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Blow up and Exponential Growth to a Kirchhoff-Type Viscoelastic Equation with Degenerate Damping Term

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Abstract
In this paper, we consider a Kirchhoff-type viscoelastic equation with degenerate damping term have initial and Dirichlet boundary conditions. We obtain the blow up and exponential growth of solutions with negative initial energy.

Keywords: *Blow up; degenerate damping; exponential growth; Kirchhoff-type equation; viscoelastic equation.*

AMS Subject Classification (2020): *35B40; 35B44; 35G61; 35L75.*

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1. Introduction

We deal with the following nonlinear Kirchhoff-type viscoelastic problem:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int\limits_0^t g(t-s)\,\Delta u(s)\,ds + |u|^v\,\partial j(u_t) = |u|^{r-1}u \ \text{ in } \ \Omega\times(0,+\infty), \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \qquad \text{on } \ x\in\Omega, \\ u(x,t) = 0 \qquad \text{on } \ x\in\partial\Omega, \end{cases} \tag{1.1}$$

here $\partial j(s)$ denotes the sub-differential $j(s)$ [1], Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. $M(\alpha)$ is a nonnegative C^1 function for $\alpha \geq 0$ satisfying

$$M(\alpha) = 1 + \alpha^{\kappa}, \ \kappa > 0.$$

The Kirchhoff type equations orginated from the nonlinear vibration of an elastic string and was firstly considered by Kirchhoff for $f = g = \delta = 0$:

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta\frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.2}$$

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where $0 < x < L, t \geq 0.$

Now, we focus on a chronological literature overview. Eq. (1.2) with $f = g = 0$ was investigated by Nishihara and Yamada [2]. The author studied the global solvability of solution for non-analytic initial data. In [3], Ikehata and Matsuyama investigated Eq.(1.2) with $\delta = 0$, $g = \delta |u_t|^{p-1} u_t$ and $f = |u|^{r-1} u$, and employed the global solvability and the energy decay of solution. Moreover, Ono [4] studied Eq. (1.2) with $g = 0$, the author employed the local and the global existence, decay properties of solutions for degenerate and non-degenerate equations with a dissipative term. Also, the author studied the blow up of solution with nonpositive and positive initial energy. The other work related to Kirchhoff type equations is Taniguchis's work. Taniguchi [5] considered the existence of local solution, also discuss the global existence and exponential asymptotic behaviour of solutions for weakly damped Eq. (1.2).

In case of $M \equiv 1$, the problem (1.1) discussed by Han and Wang [6] and the authors proved the global existence of generalized solutions, weak solutions. Moreover, they studied finite time blow-up of solutions with negative initial energy.

Furthermore, in case of $M \equiv 1$ and $g = 0$, the problem (1.1) becomes the following form

$$u_{tt} - \Delta u + |u|^{\upsilon} \partial j(u_t) = |u|^{r-1} u,$$

has been studied by some authors see [7–11].

In [12], Ekinci and Pişkin studied following equation:

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u + |u|^{\upsilon} \partial j(u_t) = |u|^{r-1} u, \tag{1.3}$$

with initial and boundary conditions. They studied blow up of solutions with arbitrary positive initial energy by constructing a energy perturbation function.

In the work [13], Piskin investigated the following equation:

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u + |u_t|^{p-1} u_t = |u|^{r-1} u \tag{1.4}$$

and proved the existence, decay and blow up of the solution. Then, Pişkin and Irkıl [14] investigated the same problem treated in [13] and studied blow up results for positive initial energy. In 2018, Pişkin and Yüksekkaya [15] considered problem (1.4) in case $p = 1$ and proved the blow up of solutions with positive and negative initial energy. Furthermore, Periera et al. [16] discussed problem (1.4) in case $p = 1$ and studied existence of the global solutions via the Faedo-Galerkin method. The authors also obtained the asymptotic behavior via the Nakao method. Then, in 2021, Periera et al. [17] investigated the existence and the energy decay estimate of global solutions for problem (1.4) in case $p \geq 1$.

The hyperbolic models with degenerate damping also are of much interest in material science and physics. It particularly appears in physics when the friction is modulated by the strains. There are a lot of studies have Kirchhoff-type viscoelastic problem with degenerate damping term. But, most of these studies are system problem. For instance, Pişkin and Ekinci [18] investigated the following system

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + \left(|u|^k + |v|^l\right) |u_t|^{p-1} u_t = f_1(u, v), \\ v_{tt} - M(\|\nabla v\|^2)\Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + \left(|v|^\theta + |u|^\varrho\right) |v_t|^{q-1} v_t = f_2(u, v), \end{cases} \tag{1.5}$$

in $\Omega \times (0, T)$. The authors discussed global existence, general decay and blow up results of solutions. Recently, Piskin and Ekinci [19] considered same problem and proved local existence result. In [20], the author studied blow up of solutions with positive initial energy for problem (1.5) without viscoelastic term. In addition, they gave some estimates for lower bound of the blow up time. On the other hand, the other some studies with degenerate damping terms are see (see [21–29]).

The equation (1.5) in case $M \equiv 1$, Pişkin et al. [29] studied local existence and uniqueness of the solution by using the Faedo-Galerkin method. Furthermore, they proved the blow up of weak solutions.

To the best of our knowledge too many system problems with Kirchhoff type and degenerate damping terms. But there are a few studies as single equation with degenerate damping and Kirchhoff type. Motivated by previous works, we prove several results concerning the blow up and exponential growth of solution for the problem (1.1).

To analyze the blow up and growth of solution for problem (1.1), we are interested in effect caused by the source term $|u|^{r-1} u$, memory $\int_0^t g(t-s)\Delta u(s)\, ds$ and degenerate damping $|u|^\upsilon \partial j(u_t)$. In our problem is that the source term of type $|u|^{r-1}u$ overcomes the stabilizing mechanisms, memory $\int_0^t g(t-s)\Delta u(s)\,ds$ and degenerate damping $|u|^{\upsilon}\partial j(u_t)$, thus causing a destabilization of the model with the blow up of the solution at a finite time [30].

The remaining part of this paper is organized as follows: In the next section, we introduce some assumptions, notations and present a lemma needed in the proof of our results. In Section 3, we prove the blow up of solution with negative initial energy. In Section 4, we prove the exponential growth of solution with negative initial energy.

2. Preliminaries

Now, we present some preliminary material which will be helpful in the proof of our results. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^q(\Omega)$ norm $\|.\|_q = \|.\|_{L^q(\Omega)}$.

$(\mathbf{A1})$ $\upsilon, p \geq 0$, $r > 1$; $\upsilon \leq \frac{n}{n-2}$, $r+1 \leq \frac{2n}{n-2}$ if $n \geq 3$. There exists positive constants c, c_0, c_1 such that for all $s, k \in R$ $j(s) : R \to R$ be a C^1 convex real function satisfies

- $j(s) \geq c|s|^{p+1}$,
- $j'(s)$ is single valued and $|j'(s)| \leq c_0 |s|^p$,
- $(j'(s) - j'(k))(s-k) \geq c_1 |s-k|^{p+1}$.

$(\mathbf{A2})$ $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$.

$(\mathbf{A3})$ Assume $g(\tau) : R^+ \to R^+$ satisfies

$$g(\tau) \geq 0,\ g'(\tau) \leq 0,$$

for all $\tau \in R^+$ and

$$\int_0^t g(\tau)\,d\tau < 1.$$

$(\mathbf{A4})$ $\int_0^t g(s)\,ds < \frac{r-1}{r+1}$.

We use the following notations

$$\begin{aligned} l &= 1 - \int_0^t g(\tau)\,d\tau, \\ (g \diamond \theta)(t) &= \int_0^t g(t-\tau) \int_\Omega |\theta(t) - \theta(\tau)|\,dx d\tau. \end{aligned}$$

Lemma 2.1. *Suppose that $(A1)$, $(A2)$ and $(A3)$ hold. Let u be a solution of (1.1). Then, $E(t)$ is nonincreasing, namely,*

$$E'(t) \leq 0.$$

Proof. A multiplication of Eq.(1.1) by u_t and integration over Ω give

$$E'(t) = -\frac{1}{2} g(t) \|\nabla u\|^2 + \frac{1}{2}(g' \diamond \nabla u)(t) - \int_0^t \int_\Omega |u(\tau)|^{\upsilon} j(u_t)(\tau)\,dx d\tau \leq 0, \tag{2.1}$$

where

$$\begin{aligned} E(t) &= \frac{1}{2}\left[\|u_t\|^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2\right] \\ &\quad + \frac{1}{2}\left[\frac{1}{\kappa+1}\|\nabla u\|^{2(\kappa+1)} + (g \diamond \nabla u)(t)\right] - \frac{1}{r+1}\|u\|_{r+1}^{r+1} \end{aligned} \tag{2.2}$$

Thus, we have

$$E(t) \leq E(0). \tag{2.3}$$

□

3. Blow up

In this section, we shall prove the blow up of solutions for problem (1.1).

Theorem 3.1. *Let $(A1)$-$(A4)$ hold. u be a any solution to (1.1) on the interval $[0,T]$. Assume further that $r > \upsilon + p$, $E(0) < 0$ and*

$$\int_0^t g(s)\,ds \geq \frac{\kappa}{\kappa+1}.$$

Then T is necessarily finite, i.e. u can't be continued for all $t > 0$.

Proof. Set

$$H(t) = -E(t). \tag{3.1}$$

By using (2.1), we have

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \frac{1}{2}g(t)\|\nabla u\|^2 - \frac{1}{2}(g' \diamond \nabla u)(t) + \int_\Omega |u(t)|^\upsilon j(u_t)u_t dx \\ &\geq \int_\Omega |u(t)|^\upsilon j(u_t)u_t dx \\ &\geq c_0 \int_\Omega |u(t)|^\upsilon |u_t|^{p+1} dx. \end{aligned} \tag{3.2}$$

Thus, we arrive at

$$0 < H(0) \leq H(t) \leq \frac{1}{r+1}\|u\|_{r+1}^{r+1}, \quad t \geq 0. \tag{3.3}$$

Now, we define

$$L(t) = H^{1-\rho}(t) + \varepsilon \int_\Omega u u_t dx,$$

where $\rho = \min\left\{\frac{r-p-\upsilon}{p(r+1)}, \frac{r-1}{2(r+1)}\right\}$ and ε is a positive constant.

By derivating $L(t)$ and using Eq.(1.1), we obtain

$$\begin{aligned} L'(t) &= (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|^{2(\kappa+1)} \\ &\quad +\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\,dxds \\ &\quad -\varepsilon\int_\Omega |u(t)|^\upsilon u(t)\partial j(u_t)(t)\,dx + \varepsilon\|u\|_{r+1}^{r+1} \\ &= (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 \\ &\quad -\varepsilon\|\nabla u\|^{2(\kappa+1)} - \varepsilon\int_0^t g(s)\,ds\|\nabla u\|^2 \\ &\quad +\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t))\,dxds \\ &\quad -\varepsilon\int_\Omega |u(t)|^\upsilon u(t)\partial j(u_t)(t)\,dx + \varepsilon\|u\|_{r+1}^{r+1}. \end{aligned} \tag{3.4}$$

By applying Young's inequality to estimate the fifth term of (3.4) as follows

$$\begin{aligned} &\left|\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t))\,dxds\right| \\ \leq\ &\int_0^t g(s)\,ds\|\nabla u\|^2 + \frac{1}{4}(g \diamond \nabla u)(t). \end{aligned}$$

From $(A3)$, since $0 < l \leq 1$. Then it follows from the definition of $H(t)$ that

$$
\begin{aligned}
-\|\nabla u\|^{2} &= \frac{2}{l}H(t) + \frac{1}{l}\|u_t\|^2 + \frac{1}{l}(g\diamond\nabla u)(t) \\
&\quad + \frac{1}{l(\kappa+1)}\|\nabla u\|^{2(\kappa+1)} - \frac{2}{l(r+1)}\|u\|_{r+1}^{r+1}. \qquad (3.5)
\end{aligned}
$$

Combining (3.4)-(3.5), we obtain

$$
\begin{aligned}
L'(t) &\geq (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\left(1+\frac{1}{l}\right)\|u_t\|^2 \\
&\quad + \varepsilon\frac{2}{l}H(t) + \varepsilon\left(\frac{1}{l}-\frac{1}{4}\right)(g\diamond\nabla u)(t) + \varepsilon\left(\frac{1}{l(\kappa+1)}-1\right)\|\nabla u\|^{2(\kappa+1)} \\
&\quad - \varepsilon\int_\Omega |u(t)|^{\upsilon}u(t)\partial j(u_t)(t)\,dx + \varepsilon\left(1-\frac{2}{l(r+1)}\right)\|u\|_{r+1}^{r+1}. \qquad (3.6)
\end{aligned}
$$

By condition $\int_0^t g(s)\,ds < \frac{r-1}{r+1}$, we have $1-\frac{2}{l(r+1)} > 0$.

In order to estimate fifth term in (3.6), since $r > \upsilon + p$, from assumption $(A1)$ and thanks to Holder's inequality and Young's inequality, we get

$$
\begin{aligned}
&\left|\int_\Omega |u(t)|^{\upsilon}u(t)\partial j(u_t)(t)\,dx\right| \\
\leq\ & \int_\Omega |u(t)|^{\upsilon+1-\frac{\upsilon+p+1}{p+1}}|u(t)|^{\frac{\upsilon+p+1}{p+1}}|u_t(t)|^{p}\,dx \\
\leq\ & C_0\left(\int_\Omega |u(t)|^{\upsilon}|u_t(t)|^{p+1}dx\right)^{\frac{p}{p+1}}\left(\int_\Omega |u(t)|^{\upsilon+p+1}dx\right)^{\frac{1}{p+1}} \\
\leq\ & C_0|\Omega|^{\frac{r-\upsilon-p}{r+1}}\left(\int_\Omega |u(t)|^{\upsilon}|u_t(t)|^{p+1}dx\right)^{\frac{p}{p+1}}\|u(t)\|_{r+1}^{\frac{\upsilon+p+1}{p+1}} \\
\leq\ & \beta(H'(t))^{\frac{p}{p+1}}\|u(t)\|_{r+1}^{\frac{\upsilon+p+1}{p+1}} \\
\leq\ & \beta\left(\delta^{-\frac{1}{p}}H'(t) + \delta\|u(t)\|_{r+1}^{\upsilon+p+1}\right), \qquad (3.7)
\end{aligned}
$$

where constant $\delta > 0$ is specified later and $\beta = C_0C_1^{-\frac{p}{p+1}}|\Omega|^{\frac{r-\upsilon-p}{r+1}}$.

Hence, (3.6) becomes

$$
\begin{aligned}
L'(t) &\geq \left[(1-\rho)H^{-\rho}(t) - \varepsilon\beta\delta^{-\frac{1}{p}}\right]H'(t) \\
&\quad + \varepsilon\left(1+\frac{1}{l}\right)\|u_t\|^2 + \varepsilon\left(\frac{1}{l(\kappa+1)}-1\right)\|\nabla u\|^{2(\kappa+1)} \\
&\quad + \varepsilon\frac{2}{l}H(t) + \varepsilon\left(\frac{1}{l}-\frac{1}{4}\right)(g\diamond\nabla u)(t) \\
&\quad + \varepsilon\left(1-\frac{2}{l(r+1)}\right)\|u\|_{r+1}^{r+1} - \varepsilon\beta\delta\|u(t)\|_{r+1}^{\upsilon+p+1}. \qquad (3.8)
\end{aligned}
$$

The choice of δ $\left(\text{i.e. } \delta = \frac{1}{\beta}\left(\frac{1}{2}-\frac{1}{l(r+1)}\right)\|u\|_{r+1}^{r-\upsilon-p}\right)$, then

$$
\varepsilon\beta\delta\|u(t)\|_{r+1}^{r+p+1} = \varepsilon\left(\frac{1}{2}-\frac{1}{l(r+1)}\right)\|u\|_{r+1}^{r+1}.
$$

Furthermore, since $\|u\|_{r+1} \geq [(r+1)H(0)]^{\frac{1}{r+1}}$ by (3.3) and $\upsilon + p - r + p(r+1)\rho \leq 0$, then

$$\begin{aligned} & (1-\rho)H^{-\rho}(t) - \varepsilon\beta\delta^{-\frac{1}{p}} \\ = \; & H^{-\rho}(t)\left[1-\rho-\varepsilon\beta\delta^{-\frac{1}{p}}H^{\rho}(t)\right] \\ \geq \; & H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\left(\frac{1}{2}-\frac{1}{l(r+1)}\right)^{-\frac{1}{p}}(r+1)^{-\rho}\|u\|_{r+1}^{\frac{p+\upsilon-r+p(r+1)\rho}{p}}\right] \\ \geq \; & H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\left(\frac{1}{2}-\frac{1}{l(r+1)}\right)^{-\frac{1}{p}}(r+1)^{-\rho-\frac{r-p-\upsilon}{p(r+1)}}H(0)_{r+1}^{\rho-\frac{r-\upsilon-p}{p(r+1)}}\right] \\ \geq \; & H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\chi\right], \end{aligned} \tag{3.9}$$

where $\chi = \left(\frac{1}{2}-\frac{1}{l(r+1)}\right)^{-\frac{1}{p}}(r+1)^{\rho-\frac{r-p-\varrho}{p(r+1)}}H(0)_{r+1}^{\rho-\frac{r-\upsilon-p}{p(r+1)}}$. Now, we choose ε to be sufficiently small such that

$$1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\chi > 0.$$

Then (3.9) and (3.8) yield

$$L'(t) \geq \varepsilon C\left[H(t) + \|u_t(t)\|^2 + \|u\|_{r+1}^{r+1} + (g \diamond \nabla u)(t)\right], \tag{3.10}$$

where $C > 0$ is a constant that does not depended on ε. Especially, (3.10) means that $L(t)$ is increasing on $[0, T)$, with

$$L(t) = H^{1-\rho}(t) + \varepsilon\int_\Omega uu_t dx \geq H^{1-\rho}(0) + \varepsilon\int_\Omega u_0u_1 dx.$$

We also select ε to be sufficiently small such that $L(0) > 0$, thus $L(t) \geq L(0) > 0$ for $t \geq 0$.

Let $\eta = \frac{1}{1-\rho}$. Since $0 < \rho < \frac{1}{2}$, it is evident that $2 > \eta > 1$. By using the following inequality

$$|x+y|^\eta \leq 2^{\eta-1}\left(|x|^\eta + |y|^\eta\right) \text{ for } \eta \geq 1,$$

applying Young's inequality, we get

$$\begin{aligned} L^\eta(t) & \leq 2^{\eta-1}\left(H(t) + \varepsilon\|u(t)\|^\eta\|u_t(t)\|^\eta\right) \\ & \leq C\left(H(t) + \|u_t(t)\|^2 + \|u(t)\|_{r+1}^{\frac{1}{\frac{1}{2}-\rho}}\right). \end{aligned} \tag{3.11}$$

By the choice of ρ, we have $\frac{1}{2} - \rho > \frac{1}{r+1}$. Now applying the inequality

$$a^\sigma \leq \left(1+\frac{1}{b}\right)(b+a), \quad a \geq 0, \quad 0 \leq \sigma \leq 1, \quad b > 0,$$

and taking $a = \|u(t)\|_{r+1}^{r+1}$, $\eta = \frac{1}{\left(\frac{1}{2}-\rho\right)(r+1)} < 1$, and $b = H(0)$, we obtain

$$\begin{aligned} \|u(t)\|_{r+1}^{\frac{1}{\frac{1}{2}-\rho}} & \leq \left(1+\frac{1}{H(0)}\right)\left(H(0) + \|u(t)\|_{r+1}^{r+1}\right) \\ & \leq C\left(H(t) + \|u(t)\|_{r+1}^{r+1}\right). \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) gives the inequality

$$\begin{aligned} L^\eta(t) & \leq C\left(H(t) + \|u_t(t)\|^2 + \|u(t)\|_{r+1}^{r+1}\right) \\ & \leq C\left(H(t) + \|u_t(t)\|^2 + \|u(t)\|_{r+1}^{r+1} + (g \diamond \nabla u)(t)\right). \end{aligned} \tag{3.13}$$

Thus, (3.10) and (3.13) arrive at

$$L'(t) \geq CL^\eta(t), \quad t \in [0, T]. \tag{3.14}$$

In the end, from (3.14) and $\eta = \frac{1}{1-\rho} > 1$, we see that $L(t) = H^{1-\rho}(t) + \varepsilon\int_\Omega uu_t dx$ blow up in finite time. This completes the proof. □

4. Exponential growth

In this section, we aim to indicate that the energy grow up as an exponential function as time as goes to infinity.

Theorem 4.1. *Let $(A1)$-$(A3)$ hold. u be a any solution to (1.1). Suppose further that $r > \upsilon + p$ and $E(0) < 0$ and*

$$\int_0^t g(s)\,ds \geq \frac{\kappa}{\kappa + 1/2}$$

Then, the solution to (1.1) grows exponentially.

Proof. We define

$$F(t) = H(t) + \varepsilon \int_\Omega u u_t dx, \tag{4.1}$$

where $H(t) = -E(t)$ and choose $0 < \varepsilon \leq 1$ in this interval to obtain small perturbation of $E(t)$ and we will indicate that $F(t)$ grows exponentialy, namely $F(t)$ satisfies a differential inequality of the form

$$\frac{dF(t)}{dt} \geq \Gamma F(t).$$

By derivating (4.1) and using Eq.(1.1), we have

$$\begin{aligned}
F'(t) &= H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\kappa+1)} \\
&\quad + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(s) \nabla u(t)\,dxds \\
&\quad - \varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx + \varepsilon \|u\|_{r+1}^{r+1} \\
&= H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 \\
&\quad + \varepsilon \int_0^t g(s)\,ds \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\kappa+1)} \\
&\quad + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) (\nabla u(s) - \nabla u(t))\,dxds \\
&\quad - \varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx + \varepsilon \|u\|_{r+1}^{r+1}.
\end{aligned} \tag{4.2}$$

Terms in (4.2) is estimated as follows:

$$\begin{aligned}
&\left| \int_0^t g(t-s) \int_\Omega \nabla u(t) (\nabla u(s) - \nabla u(t))\,dxds \right| \\
\leq\ & \frac{1}{2} \int_0^t g(s)\,ds \|\nabla u\|^2 + \frac{1}{2} (g \diamond \nabla u)(t).
\end{aligned}$$

$$\begin{aligned}
F'(t) &\geq H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \left(1 - \frac{1}{2}\int_0^t g(s)\,ds\right) \|\nabla u\|^2 \\
&\quad -\varepsilon \|\nabla u\|^{2(\kappa+1)} - \varepsilon \frac{1}{2}(g \diamond \nabla u)(t) \\
&\quad -\varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx + \varepsilon \|u\|_{r+1}^{r+1} \\
&\geq H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \left(\frac{1 - \frac{1}{2}\int_0^t g(s)\,ds}{1 - \int_0^t g(s)\,ds}\right) l \|\nabla u\|^2 \\
&\quad -\varepsilon \|\nabla u\|^{2(\kappa+1)} - \varepsilon \frac{1}{2}(g \diamond \nabla u)(t) \\
&\quad -\varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx + \varepsilon \|u\|_{r+1}^{r+1}. \\
&\geq H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \zeta l \|\nabla u\|^2 \\
&\quad -\varepsilon \|\nabla u\|^{2(\kappa+1)} - \varepsilon \frac{1}{2}(g \diamond \nabla u)(t) \\
&\quad -\varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx + \varepsilon \|u\|_{r+1}^{r+1}, \qquad (4.3)
\end{aligned}$$

where $\zeta = \frac{1 - \frac{1}{2}\int_0^t g(s)ds}{1 - \int_0^t g(s)ds}$.

By using the assumption $(A3)$ and the definition $H(t)$, we have $0 < l \leq 1$ and

$$\begin{aligned}
F'(t) &\geq H'(t) + \varepsilon (1+\zeta) \|u_t\|^2 + \varepsilon \left(\frac{\zeta}{\kappa+1} - 1\right) \|\nabla u\|^{2(\kappa+1)} \\
&\quad +\varepsilon \left(\zeta - \frac{1}{2}\right)(g \diamond \nabla u)(t) + \varepsilon \left(1 - \frac{2\zeta}{\gamma+1}\right) \|u\|_{r+1}^{r+1} \\
&\quad +2\varepsilon\zeta H(t) - \varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx.
\end{aligned}$$

By using (3.7), we get

$$\begin{aligned}
F'(t) &\geq \left[1 - \varepsilon\beta\delta^{-\frac{1}{p}}\right] H'(t) + \varepsilon (1+\zeta) \|u_t\|^2 \\
&\quad \varepsilon \left(\frac{\zeta}{\kappa+1} - 1\right) \|\nabla u\|^{2(\kappa+1)} \\
&\quad +2\varepsilon\zeta H(t) + \varepsilon \left(\zeta - \frac{1}{2}\right)(g \diamond \nabla u)(t) \\
&\quad +\varepsilon \left(1 - \frac{2\zeta}{r+1}\right) \|u\|_{r+1}^{r+1} - \varepsilon\beta\delta \|u(t)\|_{r+1}^{\upsilon+p+1}. \qquad (4.4)
\end{aligned}$$

The choice of δ $\left(\text{i.e. } \delta = \frac{1}{\beta}\left(\frac{1}{2} - \frac{\zeta}{r+1}\right)\|u\|_{r+1}^{r-\upsilon-p}\right)$, then

$$\varepsilon\beta\delta \|u(t)\|_{r+1}^{\upsilon+p+1} = \varepsilon \left(\frac{1}{2} - \frac{\zeta}{r+1}\right) \|u\|_{r+1}^{r+1}.$$

Furthermore, since $\|u\|_{r+1} \geq [(r+1)H(0)]^{\frac{1}{r+1}}$ by (3.3) and assumption $\upsilon + p - r \leq 0$, then

$$\begin{aligned}
&1 - \varepsilon\beta\delta^{-\frac{1}{p}} \\
\geq\ & 1 - \varepsilon\beta^{1+\frac{1}{p}} \left(\frac{1}{2} - \frac{\zeta}{r+1}\right)^{-\frac{1}{p}} (r+1)^{-\frac{r-p-\upsilon}{p(r+1)}} H(0)_{r+1}^{-\frac{r-\upsilon-p}{p(r+1)}} \\
\geq\ & 1 - \varepsilon\beta^{1+\frac{1}{p}} K,
\end{aligned}$$

where $K = \left(\frac{1}{2} - \frac{\zeta}{r+1}\right)^{-\frac{1}{p}} (r+1)^{-\frac{r-p-\upsilon}{p(r+1)}} H(0)_{r+1}^{-\frac{r-\upsilon-p}{p(r+1)}}$. Now, we choose ε to be sufficiently small such that

$$1 - \varepsilon\beta^{1+\frac{1}{p}} K > 0.$$

Thus,

$$F'(t) \geq \varepsilon C\left[H(t) + \|u_t(t)\|^2 + \|u\|_{r+1}^{r+1} + (g \diamond \nabla u)(t)\right] \tag{4.5}$$

where $C > 0$ is a constant that does not depended on ε.

Now, applying Young's inequality, and Sobolev Poincare inequality we have

$$\begin{aligned} F(t) &\leq H(t) + \varepsilon \|u\| \|u_t\| \\ &\leq C\left(H(t) + \|u_t\|^2 + \|u\|^2\right). \end{aligned}$$

Now, in order the estimate $\|u\|^2$ term we apply the inequality $a^l \leq (a+1) \leq (1+\frac{1}{b})(a+b)$ for $a = \|u\|_{r+1}^{r+1}$, $l = 2/r+1 < 1$, $b = H(0)$, we have

$$\begin{aligned} \|u\|^2 &\leq C\|u\|_{r+1}^2 \\ &= C\left(\|u\|_{r+1}^{r+1}\right)^{\frac{2}{r+1}} \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|u\|_{r+1}^{r+1} + H(0)\right) \\ &\leq C\left(\|u\|_{r+1}^{r+1} + H(t)\right). \end{aligned} \tag{4.6}$$

Thus,

$$F(t) \leq C\left[H(t) + \|u_t(t)\|^2 + \|u\|_{r+1}^{r+1} + (g \diamond \nabla u)(t)\right]. \tag{4.7}$$

Therefore, (4.5) and (4.7) arrive at

$$F'(t) \geq \xi F(t), \quad t \geq 0$$

This completes the proof. □

5. Conclusion

As far as we know, there is not any blow up and exponential growth results in the literature known for viscoelastic Kirchhoff type equation with degenerate damping term. Our work extends the works for some viscoelastic Kirchhoff type equations treated in the literature to the viscoelastic Kirchhoff equations with degenerate damping terms.

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