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On Exact and Numerical Solutions to the Burgers' and Coupled Burgers' Equation

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Abstract

In this work, one dimensional Burgers' equation and coupled Burgers' equation are solved via Homotopy perturbation method (HPM). Solutions two and three-dimensional graphics and tables of some obtained results are constructed with the help of a ready-made package program. All solutions found in this study validate the efficiency of the method. According to the results, we have found out that our gained solutions convergence very speedily to the analytical solutions. In conclusion, we can say that the present method can also be applied for the solutions of a wide range of nonlinear problems.

Keywords: one dimensional Burgers' equation, Coupled Burgers' equation, Homotopy perturbation method, embedding parameter

Burgers ve Coupled Burgers Denklemlerinin Tam ve Nümerik Çözümleri Üzerine

Öz

Bu çalışmada, bir boyutlu Burgers denklemi ve Burgers denklemler sistemi Homotopi pertürbasyon metodu (HPM) ile çözülmüştür. Elde edilen çözümlerin iki ve üç boyutlu grafikleri ve tabloları hazır paket programı yardımıyla oluşturulmuştur. Bu çalışmada bulunan tüm çözümler metodun etkinliğini doğrulamaktadır. Sonuçlara göre, elde ettiğimiz çözümlerin analitik çözümlere çok hızlı bir şekilde yakınsadığı ortaya çıkarılmıştır. Sonuç olarak, sunulan metodun geniş aralıktaki lineer olmayan problemlerin çözümleri için uygulanabilir olduğunu ifade etmemiz mümkündür.

Anahtar Kelimeler: bir boyutlu Burgers denklemi, Coupled Burgers denklemi, Homotopi pertürbasyon metodu, yerleştirilen parametre

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Introduction

In most cases it is difficult to solve nonlinear problems, especially analytically. Several techniques have been developed to solve these problems (Amirov & Ergun, 2020; Ergun, 2019; 2020). Perturbation techniques (Cole, 1968; He, 1999; Nayfeh, 2000) were among the popular ones and are based on the existence of small or large parameters, namely the perturbation quantities. Unfortunately, many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Hence, some non-perturbative technique (Adomian, 1994; Lyapunov, 1992; Wazwaz, 2002) have been developed, in which these techniques are independent upon small parameters. However, both perturbative and non-perturbative techniques cannot provide a simple way to adjust or control the convergence region and the rate of given approximate series (Liao, 1992; 2004).

To overcome such problems, the Homotopy perturbation method (HPM) is constructed and proposed. The method is powerful and has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors. In this work, we implement the Homotopy perturbation method (HPM) in order to obtain the analytic solutions of one dimensional Burgers' equation and coupled Burgers' equation.

Material and Method

Homotopy Perturbation Method

To explain the method, we take into consideration the subsequent nonlinear equation:

$$A(u)-f(r)=0, \quad r\in\Omega, \tag{2.1}$$

with the boundary condition

$$B(u,\partial u/\partial n)=0, \quad r\in\Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω (He, 1999). A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Then, equation (2.1) can be rewritten as the following form

$$L(u)+N(u)-f(r)=0.$$

By means of homotopy technique, we construct a homotopy

$$V(r,p):\Omega\times[0,1]\rightarrow\mathbb{R}$$

which satisfies

$$H(V,p)=(1-p)\left[L(V)-L(u_0)\right]+p\left[A(V)-f(r)\right]=0, \quad p\in[0,1], \; r\in\Omega, \tag{2.2}$$

or

$$H(V,p)=L(V)-L(u_0)+pL(u_0)+p\left[N(V)-f(r)\right]=0, \tag{2.3}$$

where $p\in[0,1]$ is an embedding parameter, u_0 is the initial approximation of equation (2.1). Distinctly, we get from equations (2.2) and (2.3),

$$H(V,0)=L(V)-L(u_0)=0, \tag{2.4}$$

$$H(V,1)=A(V)-f(r)=0. \tag{2.5}$$

The changing process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and also, $L(V)-L(u_0)$ and $A(V)-f(r)$ are called homotopy (He, 1999).

With respect to homotopy perturbation method, we can firstly use the embedding parameter p as small parameter, we will assume that the solution of equations (2.2) and (2.3) can be rewritten as a power series of p,

$$u=\lim_{p\to 1} V=V_0+V_1+V_2+\dots$$

Results and Discussion

Application of the Present Method and Numerical Results

In this section, the homotopy perturbation method (HPM) is developed to acquire the approximate solutions of the one dimensional Burgers' equation and coupled Burgers' equation is used. Primarily, we consider the one dimensional Burgers' equation is as follows,

$$u_t+u\,u_x-vu_{xx}=0,\quad t\geq 0\,, \tag{3.1}$$

$$u(x,0)=\frac{\alpha+\beta+(\beta-\alpha)e^{\gamma}}{1+e^{\gamma}}, \tag{3.2}$$

where $\gamma=\alpha(x/v)$ and parameters α , β and v are arbitrary constants. When we apply the method to the Burgers' equation, one has

$$(1-p)\left[\dot{Y}-\dot{u}_0\right]+p\left[\dot{Y}+YY'-vY''\right]=0, \tag{3.3}$$

where $\dot{Y}=\dfrac{dY}{dt},\ Y'=\dfrac{dY}{dx},\ Y''=\dfrac{d^2Y}{dx^2}$ and $p\in[0,1]$. By arranging equality (3.3), substituting the initial guess (3.2) identifying the zeroth component and

$$Y=Y_0+pY_1+p^2Y_2+\cdots\ ,$$

$$\dot{Y}=\dot{Y}_0+p\dot{Y}_1+p^2\dot{Y}_2+\cdots\ ,$$

$$Y' = Y_0' + pY_1' + p^2 Y_2' + \cdots \ ,$$

$$Y'' = Y_0'' + pY_1'' + p^2 Y_2'' + \cdots \ ,$$

we have computed the remaining components $Y_1, Y_2, Y_3, Y_4, Y_5, \ldots$ etc. via recursive scheme as follows

$$Y_0 = \frac{\alpha + \beta + (\beta - \alpha) e^{\alpha \frac{x}{v}}}{1 + e^{\alpha \frac{x}{v}}},$$

$$Y_1 = \frac{2 e^{\alpha x/v} t \alpha^2 \beta}{\left(1 + e^{\alpha x/v}\right)^2 v},$$

$$Y_2 = \frac{e^{\alpha x/v} \left(-1 + e^{\alpha x/v}\right) t^2 \alpha^3 \beta^2}{\left(1 + e^{\alpha x/v}\right)^3 v^2},$$

$$\vdots$$

In this way, we obtain the approximate solution of initial value problem (3.1)-(3.2) in series form by

$$u(x,t) = \lim_{p \to 1} \sum_{k=0}^{\infty} p^k Y_k = Y_0 + Y_1 + Y_2 + \cdots \ , \tag{3.4}$$

Also, it should be expressed that the equation has the analytic solution given by

$$u(x,t) = \frac{\alpha + \beta + (\beta - \alpha) e^{\frac{\alpha}{v}(x - \beta t)}}{1 + e^{\frac{\alpha}{v}(x - \beta t)}} \ , \tag{3.5}$$

Numerical assessments of one dimensional Burgers' equation are given in Figure 1-2 and Table 1.

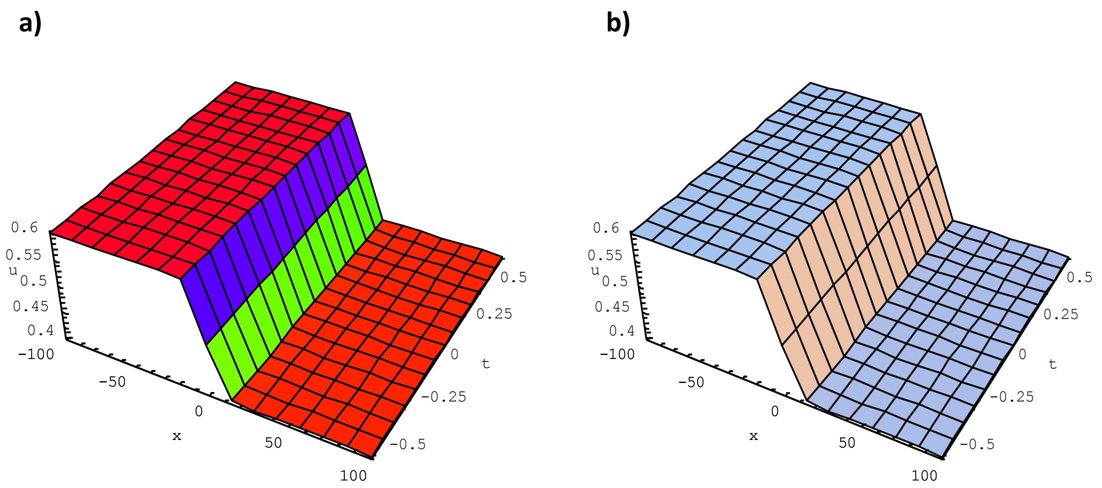

Figure 1. The 3D Graph of Approximate and Analytic Solution of One Dimensional Burgers' Equation when $\alpha = 0.1$, $v = 0.3$, $\beta = 0.5$

a) The 3D Graph of Approximate Solution (3.4) for $\phi_3(x,t)$ *via HPM b) The 3D Graph of Analytic Solution (3.5)*

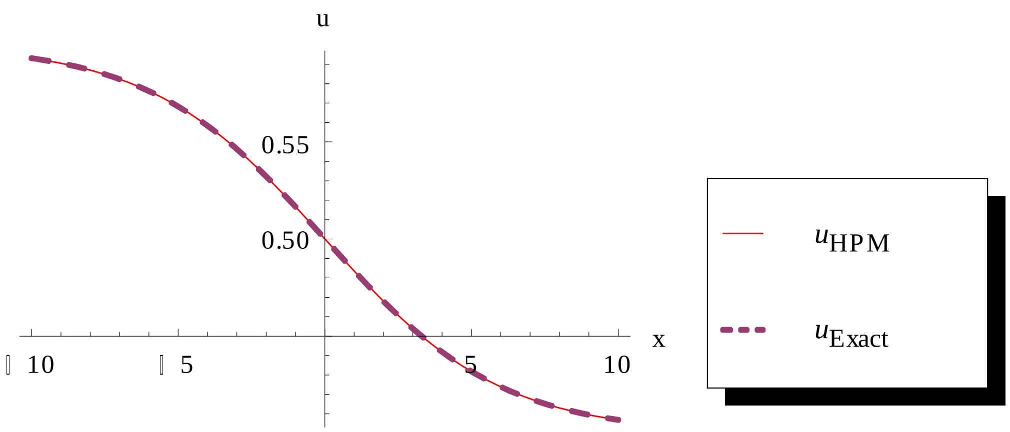

Figure 2. Comparison between HPM and Analytic Solution of One Dimensional Burgers' Equation when $\alpha = 0.1$, $v = 0.3$, $\beta = 0.5$

Table 1. Absolute Error of One Dimensional Burgers' Equation Attained by HPM for $\phi_3(x,t)$ $(\alpha = 0.1 , v = 0.3 , \beta = 0.5)$

$t_i \mid x_i$	0.1	0.2	0.3	0.4	0.5
0.1	1.92735×10^{-8}	1.54218×10^{-7}	5.20559×10^{-7}	1.23402×10^{-6}	2.41026×10^{-6}
0.2	1.92147×10^{-8}	1.5379×10^{-7}	5.19258×10^{-7}	1.23128×10^{-6}	2.40558×10^{-6}
0.3	1.91133×10^{-8}	1.53022×10^{-7}	5.16808×10^{-7}	1.22581×10^{-6}	2.39557×10^{-6}
0.4	1.897×10^{-8}	1.51917×10^{-7}	5.13221×10^{-7}	1.21764×10^{-6}	2.38027×10^{-6}
0.5	1.87853×10^{-8}	1.5048×10^{-7}	5.08512×10^{-7}	1.20681×10^{-6}	2.35977×10^{-6}

In addition, we focus on the coupled Burgers' equation has the following form:

$$u_t - u_{xx} - 2u u_x + (uv)_x = 0 \ ,$$

$$v_t - v_{xx} - 2v v_x + (uv)_x = 0 \ , \tag{3.6}$$

with the initial conditions

$$u(x,0) = \sin x \ , \ v(x,0) = \sin x \ , \tag{3.7}$$

Exact solutions of the system are given by

$$u(x,t) = v(x,t) = e^{-t} \sin x \ , \tag{3.8}$$

In order to solve numerically system (3.6) using HPM, we construct homotopy for this system

$$\begin{aligned} &(1-p)\left[\dot{Y} - \dot{u}_0\right] + p\left[\dot{Y} - Y'' - 2YY' + (YT)'\right] = 0 \ , \\ &(1-p)\left[\dot{T} - \dot{V}_0\right] + p\left[\dot{T} - T'' - 2TT' + (YT)'\right] = 0 \ , \end{aligned} \tag{3.9}$$

where $\dot{Y} = \dfrac{dY}{dt},\ Y' = \dfrac{dY}{dx},\ Y'' = \dfrac{d^2Y}{dx^2};\quad \dot{T} = \dfrac{dV}{dt},\ T' = \dfrac{dV}{dx},\ T'' = \dfrac{d^2T}{dx^2}$ and $p \in [0,1]$. By rearranging equalities (3.9), substituting the initial guesses (3.7) and the values

$$Y = Y_0 + pY_1 + p^2 Y_2 + \cdots \ ,$$

$$\dot{Y} = \dot{Y}_0 + p\dot{Y}_1 + p^2 \dot{Y}_2 + \cdots \ ,$$

$$Y' = Y_0' + pY_1' + p^2 Y_2' + \cdots \ ,$$

$$Y'' = Y_0'' + pY_1'' + p^2 Y_2'' + \cdots \ ,$$

and

$$T = T_0 + pT_1 + p^2 T_2 + \cdots \ ,$$

$$\dot{T} = \dot{T}_0 + p\dot{T}_1 + p^2 \dot{T}_2 + \cdots \ ,$$

$$T' = T_0' + pT_1' + p^2 T_2' + \cdots \ ,$$

$$T'' = T_0'' + pT_1'' + p^2 T_2'' + \cdots \ ,$$

we have gained the subsequent components, respectively

$$Y_0 = \sin x \ ,$$

$$Y_1 = -t \sin x \ ,$$

$$Y_2 = \frac{t^2}{2} \sin x \ ,$$

$$\vdots$$

and

$$T_0 = \sin x \ ,$$

$$T_1 = -t \sin x \ ,$$

$$T_2 = \frac{t^2}{2} \sin x \ ,$$

$$\vdots$$

It is possible to add more components of approximation. By means of the above obtained components, the approximate solutions of coupled Burgers' equation (3.6) have the form,

$$\begin{aligned} u(x,t) &= \lim_{p \to 1} \sum_{k=0}^{\infty} p^k Y_k = Y_0 + Y_1 + Y_2 + \cdots \ , \\ v(x,t) &= \lim_{p \to 1} \sum_{k=0}^{\infty} p^k T_k = T_0 + T_1 + T_2 + \cdots \ , \end{aligned} \tag{3.10}$$

and we have attained the subsequent numerical evaluations in Figure 3-4 and Table 2 :

a) **b)**

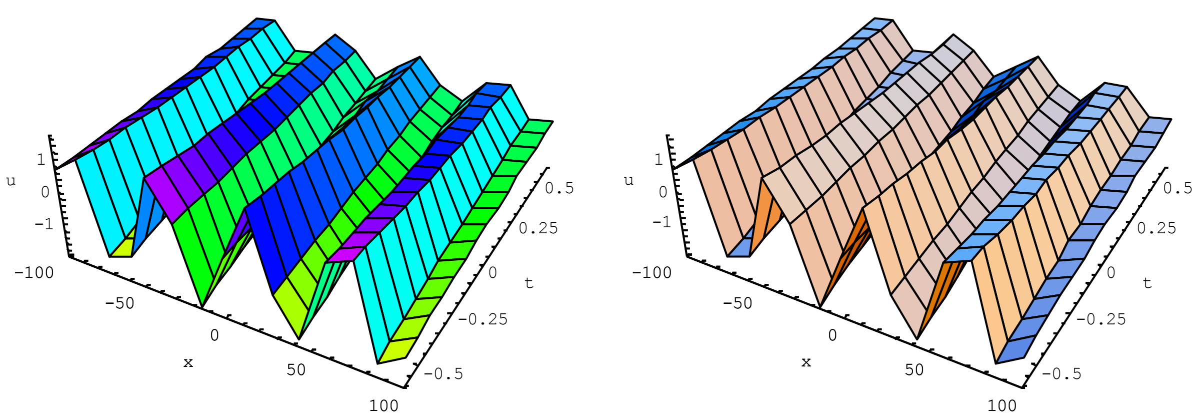

Figure 3. The 3D Graph of Approximate and Analytic Solution of Coupled Burgers' Equation *a) The 3D Graph of Approximate Solution (3.10) for* $\phi_{10}(x,t)$ *via HPM b) The 3D Graph of Analytic Solution (3.8)*

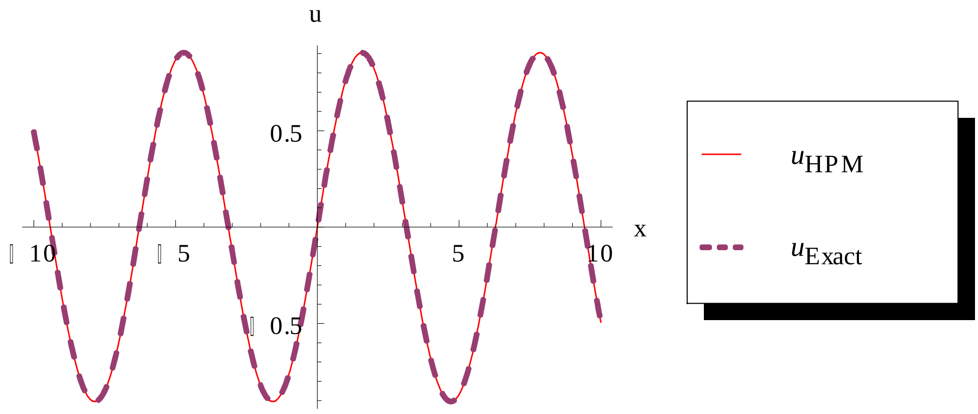

Figure 4. Comparison between HPM and Analytic Solution of Coupled Burgers' Equation

Table 2. Absolute Error of Approximate Solution of Coupled Burgers' Equation Acquired by HPM for $\phi_{10}(x,t)$

t_i x_i	0.1	0.2	0.3	0.4	0.5
0.1	0.	2.76168×10^{-15}	1.5811×10^{-13}	2.78327×10^{-12}	2.56944×10^{-11}
0.2	2.77556×10^{-17}	5.52336×10^{-15}	3.14665×10^{-13}	5.53874×10^{-12}	5.11321×10^{-11}
0.3	0.	8.18789×10^{-15}	4.68153×10^{-13}	8.23891×10^{-12}	7.60589×10^{-11}
0.4	0.	1.07692×10^{-14}	6.1684×10^{-13}	1.08568×10^{-11}	1.00226×10^{-10}
0.5	0.	1.32672×10^{-14}	7.59393×10^{-13}	1.3366×10^{-11}	1.23391×10^{-10}

In this sub-section, we explain and discuss numerical results indicated above.

Initially, we will evaluate the results for one dimensional Burgers' equation (3.1). Numerical results attained by HPM for one dimensional Burgers' equation are demonstrated above in Figure 1-2 and Table 1. In Figure 1-2, we compare the approximate series solutions with analytic solution and show very good agreement between HPM and analytic solution. In Table 1, it is seen that even three components of the approximate series solution gained by HPM is much close to the analytic solution.

Additionally, numerical results acquired by means of HPM for coupled Burgers' equation (3.6) are constructed above in Figure 3-4 and Table 2. We illustrated the simulation of numerical values for the system which are cited in Figures 3 and 4 in terms of 3D and 2D plots, respectively. It is seen that there are very good agreement with approximate series solutions and analytic solutions. From Table 2, it is shown that numerical solutions also obtained for ten components by means of HPM are very convergent to analytic solutions. We can observe from the cited table, the hired solution algorithm is efficient and accurate. Furthermore, all gained results indicate the validity, effectiveness and applicability of the present method.

Conclusion

In this paper, we investigate the solution for one dimensional Burgers' and coupled Burgers' equations and their corresponding consequences using efficient analytical algorithm namely, HPM.

Our obtained numerical results verify and indicate the success of the method for these equations. Numerical approximations demonstrate a high degree of accuracy of the applied method. Based on the current results and findings presented in all figures and tables, implementation of the method in this way has proved an efficient means for solving such physical and mathematical models. It will be encouraging for further studies.

Consequently, the HPM is effective and powerful by determining analytic solutions for these kind of problems in science and engineering (Abbasbandy, 2007; Ganji & Rafei, 2006; Ganji & Rajabi, 2006; He, 1999). The solutions are very rapidly convergent by performing the present method.

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Ethics

There are no ethical issues related to the publication of this article.

Conflict of Interest

The author state that there is no conflict of interest.

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