha \end{pmatrix}, \\ \kappa_\psi &= \frac{\sqrt{6}}{3}\kappa_\alpha\sqrt{1-f+f^2}, \\ \tau_\psi &= \frac{1}{\sqrt{3}}\kappa_\alpha(1+f) + \frac{\sqrt{3}}{2}\frac{f'}{(1-f+f^2)}. \end{aligned} \tag{3.8}$$

Proof. The relation between Frenet-Serret invariants of TNB-adjoint curve and α can be easily obtained by using equation (3.4) and Frenet formulas of α. □

Corollary 3.14. *Let a Frenet curve $\alpha : I \to \mathbb{E}^3$ be general helix. Then, the Darboux vectors of both α and TNB-adjoint curve of α are equal.*

Proof. By using equations (2.3) and (3.8), we have

$$\begin{aligned} W_\psi &= \tau_\psi(s)T_\psi(s) + \kappa_\psi(s)B_\psi(s) \\ &= \left(\frac{1}{\sqrt{3}}\kappa_\alpha (1+f) + \frac{\sqrt{3}}{2}\frac{f'}{(f^2-f+1)}\right)\left(\frac{1}{\sqrt{3}}(T_\alpha + N_\alpha + B_\alpha)\right) \\ &\quad + \left(\frac{\sqrt{6}}{3}\kappa_\alpha\sqrt{f^2-f+1}\right)\left(\frac{1}{\sqrt{6}}\frac{1}{\sqrt{f^2-f+1}}\left(\begin{array}{c} (2f-1)\,T_\alpha \\ -(1+f)\,N_\alpha + (2-f)\,B_\alpha \end{array}\right)\right). \end{aligned}$$

Then, we make some appropriate calculations and get

$$\begin{aligned} W_\beta &= \tau_\alpha(s)T_\alpha(s) + \kappa_\alpha(s)B_\alpha(s) + \frac{f'}{2\,(f^2-f+1)}(T_\alpha + N_\alpha + B_\alpha) \\ &= W_\alpha + \frac{f'}{2\,(f^2-f+1)}(T_\alpha + N_\alpha + B_\alpha)\,. \end{aligned}$$

In the last equation since α is general helix, we have $f' = 0$ which means the Darboux vectors of α and ψ are equal. □

Corollary 3.15. *Let a Frenet curve $\alpha : I \to \mathbb{E}^3$ be general helix. Then, TNB-adjoint curve of α is general helix.*

Proof. If we compute the quotient of the torsion and curvature of TNB-adjoint curve of α which are in Theorem 3.13, we have

$$\frac{\tau_\psi}{\kappa_\psi} = \frac{\frac{1}{\sqrt{3}}\kappa_\alpha(1+f) + \frac{\sqrt{3}}{2}\frac{f'}{(1-f+f^2)}}{\frac{\sqrt{6}}{3}\kappa_\alpha\sqrt{1-f+f^2}}.$$

Assuming that α is general helix, then $f' = 0$. So, we get

$$\frac{\tau_\psi}{\kappa_\psi} = \frac{1}{\sqrt{2}}\frac{(1+f)}{\sqrt{1-f+f^2}}$$

which means $\frac{\tau_\psi}{\kappa_\psi}$ is constant. □

Example 3.16. Consider the unit speed curve $\alpha : (-\pi, \pi) \to E^3$given by

$$\alpha(s) = \frac{3}{4}\left(\cos(s) + \frac{\cos(3s)}{9}, \sin(s) + \frac{\sin(3s)}{9}, -\frac{2\cos(s)}{\sqrt{3}}\right).$$

The tangent, principal, binormal vectors, the curvature and the torsion were found in [7]. Now let's find TN, TB, NB, TNB adjoint curves of α. By using the definitions of these curves, we obtained respectively;

$$\beta(s) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} \frac{3}{4}\cos(s) + \frac{1}{12}\cos(3s) - \frac{\sqrt{3}}{4}\sin(2s)\,, \\ \frac{3}{4}\sin(s) + \frac{1}{12}\sin(3s) + \frac{\sqrt{3}}{4}\cos(2s)\,, \\ -\frac{2\cos(s)}{\sqrt{3}} + \frac{s}{2} \end{array}\right) + (c_1, c_2, c_3),$$

$$\gamma(s) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} \frac{3}{4}\cos(s) + \frac{1}{12}\cos(3s) + \frac{3}{4}\sin(s) - \frac{1}{12}\sin(3s)\,, \\ \sin(s) + \frac{1}{9}\sin(3s) - \cos(s) + \frac{\cos^3(s)}{3}, \\ -\frac{2\cos(s)}{\sqrt{3}} + \frac{\sqrt{3}}{2}\sin(s) \end{array}\right) + (c_1, c_2, c_3),$$

$$\zeta(s) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} -\frac{\sqrt{3}}{4}\sin(2s) + \frac{3}{4}\sin(s) - \frac{1}{12}\sin(3s)\,, \\ \frac{\sqrt{3}}{4}\cos(2s) - \cos(s) + \frac{\cos^3(s)}{3}, \\ \frac{\sqrt{3}}{2}\sin(s) + \frac{s}{2} \end{array}\right) + (c_1, c_2, c_3),$$

$$\psi(s) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} \frac{3}{4}\cos(s) + \frac{1}{12}\cos(3s) + \frac{\sqrt{3}}{4}\sin(2s) + \frac{3}{4}\sin(s) - \frac{1}{12}\sin(3s)\,, \\ \frac{3}{4}\sin(s) + \frac{1}{12}\sin(3s) + \frac{\sqrt{3}}{4}\cos(2s) + \frac{\cos^3(s)}{3} - \cos(s)\,, \\ -\frac{2\cos(s)}{\sqrt{3}} + \frac{s}{2} + \frac{\sqrt{3}}{2}\sin(s) \end{array}\right) + (c_1, c_2, c_3),$$

where c_1, c_2, c_3 are constants. The curve α and TN, TB, NB, TNB adjoint curves of α are shown in the following computer generated graphs. In the graphs, red, green, yellow, blue and purple curves indicate α, TN, TB, NB, TNB adjoint curves of α, respectively.

In Figure 1, we have shown α and the TN-adjoint curve of α. In Figure 2, we have shown α and the TB-adjoint curve of α. In Figure 3, we have shown α and the NB-adjoint curve of α. In Figure 4, we have shown α and the TNB-adjoint curve of α. In Figure 5, we have shown α and the adjoint curves of α.

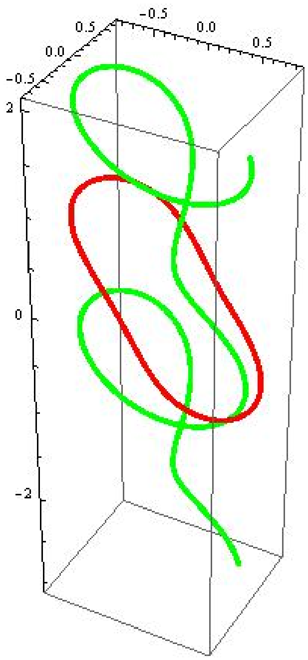

FIGURE 1. α and the TN-adjoint curve of α

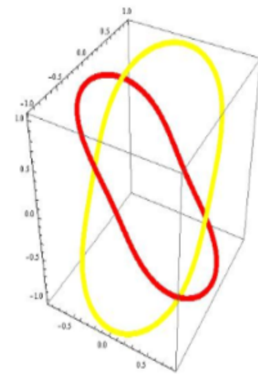

FIGURE 2. α and the TB-adjoint curve of α

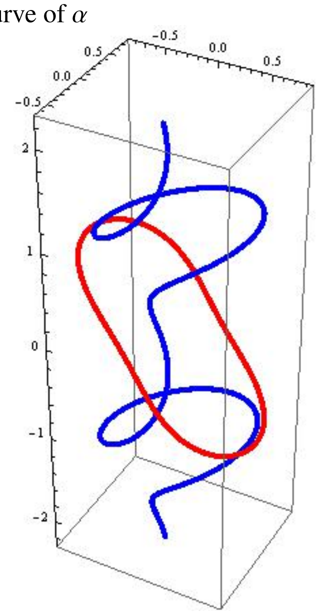

FIGURE 3. α and the NB-adjoint curve of α

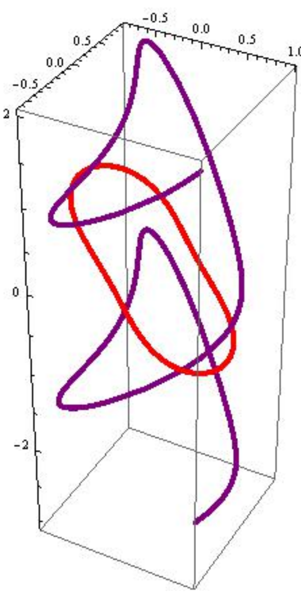

FIGURE 4. α and the TNB-adjoint curve of α

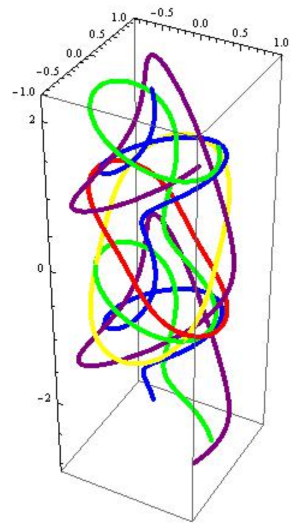

FIGURE 5. α and its adjoint curves of α

4. Conclusion

In this study, we construct new adjoint curves by combining special Smarandache curves and integral curves. We call these new curves TN-adjoint curve, TB-adjoint curve, NB-adjoint curve and TNB-adjoint curve. We establish some relations between a Frenet curve and these curves. Furthermore, the idea analyzing the trajectory ruled surfaces that are examined with the help of new adjoint curves which still is an open problem.

Acknowledgement

We would like to thank the reviewers for their detailed comments and suggestions for the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

References

[1] Abdel-Aziz, H.S., Khalifa Saad, M., *Computation of Smarandache curves according to Darboux frame in Minkowski 3-space*, J. of Egyptian Math. Society, **25**(4), (2017), 382–390.

[2] Ali, A.T., *Special Smarandache curves in the Euclidean space*, International J. Math. Combin., **2**(2010), 30–36.

[3] Barros, M., *General helices and a theorem of Lancret*, Proc. Am. Math. Society, **125**(5)(1997), 1503–1509.

[4] Camcı, Ç. , İlarslan, K., Kula, L., Hacısalihoğlu, H.H., *Harmonic curvatures and generalized helices in E^n*, Chaos- Solutions and Fractals, **40**(5)(2009), 2590–2596.

[5] Choi, J.H., Kim, Y.H., *Associated curves of a Frenet curve and their applications*, Applied Math. and Comp.,**218**(2012), 9116–9124.

[6] Choi, J.H., Kim, Y.H., Ali, A.T., *Some associated curves of Frenet non-lightlike curves in E_1^3* , J. Math. Anal. Appl., **394**(2012), 712–723.

[7] Deshmukh, S., Chen, B.Y., Algehanemi, A., *Natural Mates of Frenet Curves in Euclidean 3-space*, Turk. J. of Math.,**42**(2018), 2826–2840.

[8] Elzawy, M., Mosa, S., *Smarandache curves in the Galilean 4-space G_4* , J. of Egyptian Math. Society, **25**(1)(2017), 53–56.

[9] Hayden, H.A., *On a general helix in Riemannian n-space*, Proc. London Math. Society, **32**(2)(1931), 37–45.

[10] Izumiya, S., Takeuchi, N., *Special curves and ruled surfaces*, Beitrage zur Alg. und Geo. Contributions to Alg. and Geo., **44**(1)(2003), 203–212.

[11] Izumiya, S., Takeuchi, N., *New special curves and developable surfaces*, Turk J. Math., **28**(2)(2004), 153–163.

[12] Kahraman, T., Ugurlu, H.H., *Dual Smarandache curves and Smarandache ruled surfaces*, Mathematical Sciences and Applications E-Notes, **2**(1)(2014), 83–98.

[13] Millman, R.S., Parker, G.D., Elements of Differential Geometry, Englewood Cliffs, NJ, USA, Prentice Hall, 1977.

[14] Nurkan, S.K., Güven, İ.A., Karacan, M.K., *Characterizations of adjoint curves in Euclidean 3-space*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, **89**(1)(2019), 155–161.

[15] Solouma, E.M., *Special equiform Smarandache curves in Minkowski space-time*, J. of Egyptian Math. Society, **25**(3)(2017), 319–325.
[16] Şenyurt, S., Çalışkan, A., *Smarandache curves of Mannheim curve couple according to Frenet frame* , Mathematical Sciences and Applications E-Notes, **5**(1)(2017), 122–136.
[17] Taşköprü, K., Tosun, M., *Smarandache curves on* S^2, Bol. Soc. Paran. Mat., **32**(1)(2014), 51–59.
[18] Turgut, M., Yılmaz, S., *Smarandache curves in Minkowski space time*, International J. Math. Combin., **3**(2008), 51–55.