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Research Article

FIXED POINTS OF SOFT SET-VALUED MAPS WITH APPLICATIONS TO DIFFERENTIAL INCLUSIONS

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ABSTRACT

In this paper, a notion of soft set-valued maps in Hausdorff fuzzy metric space is introduced. To this end, we establish fixed point theorems of set-valued mappings whose range set lies in a family of soft sets. Consequently, a few significant fixed point results of fuzzy, multivalued and single-valued mappings are pointed out and discussed. Some illustrative nontrivial examples which dwell upon the generality of our results are also provided. As an application, sufficient conditions for solvability of multi-valued boundary value problems involving both Riemann-Liouville and Caputo fractional derivatives with non-local fractional integro-differential boundary conditions are investigated to indicate a usability of the ideas presented herein.
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1. INTRODUCTION

The evolvement of fuzzy mathematics started with the introduction of the concepts of fuzzy sets by Zadeh [50] in 1965. Fuzzy set theory is now well-known as one of the mathematical tools for handling situations that are uncertain in nature. As a result, the theory of fuzzy sets has gained great applications in diverse domains such as management sciences, engineering, environmental sciences, medical sciences and in other emerging fields. Meanwhile, the basic notions of fuzzy sets have been modified and improved in different directions; for example, see [3, 12, 33, 34]. In order to apply the idea of fuzzy sets to the classical concepts of metric spaces, Kramosil and Michalek [23] introduced the notion of fuzzy metric spaces with the aid of continuous triangular norm originally defined by Schweizer and Sklar [44] in their study of statistical metric spaces.

Thereafter, George and Veeramani [16] modified the idea of fuzzy metric space due to Kramosil and Michalek [23], thereby, defining a Hausdorff topology on the new fuzzy metric space. Not long ago, Gregori et al. [13] provided many examples of fuzzy metrics in the sense of George and Veeramani [16] and also presented some applications of these metrics in the area of color image processing.

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Along the line, the notion of fuzzy metric spaces was first time extended to fixed point theory by Grabiec [17]. Thereafter, Gregori and Sapena [15] established another notion of fuzzy contractive mapping and studied its application to fixed point theorem in fuzzy metric spaces in the sense of both Kramosil and Michalek [23], George and Veeramani [16]. Mihet [27] initiated the concept of fuzzy ψ–contractive mapping in non-Archimedean fuzzy metric spaces, which extended the category of fuzzy contractive mapping due to Gregori and Sapena [13]. In 2016, Gregori and Minana [14] presented a fixed point theorem under fuzzy ψ-contraction and obtained the result of Wardowski [48] as a consequence. Later on, several authors studied different fixed point theorems in fuzzy metric spaces, see, for instance, [1, 20, 25, 29, 39, 40, 45] and the references therein. In 2004, Rodriguez and Romaguera [42] initiated a method for constructing Hausdorff fuzzy metrics on the set of nonempty compact subsets of a fuzzy metric space. This development paved fruitful way for the study of fixed point theorems of point-to-set valued mappings in the literature; see, for instance, [21,38,46].

As a further improvement of the notions of fuzzy sets, Molodstov [34] initiated the concept of soft set theory (SST) with the aim of handling phenomena and notions of ambiguous, undefined and imprecise environments in which the applications of fuzzy sets have been incapacitated. In particular, SST does not need the pre-specifications of a parameter inherent with fuzzy sets, rather, it accommodates approximate descriptions of objects. In other words, one can use any suitable parametrization tool with the help of words, sentences, real numbers, mappings, and so on; thereby, making SST an adequate formalism for approximate reasoning. Consequently, the area of applied mathematics gained huge development as a result of the introduction of soft set. Recall that in classical mathematics, to describe any system or object, we first construct its mathematical model and then attempt to obtain the exact solution. If the exact solution is too complicated, then we define the notion of approximate solution. On the other hand, in soft set theory, the initial description of an object takes an approximate nature with no restriction, and the notion of exact solution is not essential. In [34], Moldstov pointed out several directions for possible applications of soft set, such as in smoothness of functions, game theory, Riemann-integration, operation research, probability and so on. Presently, the concept of soft set is receiving more than a handful of extensions in different perspectives. For example, see [7, 10, 32, 41] and the references therein.

It is well-known that set-valued analysis has enormous applications in control theory,game theory, biomathematics, qualitative physics, viability theory, and so on. With this motivation, recently, Mohammed and Azam [30,31] studied the concept of soft setvalued maps, that is, a map whose range set lies in a family of soft sets; and introduced the notions of *e*-soft fixed points and *E*-soft fixed points. They ([30, 31]) applied these new ideas to propose a game theoretic approach in decision making problems and in the investigation of existence of solutions to some integro-differential equations. Moreover, it is shown in [30] that every fuzzy mapping is a special kind of soft set-valued map. Since every fuzzy mapping has its corresponding multifunction analogue (see [11, Theorem 2.2]), hence, the idea of *e*-soft fixed point theorems is a generalization of the concept of fuzzy fixed points and fixed points of multi-valued mappings.

To the best of our knowledge, there is no contribution in the literature so far concerning fixed point theorems of soft set-valued maps in Hausdorff fuzzy metric spaces. To this end, the main aim of this paper is to initiate the idea of *e*-soft fixed point theorems of soft set-valued maps in Hausdorff fuzzy metric spaces. Furthermore, we also present some connections of soft set-valued maps with regards to fuzzy and multivalued mappings. Consequently, a few results in the latter mappings are deduced as corollaries. In addition, fractional calculus is not only a growing and a productive field for its own sake, but rather represents a modern philosophy dealing with how to construct and apply certain non-local operators to real-life problems. With this incentive, one of our results is applied to analyze some sufficient conditions for the existence of solutions of mixed non-convex Riemann-Liouville and Caputo fractional differential inclusions with non-local fractional integro-differential boundary conditions. Finally, nontrivial examples are provided to support the hypotheses and usability of our results.

2. PRELIMINARIES

In this section, we recall some requisite concepts of fuzzy sets, fuzzy metric spaces, soft sets and soft set-valued maps. Throughout this article, denote by R, R+ and N is the set of real, non-negative real and natural numbers, respectively.

Definition 2.1. *[50] Let X be a nonempty set. A fuzzy set in X is a function with domain X and values in [0; 1]. If A is a fuzzy set in X and $x \in X$, then the function value A(x) is called the degree of membership of x in A. The α-level set of a fuzzy set A, denoted by $[A]_\alpha$, is defined as*

$$[A]_\alpha = \begin{cases} \overline{\{x \in X : A(x) > 0\}}, & \text{if } \alpha = 0 \\ \{x \in X : A(x) \geq \alpha\}, & \text{if } \alpha \in (0,1]. \end{cases}$$

where M represents the closure of the crisp set M. We denote the family of fuzzy sets in X by I^X.

Definition 2.2. [*18] Let X be an arbitrary set and Y a metric space. A mapping $T : X \to I^Y$ is called a fuzzy mapping. A point $u \in X$ is said to be a fuzzy fixed point of a fuzzy mapping T if there exists an $\alpha \in (0; 1]$ such that $u \in [Tu]_\alpha$.*

Definition 2.3. *[44] A binary operation $*: [0; 1]^2 \to [0; 1]$ is called a continuous t-norm if ([0; 1]; *) is an Abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ ($a; b; c; d \in [0; 1]$).*

Common examples of continuous *t*-norm are :

(i) $a * b = \min\{a, b\}$ (minimum *t*-norm).
(ii) $a * b = ab$ (product *t*-norm).
(iii) $a * b = \max\{a + b - 1; 0\}$ (Lukasievicz *t*-norm).

Following Kramosil and Michalek [23, Definition 7], in order to obtain a Hausdorff topology on fuzzy metric space, George and Veeramani [16] defined fuzzy metric space as follows.

Definition 2.4. *[16] The 3-tuple (X, M, *) is a called a fuzzy metric space if X is an arbitrary nonempty set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,*

(FM1) $M(x, y, t) > 0$,
(FM2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(FM3) $M(x, y, t) = M(y, x, t)$,
(FM4) $M(x, y, t) * M(y, z, s) \geq M(x, z, t + s)$,
(FM5) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous*.

Remark 2.5. *It is noteworthy that* $0 < M(x, y, t) < 1$ *for all* $t > 0$, *provided* $x \neq y$ *(cf. [28]).*

Example 2.6. *[16] Let (X, σ) be a metric space and $a * b = ab$ (or $a * b = \min\{a, b\}$) for all $a, b \in [0, 1]$. Define $M : X^2 \times (0, \infty) \to [0, 1]$ as*

$$M(x, y, t) = \frac{t}{t + \sigma(x, y)},$$

*for all $x, y \in X$ and $t > 0$. Then (X, M, *) is a fuzzy metric space. This fuzzy metric M induced by the metric σ is called the standard fuzzy metric.*

Example 2.7. *[13] Let X be a nonempty set, $f : X \to \mathbb{R}^+$ be a one-one function and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. Fix $\alpha, \beta > 0$ and define $M : X^2 \times (0, \infty) \to [0, 1]$* as

$$M(x,y,t) = \left(\frac{(\min\{f(x), f(y)\})^{\alpha} + g(t)}{(\max\{f(x), f(y)\})^{\alpha} + g(t)}\right)^{\beta},$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space, where $*$ is the product t-norm.

Example 2.8. *[13] Let (X, σ) be a bounded metric space and suppose there exists $\eta \in (0, \infty)$ such that $g : \mathbb{R}^+ \to (\eta, \infty)$ is an increasing continuous function. Define $M : X^2 \times (0, \infty) \to [0, 1]$ by*

$$M(x,y,t) = 1 - \frac{\sigma(x,y)}{g(t)}.$$

Then $(X,M, *)$ is a fuzzy metric space, where $*$ is the Lukasievicz t-norm. For further examples of fuzzy metric spaces, we refer the interested reader to Gregori et al. [13].

Definition 2.9. *[16] Let $(X,M, *)$ be a fuzzy metric space.*

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to a point $x \in X$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t, p > 0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

(iv) A subset A of X is said to be closed if for each convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in A$ and $x_n \to x$, we have $x \in A$.

(v) A subset A of X is said to be compact if every sequence in X has a convergent subsequence.

Throughout this article, denotes the family of nonempty compact subsets of X.

Definition 2.10. *[21] Let $(X, M, *)$ be a fuzzy metric space. The function M is said to be continuous on $X^2 \times (0, \infty)$ if*

$$\lim_{n\longrightarrow\infty} M(x_n, y_n, t_n) = M(x,y,t),$$

whenever $\{(x_n, y_n, t_n)\}_{n\in\mathbb{N}}$ is a sequence in $X^2 \times (0, \infty)$ which converges to $(x, y, t) \in X^2 \times (0, \infty)$; that is,

$$\lim_{n\longrightarrow\infty} M(x_n, x, t) = \lim_{n\longrightarrow\infty} M(y_n, y, t) = 1$$

Lemma 2.11. *[42] If $(X, M, *)$ is a fuzzy metric space, then M is a continuous function on $X^2 \times (0, \infty)$.*

In 2004, Rodriguez and Romaguera [42] proposed a method for constructing Hausdorff fuzzy metric on the set of nonempty compact subsets of a given fuzzy metric space and introduced the following definition.

Definition 2.12. *[42] Let $(X, M, *)$ be a fuzzy metric space. For each $A, B \in$ and $t > 0$, the Hausdorff fuzzy metric $H_M : K^2_X \times (0, \infty) \to \mathbb{R}^+$ is defined as*

$$H_M(A,B,t) = \min\left\{ \inf_{x\in A} \sup_{y\in B} M(x,y,t), \inf_{y\in B} \sup_{x\in A} M(x,y,t) \right\}.$$

Let $(X, M, *)$ be a fuzzy metric space and N_X be the family of nonempty subsets of X. For $x \in X$, $A \in N_X$ and all $t > 0$, the function $M(x, A, t)$ is defined as

$$M(x,A,t) = \sup\{M(x,a,t) : a \in A\}.$$

Lemma 2.13. *[42] Let $(X, M, *)$ be a fuzzy metric space. If $A, B \in$ and $x \in A$, then there exists $y \in B$ such that*

$$M(x,y,t) = \sup_{b\in B} M(x,b,t).$$

Lemma 2.14. *[42] Let $(X, M, *)$ be a fuzzy metric space. Then, for each $x \in X$, $B \in$ and $t > 0$, there exists $b \in B$ such that*

$$M(x,B,t) = M(x,b,t).$$

Let (X, σ) be a metric space. For $A, B \in$, the Hausdorff metric $H : K^2{}_X \to \mathbb{R}^+$ is defined as

$$H(A,B) = \max\{\sup_{x\in A} \sigma(x,B), \sup_{y\in B} \sigma(A,y)\}.$$

From Proposition 3 in [42], the following relationship between the Hausdorff metric H and Hausdorff fuzzy metric H_M is established.

Lemma 2.15. *[42] Let $(X,M *)$ be a fuzzy metric space, where M is the standard fuzzy metric induced by the metric σ with $a * b = ab$. Then, for each $t > 0$ and $A, B \in$, we have*

$$H_M(A,B,t) = \frac{t}{t + H(A,B)}.$$

Let E be the universal set of parameters, $A \subseteq E$ and $P(X)$ denotes the power set of an initial universe of discourse X. Molodstov [34] initiated the concept of soft sets with the following definition.

Definition 2.16. *[34] The pair (F, A) is called a soft set over X under E, where $A \subseteq E$ and F is a set-valued mapping $F : A \to P(X)$.*

In other words, a soft set over X is a parameterized family of subsets of X. For each $e \in E$, $F(e)$ is considered as the set of e-approximate elements of (F,A).

Example 2.17. *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the universal set of movies and $A =\{e_1, e_2\}$, where e_1 = 3D image and e_2 = not suitable for children under the age of 12. Then $F(e_1) = \{x_4, x_5\}$ means that the only 3D movies are x_4 and x_5; $F(e_2) = \{x_1, x_2, x_3\}$ means that the movies x_1, x_2 and x_3 are not suitable for children aged below 12.*

For further examples of soft sets, the interested reader is referred to [30, 31, 34]. Hereafter, we shall represent the family of soft sets over X under E by $[P(X)]^E$.

Mohammed and Azam [30] introduced the notion of soft set-valued maps and e-soft fixed points in the following manner.

Definition 2.18. *[30] A mapping $T : X \to [P(X)]^E$ is called a soft set-valued map. A point $u \in X$ is said to be an e-soft fixed point of T if $u \in (Tu)(e)$, for some $e \in E$. If for each $x \in X$, $DomTx = E$*

and $u \in (Tu)(e)$ *for all* $e \in E$*, then u is known as E-soft fixed point of T. Here, the domain of T, written as DomT, is given as*

$$DomT = \{x \in X : (Tx)(e) \subseteq X, e \in E\}.$$

Denote by $E_{Fix(T)}$, the set of all e-soft fixed points of a soft set-valued map T. Notice that if $T : X \to [P(X)]^E$ is a soft set-valued map, then the pair (Tx,E) is a soft set over X, for all $x \in X$. Throughout this article, the set $(Tx)(e)$ shall be written as $(T_e x)$ for all $x \in X$ and $e \in E$.

Several examples of soft set-valued maps have been provided in [30, 31]. However, we give additional examples as follows.

Example 2.19. *Let* $X = \{1, 2, 3\}$ *and* $E = \{1, 2\}$. *Define* $T : X \to [P(X)]^E$ as follows:

$$(T_e x) = \begin{cases} \{1,3\}, & \text{if } e = 1 \\ \{2,3\}, & \text{if } e = 2. \end{cases}$$

Then T is a soft set-valued map. Notice that $1 \in (T_e 1)$ *for* $e = 1$ *and* $2 \in (T_e 2)$ *for* $e = 2$; *hence, 1 and 2 are e-soft fixed points of T. But,* $2 \notin (T_e 2)$ *and* $1 \notin (T_e 1)$ *for* $e = 1$ *and* $e = 2$, *respectively. It follows that 1 and 2 are not E-soft fixed points of T. On the other hand,* $3 \in (T_e 3)$ *for all* $e \in E$*; thus, the set of all E-soft fixed points of T*is *given by* $E_{Fix(T)} = \{3\}$*. The map T can be represented as in Figure 1. Notice that in Figure 1, the dots represent other subsets of X.*

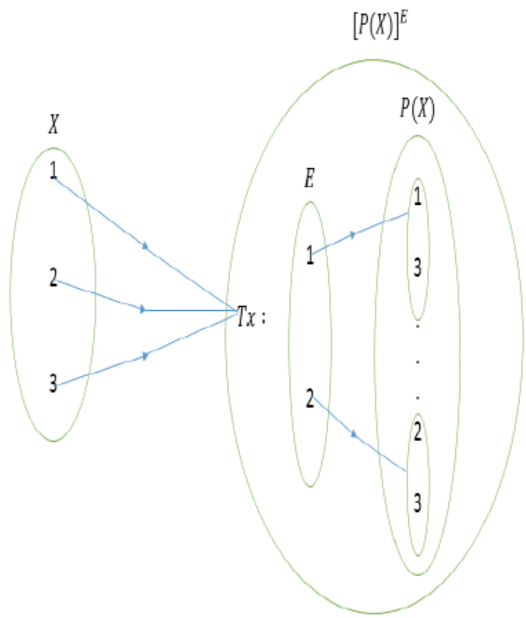

Figure 1. Graphical representation of the soft set-valued map in Example 2.19

Example 2.20. *[30] Let* $X = [0, 1]$ *and* $E = [0, 1]$.

Define $T : X \to [P(X)]^E$ by

$$(T_e x) = \begin{cases} \left[0, \frac{x}{3}\right], & \textit{if } e_1 \in E \\ \left(0, \frac{x}{6}\right) \bigcup \left(\frac{x}{4}, \frac{x}{3}\right], & \textit{if } \ e_2 \in E \\ \emptyset, \quad \textit{otherwise} & \end{cases}$$

Then T is a soft set-valued map.

Figure 2 is a three-dimensional (3D) graphical representation of the soft set-valued map in Example 2.20.

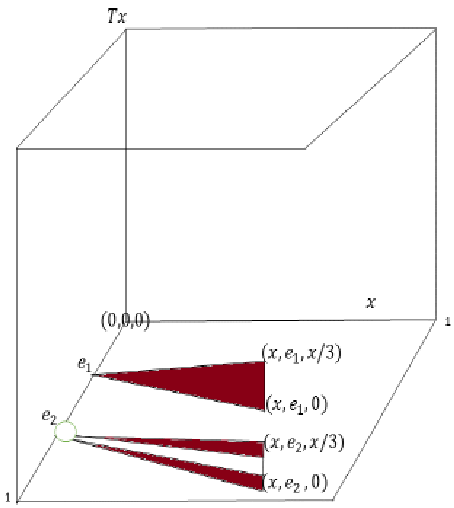

Figure 2. Graphical representation of the soft set-valued map in Example 2.20

Example 2.21. *Let* $X = \mathbb{R}$ *and* $E = [0, 10]$. *Define* $T : X \to [P(X)]^E$ *by*

$$(T_e x) = \begin{cases} [0, 1 + \sin^2 x], & \textit{if } 0 \leq e < 4 \\ \left(\frac{\sin x}{5}, 1\right], & \textit{if } 4 \leq e \leq 10. \end{cases}$$

Then T is a soft set-valued map. Notice that $u = 0 \in (T_e 0) = [0, 1]$, *for* $e \in [0, 4)$*; in this case, 0 is an e-soft fixed point of T. But* 0 *is not an e-soft fixed point of T for* $e \in [4, 10]$, *since* $0 \notin (T_e 0) = (0, 1]$. *Furthermore, observe that* $u = 1/2 \in (T_e 1/2)$ *for all* $e \in E$*; hence* $u = 1/2$ *is an E-soft fixed point of T.*

Remark 2.22. *It is well-known that every fuzzy set is a special kind of soft set (see [34]). In like manner, every fuzzy mapping* $A : X \to I^X$ *can be thought as a soft set-valued map* $\Lambda_A : X \to [P(X)]^{[0;1]}$, *defined by*

$$\Lambda A(x)(e) = \{t \in X : (Ax)(t) \geq e\}.$$

Notice that $X \mapsto P(X)$ *is embedding by* $x \to \{x\}$ *and* $P(X) \mapsto I^X$ *is embedding by* $V \to {}_{XV}$, *for every subset V of P(X); where* ${}_{XV}$ *is the characteristic function of the crisp set V. Similarly,* $I^X \mapsto [P(X)]^{[0;1]}$ *is embedding by* $B \to \Gamma_B$, *for each B in* I^X*; where*

$$\Gamma_B(e) = \{t \in X : B(t) \geq e\}.$$

Consequently, every fuzzy mapping is a special kind of soft set-valued map (cf. [30]).

3. MAIN RESULTS

In this section, the notion of *e*-soft fixed points of soft set-valued maps in Hausdorff fuzzy metric space is initiated. It is further shown herein that a few important fixed point results in the setting of point-to-point and point-to-set valued mappings can be deduced as special cases of our results. Meanwhile, we start with the following lemmas.

Lemma 3.1. *Let $(X, M, *)$ be a fuzzy metric space and $H_M : K^2 \times (0, \infty) \to \mathbb{R}^+$ be Hausdorff fuzzy metric on X. If $A, B \in$ and $a \in A$, then there exists $b \in B$ such that for all $t > 0$,*

$$H_M(A, B, t) \leq \sup_{b' \in B} M(a, b', t).$$

Proof. The proof is a direct consequence of the definition of H_M.

Lemma 3.2. *Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that for all $n \in \mathbb{N}$,*

$$M(x_n, x_{n+1}, \eta t) \geq M(x_n, x_{n+1}; t).$$

where $\eta \in (0, 1)$ and $t > 0$. Assume further that

$$\lim_{t\to\infty} M(x, y, jt) = 1,$$

for all $x, y \in X$, $t > 0$ and $j > 1$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Proof. The idea follows the techniques of Kiany and Harandi [21, Lemma 1].

Theorem 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to [P(X)]^E$ be a soft set-valued map. Assume that the following conditions are satisfied:*

(i) for each $x \in X$, there exists $e \in E$ such that $(T_e x)$ is a nonempty compact subset of X;
(ii) $\lim_{t\to\infty} M(x, y, jt) = 1$ for all $t > 0$ and $j > 1$;
(iii) for all $t > 0$ and $x, y \in X$, there exists $\eta \in (0, 1)$ such that

$$H_M((T_e x), (T_e y), \eta t) \geq \Delta(x, y, t), \tag{3.1}$$

where

$$\Delta(x,y,t) = \min\left\{M(x,y,t), \frac{M(y,(T_e y),t)[1+M(x,(T_e x),t)]}{1+M(x,y,t)}\right\}.$$

Then, T has an e-soft fixed point in X.

Proof. Let $x_0 \in X$, then by hypothesis, there exists $e \in X$ such that $(T_e x_0)$ is a nonempty compact subset of X. Take $x_1 \in (T_e x_0)$. For this $x_1 \in X$, by hypothesis, there exists $e \in E$ such that $(T_e x_1) \in$. If $(T_e x_0) = (T_e x_1)$ for some $e \in E$, then $x_1 \in (T_e x_1)$ and therefore, x_1 is the expected *e*-soft fixed point of T and that ends the proof. So, we presume that $(T_e x_0) \neq (T_e x_1)$ if and only $x_0 \neq x_1$. Since $x_1 \in (T_e x_0)$ and $(T_e x_1) \in$, then by lemmas 3.2 and 2.13, we can find $x_2 \in (T_e x_1)$ such that

$$\begin{aligned} M(x_1, x_2, \eta t) &= \sup_{x_2' \in (T_e x_1)} M(x_1, x_2', \eta t) \\ &\geq H_M((T_e x_0), (T_e x_1), \eta t) \\ &\geq \Delta(x_0, x_1, t). \end{aligned}$$

If $(T_e x_1) = (T_e x_2)$ for some $e \in E$, then $x_2 \in (T_e x_2)$. It follows that x_2 is an e-soft fixed point of T and that completes the proof. Similarly, suppose that $x_1 \neq x_2$ if and only if $(T_e x_1) \neq (T_e x_2)$. Since $x_2 \in (T_e x_1)$ *and* $(T_e x_2) \in$, therefore, by lemmas 3.2 and 2.13, there exists $x_3 \in (T_e x_2)$ such that

$$\begin{aligned} M(x_2, x_3, \eta t) &= \sup_{x_3' \in (T_e x_2)} M(x_2, x_3', \eta t) \\ &\geq H_M((T_e x_1), (T_e x_2), \eta t) \\ &\geq \Delta(x_1, x_2, t). \end{aligned}$$

Continuing this process recursively, we can generate a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_{n+1} \in (T_e x_n)$ and

$$\begin{aligned} M(x_n, x_{n+1}, \eta t) &= \sup_{x_{n+1}' \in (T_e x_n)} M(x_n, x_{n+1}', \eta t) \\ &\geq H_M((T_e x_{n-1}), (T_e x_n), \eta t) \\ &\geq \Delta(x_{n-1}, x_n, t), \end{aligned}$$

where

$$\Delta(x_{n-1}, x_n, t) = \min\left\{M(x_{n-1}, x_n, t), \frac{M(x_n, (T_e x_n), t)[1 + M(x_{n-1}, (T_e x_{n-1}), t)]}{1 + M(x_{n-1}, x_n, t)}\right\} \quad (3.2)$$

By Lemma 2.14, (3.2) reduces to

$$\Delta(x_{n-1}, x_n, t) = \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}. \quad (3.3)$$

Now, we evaluate $\Delta(x_{n-1}, x_n, t)$ under the following cases.

Case 1: If $\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, {}_t)\} = M(x_n, x_{n+1}, t)$, then, we have

$$M(x_n, x_{n+1}, \eta t) \geq M(x_n, x_{n+1}, t). \quad (3.4)$$

Hence, by Lemma 2.11, (3.4) implies that $\{x_n\}_{n\in_{\mathbb{N}}}$ is a Cauchy sequence in X.

Cae 2 : If $\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_{n-1}, x_n, t)$, then, we get

$$\begin{aligned} M(x_n, x_{n+1}, \eta t) &\geq M(x_{n-1}, x_n, t) \\ &\geq M(x_{n-2}, x_{n-1}, \frac{t}{\eta}) \\ &\vdots \\ &\geq M(x_0, x_1, \frac{t}{\eta^n}). \end{aligned} \quad (3.5)$$

Since $(X, M, *)$ is a fuzzy metric space, then taking $t = t/2 + t/2$ and using the inequality given in $(FM4)$ on $M(xn, xn+p, \text{t})$, for any positive integer p, we further consider the following subcases of Case 2.

Cae 2(i): If p is an odd number, say $p = 2m + 1$, $p \in \mathbb{N}$, we have

$$\begin{aligned}M(x_n, x_{n+2m+1}, t) &\geq M\left(x_n, x_{n+1}, \frac{t}{2}\right) * M\left(x_{n+1}, x_{n+2m+1}, \frac{t}{2}\right)\\ &\geq M\left(x_n, x_{n+1}, \frac{t}{2}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{2^2}\right)\\ &\quad * M\left(x_{n+2}, x_{n+2m+1}, \frac{t}{2^2}\right)\\ &\geq M\left(x_n, x_{n+1}, \frac{t}{2}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{2^2}\right)\\ &\quad * M\left(x_{n+2}, x_{n+3}, \frac{t}{2^3}\right) * \cdots * M\left(x_{n+2m}, x_{n+2m+1}, \frac{t}{2^m}\right). \end{aligned} \tag{3.6}$$

Using (3.5) on each term of (3.6), we get

$$\begin{aligned}M(x_n, x_{n+2m+1}, t) \;\geq\; & M\left(x_0, x_1, \frac{t}{\eta^n}\right) * M\left(x_0, x_1, \frac{t}{\eta^{n+1}}\right)\\ & *M\left(x_0, x_1, \frac{t}{\eta^{n+2}}\right) \cdots * M\left(x_0, x_1, \frac{t}{\eta^{n+2m}}\right).\end{aligned}$$

Case 2(ii): If p is even, say $p = 2m$; $m \in \mathbb{N}$, then

$$\begin{aligned}M(x_n, x_{n+2m}, t) \;\geq\; & M\left(x_n, x_{n+1}, \frac{t}{2}\right) * M\left(x_{n+1}, x_{n+2m}, \frac{t}{2}\right)\\ \geq\; & M\left(x_n, x_{n+2}, \frac{t}{2}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{2^2}\right)\\ & *M\left(x_{n+2}, x_{n+2m}, \frac{t}{2^2}\right)\\ \geq\; & M\left(x_n, x_{n+2}, \frac{t}{2}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{2^2}\right)\\ & *M\left(x_{n+2}, x_{n+3}, \frac{t}{2^2}\right) * M\left(x_{n+3}, x_{n+2m}, \frac{t}{2^3}\right)\\ \geq\; & M\left(x_n, x_{n+1}, \frac{t}{2}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{2^2}\right)\\ & *M\left(x_{n+2}, x_{n+3}, \frac{t}{2^3}\right) * \cdots M\left(x_{n+2m-2}, x_{n+2m}, \frac{t}{2^{2m}}\right)\\ \geq\; & M\left(x_0, x_1, \frac{t}{\eta}\right) * M\left(x_0, x_1, \frac{t}{\eta^{n+1}}\right)\\ & *M\left(x_0, x_1, \frac{t}{\eta^{n+2}}\right) * \cdots * M\left(x_0, x_1, \frac{t}{\eta^{n+2m-2}}\right).\end{aligned}$$

Therefore, from Case 1 and Case 2, together with Condition (ii), for all $p \in \mathbb{N}$, we obtain

$$\lim_{n \longrightarrow \infty} M(x_n, x_{n+p}, t) \geq 1 * 1 * \cdots * 1 = 1.$$

This shows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since $(X,M,*)$ is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. This implies that $M(x_n, u, t) \to 1$ as $n \to \infty$, for all $t > 0$.

Now, to show that u is an e-soft fixed point of T, consider

$$\begin{aligned} H_M((T_e x_n),(T_e u),\eta t) &\geq \Delta(x_n,u,t) \\ &= \min\left\{M(x_n,u,t), \frac{M(u,(T_e u),t)[1+M(x_n,(T_e x_n),t)]}{1+M(x_n,u,t)}\right\} \end{aligned} \tag{3.7}$$

Using the continuity of the fuzzy metric M in (3.7), we have

$$\lim_{n\longrightarrow\infty} H_M((T_e x_n),(T_e u),\eta t) \geq \min\{1\} = 1.$$

It follows that $\lim_{n\to\infty} \sup_{u'\in(Teu)} M(x_{n+1}, u', t) = 1$. Hence, there exists a sequence $\{u'_n\}_{n\in\mathbb{N}}$ in $(T_e u)$ such that

$$\lim_{n\longrightarrow\infty} M(x_n, u'_n, t) = 1, \text{ for each } t > 0. \tag{3.8}$$

Consequently, for all $n \in \mathbb{N}$, we get

$$M(u'_n, u, t) \geq M(u'_n, x_n, \frac{t}{2}) * M(x_n, u, \frac{t}{2}). \tag{3.9}$$

Letting $n \to \infty$ in (3.9) and using (3.8), we arrive at $\lim_{n\to\infty} M(u'_n, u, t) = 1$. This implies that $u'_n \to u$ as $n \to \infty$. Since $(T_e u) \in$ and u'_n is a sequence in $(T_e u)$, it follows that $u \in (T_e u)$ for some $e \in E$; that is, u is an e-soft fixed point of T.

The next Theorem is a simple application Theorem 3.3.

Theorem 3.4. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to [P(X)]^E$ be a soft set-valued map. Assume that the following conditions are satisfied:*

(i) for each $x \in X$, there exists $e \in E$ such that $(T_e x)$ is a nonempty compact subset of X;

(ii) $\lim_{t\to\infty} M(x, y, jt) = 1$ for all $t > 0$ and $j > 1$;

(iii) there exist a continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ with the property that $\min_{t\in\mathbb{R}^+}\varphi(t) = 0$, $\max_{t\in\mathbb{R}^+} \varphi(t) = 1$ and $\varphi(\omega) \geq \omega$ for all $\omega \in (0, 1)$;

(iv) for all $x, y \in X$ and $t > 0$, there exists $\eta \in (0, 1)$ such that

$$H_M((T_e x), (T_e y), \eta t) \geq \varphi(\Delta(x, y, t)),$$

where

$$\Delta(x,y,t) = \min\left\{M(x,y,t), \frac{M(y,(T_e y),t)[1+M(x,(T_e x),t)]}{1+M(x,y,t)}\right\}.$$

Then, T has an e-soft fixed point in X.

Proof. By condition (iii), we have

$$\begin{aligned} H_M((T_e x),(T_e y),\eta t) &\geq \varphi(\Delta(x,y,t)) \\ &\geq \Delta(x,y,t). \end{aligned}$$

From here, Theorem 3.3 can be applied to find $u \in X$ such that $u \in (T_e u)$ for some $e \in E$.

Denote the family of functions satisfying Condition (iii) of Theorem 3.4 by Ω. Now, we deduce some immediate consequences of theorems 3.3 and 3.4 as follows.

Corollary 3.5. *Let (X, M, *) be a complete fuzzy metric space and* $T : X \to [P(X)]^E$ *be a soft set-valued map. Assume that the following conditions are satisfied:*

(i) for each $x \in X$*, there exists* $e \in E$ *such that* $(T_e x)$ *is a nonempty compact subset of X;*
(ii) $\lim_{t\to\infty} M(x, y, jt) = 1$ *for all* $t > 0$ *and* $j > 1$*;*
(iii) for all $x, y \in X$ *and* $t > 0$*, there exist* $\eta \in (0, 1)$ *and* $\varphi \in \Omega$ *such that*

$$H_M((T_e x), (T_e y), \eta t) \geq \varphi(M(x, y, t)).$$

Then, T has an e-soft fixed point in X.

Corollary 3.6. *Let (X, M, *) be a complete fuzzy metric space and* $T : X \to [P(X)]^E$ *be a soft set-valued map. Assume that the following conditions are satisfied:*

(i) for each $x \in X$*, there exists* $e \in E$ *such that (Tex) is a nonempty compact subset of X;*
(ii) $\lim_{t\to\infty} M(x, y, jt) = 1$ *for all* $t > 0$ *and* $j > 1$*;*
(iii) for all $x, y \in X$ *and* $t > 0$*, there exist* $\eta \in (0, 1)$ *such that*

$$H_M((T_e x), (T_e y), \eta t) \geq (M(x, y, t)).$$

Then, T has an e-soft fixed point in X.

3.1. Consequences in fuzzy and multivalued mappings

In this subsection, we will show that there is a link between soft set-valued maps, fuzzy mappings and multivalued mappings.

Corollary 3.7. *Let* (X, σ) *be a complete metric space and* $F : X \to I^X$ *be a fuzzy mapping. Assume that the following conditions are satisfied:*

(i) for each $x \in X$*, there exist* $\alpha(x) \in (0, 1]$ *and* $\eta \in (0, 1)$ *such that* $[Fx]_{\alpha(x)}$ *is a nonempty compact subset of X;*
(ii) $H([Fx]_{\alpha(x)}, [Fy]_{\alpha(y)}) \leq \eta\sigma(x, y)$*,*

for all $x, y \in X$*. Then, there exists* $u \in X$ *such that* $u \in [Fu]_{\alpha(u)}$*.*

*Proof. Let (X, M, *)* be a standard fuzzy metric space induced by the metric σ with $a * b = ab$. Since (X, σ) is a complete metric space, then $(X, M, *)$ is complete. Let A, B be two compact subsets of X. Then, applying Lemma 2.14, we have

$$\begin{aligned} H_M(A,B,t) &= \min\left\{\inf_{x\in A}\sup_{y\in B} M(x,y,t), \inf_{y\in B}\sup_{x\in A} M(x,y,t)\right\} \\ &= \min\left\{\inf_{x\in A}\sup_{y\in B}\frac{t}{t+\sigma(x,y)}, \inf_{y\in B}\sup_{x\in A}\frac{t}{t+\sigma(x,y)}\right\} \\ &= \min\left\{\frac{t}{t+\sup_{x\in A}\inf_{y\in B}\sigma(x,y)}, \frac{t}{t+\sup_{y\in B}\inf_{x\in A}\sigma(x,y)}\right\} \\ &= \frac{t}{t+\max\left\{\sup_{x\in A}\inf_{y\in B}\sigma(x,y), \sup_{y\in B}\inf_{x\in A}\sigma(x,y)\right\}} \\ &= \frac{t}{t+H(A,B)}. \end{aligned}$$

Let $E = (0, 1]$ and define a soft set-valued map $\Theta_F : X \to [P(X)]^E$ as

$$\Theta_F x(e) = \{t \in X : (Fx)(t) \geq e\} = [Fx]_e:$$

That is, $\Theta_F x(e) = [Fx]_{\alpha(x)}$, for each $\alpha(x) = e \in (0; 1]$. Therefore, for all $x, y \in X$, we have

$$\begin{aligned} H_M(\Theta_F x(e), \Theta_F y(e), \eta t) &= H_M([Fx]_{\alpha(x)}, [Fy]_{\alpha(y)}, \eta t) \\ &= \frac{\eta t}{\eta t + H([Fx]_{\alpha(x)}, [Fy]_{\alpha(y)})} \\ &\geq \frac{\eta t}{\eta t + \eta\sigma(x,y)} \\ &\geq \frac{t}{t + \sigma(x,y)} = M(x,y,t). \end{aligned}$$

Therefore, by Theorem 3.3, there exists $u \in X$ such that $u \in \Theta_F(u) = [Fu]_{\alpha(u)}$. □

Recall that a set-valued mapping $S : X \to$ is called a multivalued mapping if $S(x) \subseteq$ for each $x \in X$. A point $u \in X$ is called a fixed point of a multivalued mapping S if $u \in Su$ (see [35]).

Corollary 3.8. *[35] Let (X, σ) be a complete metric space and $S : X \to$ be a multivalued mapping. Assume that for all $x, y \in X$, there exists $\eta \in (0, 1)$ such that*

$$H(Sx, Sy) \leq \eta\sigma(x; y):$$

Then, there exists $u \in X$ such that $u \in Su$.

Proof. For each $x \in X$ and $\alpha(x) \in (0, 1]$, consider a fuzzy mapping $F : X \to I^X$ defined by

$$(Fx)(t) = \begin{cases} \alpha(x), & \text{if } t \in Sx \\ 0, & \text{if } t \notin Sx. \end{cases}$$

It follows that

$$[Fx]_{\alpha(x)} = \{t \in X : (Fx)(t) \geq \alpha(x)\} = Sx.$$

Consequently, Corollary 3.7 can be applied to find $u \in X$ such that $u \in Su$. □

3.2. Consequences in single-valued mappings

In this subsection, we will show that results of the previous section can be applied to derive fixed point theorems of some single-valued mappings.

Corollary 3.9. *[21, Theorem 2.6] Let $(X, M, *)$ be a complete fuzzy metric space and $g : X \to X$ be a single-valued mapping. Assume that for each $x, y \in X$ and $t > 0$, there exist $\eta \in (0, 1)$, $j > 1$ such that the following conditions hold:*

(i) $M(gx, gy, \eta t) \geq M(x, y, t)$;
(ii) $\lim_{t\to\infty} M(x, y, jt) = 1$.

Then, there exists $u \in X$ such that $gu = u$.

Proof. We know that the singleton $\{x\}$ is a nonempty compact subset of X for every $x \in X$. For $e \in E$ and all $x \in X$, consider a soft set-valued map $\Theta : X \to [P(X)]^E$ defined by $(\Theta e x) = \{gx\}$. It follows that $(\Theta_e x) \in$ for all $x \in X$. Thus, all the conditions of Corollary 3.6 reduces to the hypotheses of Corollary 3.9. Therefore, by applying Corollary 3.6, we can find $u \in X$ such that $u \in (\Theta_e u)$. The definition of Θ implies that $(\Theta_e u) = \{gu\}$. Consequently, $u = gu$.

Remark 3.10. *From theorems 3.3 and 3.4, a few more results in the existing literature can be deduced as corollaries. For example, the main results of Grabiec [17], Gregori and Sapena [15], Saini and Singh [43], Supak et al [38], and some references therein.*

Now, we provide an example to support the hypotheses and generality of our results.

Example 3.11. *Let $X = \{1, 2, 3, 4\}$ and $\sigma : X \times X \to \mathbb{R}$ be defined as*

$$\sigma(x,y)=\begin{cases}0, & if\ x=y\\ 5, & if\ x\neq y \text{ and } x,y\in\{1,2\},\{1,3\} \text{ or } \{2,3\}\\ 3, & if\ x\neq y \text{ and } x,y\in\{2,4\}\\ 6, & if\ x\neq y \text{ and } x,y\in\{3,4\}\\ 7, & if\ x\neq y \text{ and } x,y\in\{1,4\}.\end{cases}$$

Then, (X, σ) is a complete metric space. Let $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x,y,t)=\frac{t}{t+\sigma(x,y)},$$

for all $x, y \in X$ and $t > 0$. Then, we find that $(X, M, *)$ is a complete fuzzy metric space. Take $E = [0, 1]$, then for each $e \in E$, consider a soft set-valued map $T : X \to [P(X)]^E$ defined by

$$(T_e x)=\begin{cases}\{3,4\}, & if\ x\in\{1,2,3\} \text{ and } 0\leq e<\frac{1}{10}\\ \{1,2\}, & if\ x=4 \text{ and } \frac{1}{10}\leq e\leq 1.\end{cases}$$

We see that for each $x \in X$, there exists $e \in E$ such that $(T_e x) \in$. For all $x \in X$, $t > 0$ and $A \in$, take

$M(x, A, t) = \sup\{ M(x, a, t) : a \in A\}$. (3.10)

In this routine calculation, we shall use Lemma 2.15 and the relation (3.10) in the following cases.

Case 1: *For $x = y$, we have $H((T_e x), (T_e y)) = 0$, therefore,*

$$H_M((T_e x),(T_e y),\eta t)=\frac{\eta t}{\eta t+0}=1,$$

$$M(x,y,t)=M(x,(T_e x),t)=M(y,(T_e y),t)=1,$$

for all $t > 0$ and $\Delta(x, y, t) = \min\{1, 1\} = 1$. Hence,

$$H_M((T_e x), (T_e y), \eta t) \geq \Delta(x, y, t),$$

for all $x, y \in X$ and $\eta \in (0, 1)$. In the remaining cases, we take $\eta = 1/2$.

Case 2: *For $x = 1$ and $y = 4$, we have $M(1, 4, t) = t / t+7$, $M(4, (T_e 4), t) = t / t+3$, $M(1, (T_e 1), t) = t / t+5$,*

$$H((T_e 1), (T_e 4)) = H(\{3, 4\}, \{1, 2\}) = 5.$$

Therefore,

$$H_M\left((T_e 1),(T_e 4),\frac{t}{2}\right)=\frac{t}{t+\frac{5}{2}}$$

and

$$\Delta(1,4,t) = \min\left\{\frac{t}{t+7}, \frac{\frac{t}{t+3}\left[1+\frac{t}{t+5}\right]}{1+\frac{t}{t+7}}\right\} = \frac{t}{t+7}.$$

Thus, for all $t > 0$, *we get*

$$H_M\left((T_e1),(T_e4),\frac{t}{2}\right) = \frac{t}{t+\frac{5}{2}} \geq \Delta(1,4,t) = \frac{t}{t+7}.$$

Case 3: *For* $x = 2$ *and* $y = 4$, *we have* $M(2, 4, t) = t / t+3$, $M(4, (T_e4), t) = t / t+3$, $M(2, (T_e2), t) = t / t+3$,

$$H((T_e2),(T_e4)) = H(\{3, 4\}; \{1, 2\}) = 5.$$

Thus,

$$H_M\left((T_e2),(T_e4),\frac{t}{2}\right) = \frac{t}{t+\frac{5}{2}}$$

and

$$\Delta(2,4,t) = \min\left\{\frac{t}{t+3}, \frac{t}{t+3}\right\} = \frac{t}{t+3}.$$

Hence, for all $t > 0$, *we have*

$$H_M\left((T_e2),(T_e4),\frac{t}{2}\right) = \frac{t}{t+\frac{5}{2}} \geq \Delta(2,4,t) = \frac{t}{t+3}.$$

Case 4: *For* $x = 3$ *and* $y = 4$, *we obtain* $M(3, 4, t) = t / t+6$, $M(4, (T_e4), t) = t / t+3$, $M(3, (T_e3), t) = 1$,

$$H((T_e2),(T_e4)) = H(\{3, 4\}, \{1, 2\}) = 5.$$

It follows that

$$H_M\left((T_e3),(T_e4),\frac{t}{2}\right) = \frac{t}{t+\frac{5}{2}}$$

and

$$\begin{aligned}\Delta(3,4,t) &= \min\left\{\frac{t}{t+6}, \left(\frac{t}{t+3}\right)\left(\frac{t+6}{t+3}\right)\right\} \\ &= \frac{t}{t+6}.\end{aligned}$$

Consequently, for all $t > 0$*, we have*

$$\begin{aligned}H_M\left((T_e3),(T_e4),\frac{t}{2}\right) &= \frac{t}{t+\frac{5}{2}} \\ &\geq \Delta(3,4,t) = \frac{t}{t+6}.\end{aligned}$$

Hence, for all $x, y \in X$ *and* $t > 0$*, there exists* $\eta = 1/2 \in (0, 1)$ *such that*

$$H_M((T_e x), (T_e y), \eta t) \geq \Delta(x, y, t).$$

Thus, all the conditions of Theorem 3.3 are satisfied. In this case, the set of all e-soft fixed points of T is given by $E_{Fix(T)} = \{3\}$.

Remark 3.12. *Notice that if the soft set-valued map T in Example 3.11 is assumed to be a multivalued mapping in the sense of Nadler [35], then taking* $x = 1$ *and* $y = 4$*, we have*

$$\begin{aligned}H(T1,T4) &= H(\{3,4\},\{1,2\}) \\ &= 5 > 7\alpha = \alpha\sigma(1,4),\end{aligned}$$

for all $\alpha \in [0, 1)$*. Therefore, Corollary 3.8 or [35, Theorem 5] cannot be applied in this case to obtain a fixed point of T.*

4. APPLICATION: AN EXISTENCE THEOREM OF MIXED NON-CONVEX FRACTIONAL DIFFERENTIAL INCLUSIONS

Fractional differential equations (FDEs) have enjoyed keen attentions of researchers due to their enormous applications in different fields of sciences and engineering. A lot of useful work is currently going on in this direction; see, for instance, Abdeljawad et al. [2], Atangana and Owolabi [4], Toufik and Atangana [47] and the references therein. For a comprehensive monograph on this matter, the interested reader is also referred to Kilbal et al. [22]. In particular, boundary value problem (BVPs) for FDEs with non-local boundary conditions (BCs) arise in various branches of applied mathematics and engineering; for example, in heat conduction, underground water ow, thermoelasticity, and many problems in plasma physics involve FDEs with non-local BCs. Non-local integral boundary conditions are widely employed where classical boundary conditions fail to produce the needed physical properties of the model being investigated. Usually, the first most concerned problem in the study of FDEs is the condition for the existence of its solution(s). In this direction, by applying different fixed point theorems such as Banach's, Krasnoselskii's and Leray-Schauder nonlinear alternative fixed point theorems, many authors have established some useful results on existence and uniqueness of solutions to BVPs for FDEs; see, for instance, [5, 6,19,24,36,49] and the references therein.

Recently, Ntouyas et al [37] introduced a mixed type BVP involving both Riemann-Liouville and Caputo fractional derivatives having non-local fractional integro-differential BCs, given as:

$$\begin{cases} {}^{RL}D^p({}^C D^r x(t)) = g(t, x(t)), & t \in (0, \delta) = J \\ x'(\tau) = \gamma\ {}^C D^\mu x(\xi), & x(\delta) = \beta I^q x(\kappa), \tau, \xi, \kappa \in J, \end{cases} \tag{4.1}$$

where ${}^{RL}D^p$ represents the Riemann-Liouville fractional derivative of order $p \in (0, 1)$, ${}^C D^r$; ${}^C D^\mu$ denote Caputo fractional derivatives of order $r \in (0, 1)$ and $\mu \in (0, p + r)$, I^q is the Riemann-Liouville fractional integral of order $q > 0$, $g : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\gamma, \beta \in \mathbb{R}$. In [37], by using Banach contraction mapping principle, Krasnoselskii fixed point theorem and non-linear alternative of Leray-Schauder type fixed point theorem, conditions for the existence and uniqueness of solution to the BVP (4.1) is investigated. They also studied the existence result for inclusion version of problem (4.1) by using fixed point theorem of multivalued maps due to Covitz and Nadler [9]. Following [37], in this section, both techniques of soft set-valued maps and fuzzy set-valued maps defined on a complete fuzzy metric space are used to discuss some sufficient conditions for the existence of solutions of the BVP (4.1). From [37], the mixed BVP is given as:

$$\begin{cases} {}^{RL}D^p({}^C D^r x(t)) \in K(t, x(t)), & t \in (0, \delta) = J \\ x'(\tau) = \gamma\ {}^C D^\mu x(\xi), & x(\delta) = \beta I^q x(\kappa), \tau, \xi, \kappa \in J, \end{cases} \tag{4.2}$$

where $K : J \times \mathbb{R} \to P(\mathbb{R})$ is a point-to-set-valued map. In our result herein, we slightly adopt the technique of [37] and consider problem (4.2) in the case of non-convex righthand side of the inclusion. First, we recall the needed concepts and results of fractional calculus as follows.

Definition 4.1. *[22] The Riemann-Liouville fractional derivative of order p of a function $g : (0, \infty) \to \mathbb{R}$ is defined as*

$${}^{RL}D^p g(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_{0+}^t (t-s)^{n-p-1} g(s) ds, \ \ p > 0, n = [p] + 1,$$

where $[p]$ denotes the integer part of the real number p and $\Gamma(.)$ is the Gamma function.

Definition 4.2. *[22] The Riemann-Liouville fractional integral of order p of a function $g : (0, \infty) \to \mathbb{R}$ is defined as*

$$I^p g(t) = \frac{1}{\Gamma(p)} \int_{0+}^t (t-s)^{p-1} g(s) ds, p > 0.$$

Definition 4.3. *[22] The fractional derivative of order p for n-times differentiable function $g : (0, \infty) \to \mathbb{R}$ in Caputo sense is defined as*

$${}^C D^p g(t) = \frac{1}{\Gamma(n-p)} \int_{0+}^t (t-s)^{n-p-1} \left(\frac{d}{ds}\right)^n g(s) ds, p > 0, n = [p] + 1.$$

Let $(J, \mathbb{R})$ be the Banach space of all continuous functions $x : J \to \mathbb{R}$ equipped with a topology of uniform convergence with the norm defined by:

$$||x|| = \sup\{|x(t)| : t \in J\}.$$

By $L'(J, \mathbb{R})$, we mean the Banach space of Lebesgue integrable functions $x : J \to \mathbb{R}$ endowed with the norm:

$$\|x\|_{L'} = \int_a^b |x(t)|dt.$$

Let

$$\Theta_1 = \frac{\Gamma(p)}{\Gamma(p+r-1)}\tau^{p+r-2} - \gamma\frac{\Gamma(p)}{\Gamma(p+r-\mu)}\xi^{p+r-\mu} \neq 0. \tag{4.3}$$

$$\Theta_2 = \beta\frac{\Gamma(p)}{\Gamma(p+r-q)}\kappa^{p+r+q-1} - \frac{\Gamma(p)}{\Gamma(p+r)}\delta^{p+r-1} \neq 0. \tag{4.4}$$

$$\Theta_3 = 1 - \beta\frac{1}{\Gamma(1+q)}\kappa^{q} \neq 0. \tag{4.5}$$

Lemma 4.4. *[37] A function $x \in C'(J, \mathbb{R})$ is a solution of the BVP* (4.2) *if there exists a function $u \in L'(J, \mathbb{R})$ with $u \in K(t, x)$ almost everywhere (a.e.) on J such that*

$$\begin{aligned} x(t) = {} & I^{p+r}u(s)(t) \\ & + \frac{1}{\Theta_1}\left[\gamma I^{p+r-\mu}u(s)(\xi) - I^{p+r-1}u(s)(\tau)\right]\left[\frac{\Gamma(p)}{\Gamma(p+r)}t^{p+r-1} - \frac{\Theta_2}{\Theta_3}\right] \\ & + \frac{1}{\Theta_3}\left[\beta I^{p+r+q}u(s)(\kappa) - I^{p+r}u(s)(\delta)\right], \end{aligned}$$

and $x'(\tau) = \gamma\, {}^{C}D^{\mu}x(\xi)$, $x(\delta) = \beta I^{q}x(\kappa)$, where Θ_1, Θ_2 and Θ_3 are given by (4.3), (4.4) *and* (4.5), *respectively.*

For computational convenience, we set the following notations from [37]:

$$\Psi_0 = \frac{\Gamma(p)}{\Gamma(p+r)}\delta^{p+r-1} + \frac{|\Theta_2|}{|\Theta_3|}. \tag{4.6}$$

$$\begin{aligned} \Psi = {} & \frac{\delta^{p+r}}{\Gamma(p+r)} \\ & + \frac{\Psi_0}{|\Theta_1|}\left[|\gamma|\frac{\xi^{p+r-\mu}}{\Gamma(p+r-\mu+1)} + \frac{\tau^{p+r-1}}{\Gamma(p+r)}\right] \\ & + \frac{1}{|\Theta_3|}\left[|\beta|\frac{\kappa^{p+r+q}}{\Gamma(p+r+q+1)} + \frac{\delta^{p+r}}{\Gamma(p+r+1)}\right]. \end{aligned} \tag{4.7}$$

Definition 4.5. *Let X be a nonempty set. A single-valued mapping $g : X \to X$ is said to be a selection of a set-valued map $K : X \to P(X)$ if $g(x) \in$ for each $x \in X$.*

For each $x \in X = C(J, \mathbb{R})$, we denote the set of all selections of K by $S_{K,x} = \{g \in L'(J, \mathbb{R}) : g(t) \in K(t, x) \text{ a.e. } t \in J\}$.

Definition 4.6. *Let X be a nonempty set. A set-valued map $K : X \to P(X)$ is called lower semi-continuous at x_0 if for any $y_0 \in K(x_0)$ and any neighborhood U of y_0, there exists a neighborhood U^* of x_0 such that $K(x_0) \cap U$ is nonempty, for all $x \in U^*$. A set-valued map K is said to be lower semi-continuous if it is so at every point $x_0 \in X$.*

Now, we present the main result of this subsection as follows.

Theorem 4.7. *Consider the BVP* (4.2). *Suppose that the following conditions are satisfied:*

(C1) The set-valued map $K : J \times \mathbb{R} \to K_{\mathbb{R}}$ is such that for each $x \in \mathbb{R}$, $K(t, x)$ is measurable and lower semi-continuous;

(C2) there exists a constant $\zeta > 0$ such that $H(K(t, x), K(t, y))\ \zeta|x - y|$, for almost all $t \in J$ and $x, y \in \mathbb{R}$;

(C3) there exists $\eta \in (0, 1)$ such that $\Psi\zeta \leq \eta$.

Then, the BVP (4.2) *has at least one solution in $X = C(J, \mathbb{R})$.*

Proof. Let (X, σ) be a metric space induced from the norm $(X, \|.\|)$ and let $(X, M, *)$ be the standard fuzzy metric space endowed with the product t-norm $a * b = ab$, for all $a, b \in [0, 1]$. Let $F : X \to I^X$ be a fuzzy mapping. Then, consider the α-level set of F, defined by

$$[Fx]_{\alpha(x)} = \left\{ \begin{array}{l} \pi \in X : \\ \pi(t) \in \Omega_K = \left\{ \begin{array}{l} I^{p+r}u(s)(t) \\ + \dfrac{1}{\Theta_1}\left[\gamma I^{p+r-\mu}u(s)(\xi) - I^{p+r-1}u(s)(\tau)\right] \\ \times \left[\dfrac{\Gamma(p)}{\Gamma(p+r)}t^{p+r-1} - \dfrac{\Theta_2}{\Theta_3}\right] \\ + \dfrac{1}{\Theta_3}\left[\beta I^{p+r+q}u(s)(\kappa) - I^{p+r}u(s)(\delta)\right] \end{array} \right\} \end{array} \right. ,$$

for $u \in S_{K,x}$. Obviously, the set of fuzzy fixed points of F is the solution set of problem (4.2). We have to show that F satisfies all the hypotheses of Corollary 3.7.

Let $x \in X$ be arbitrary. Since the set-valued map K, $J \times \mathbb{R} \to K_{\mathbb{R}}$ is lower semi continuous, it follows from Michael's selection theorem ([26, Theorem 1]) that there exists a continuous function $\varepsilon_x : J \times \mathbb{R} \to \mathbb{R}$ such that $\varepsilon_x(t, x) \in K(t, x)$ for each $(t, x) \in J \times \mathbb{R}$. Therefore, $\Omega_K \subseteq [F_x]_{\alpha(x)}$. So, $[Fx]_{\alpha(x)}$ is nonempty. Clearly, $[Fx]_{\alpha(x)}$ is compact. However, to see this, notice that since $u(t) \in K(t, x)$ almost everywhere for each $t \in J$, hence, u is continuous on J and $u(t)$ is compact for each $t \in J$. Consequently, $[Fx]_{\alpha(x)}$ is compact.

For $x \in X$, take $\pi_1 \in [Fx]_{\alpha(x)}$, for each $\alpha(x) \in (0, 1]$. Then, there exists $u_1(t) \in K(t, x)$ such that for each $t \in J$,

$$\pi_1(t) = I^{p+r}u_1(s) + \frac{1}{\Theta_1}\left[\gamma I^{p+r-\mu}u_1(s)(\xi) - I^{p+r-1}u_1(s)(\tau)\right] \times \left[\frac{\Gamma(p)}{\Gamma(p+r)}t^{p+r-1} - \frac{\Theta_2}{\Theta_3}\right] + \frac{1}{\Theta_3}\left[\beta I^{p+r+q}u_1(s)(\kappa) - I^{p+r}u_1(s)(\delta)\right]. \tag{4.8}$$

From condition (C_2), we have $H(K(t, x), K(t, y)) \leq \zeta\, |x - y|$. Hence, there exists $\omega(t) \in K(t, y(t))$ such that $|u_1(t) - \omega(t)| \leq \zeta|x(t) - y(t)|$, $t \in J$. Define $M : J \to P(\mathbb{R})$ by $M(t) = \{\omega \in \mathbb{R} : |u_1(t) - \omega(t)| \leq \zeta\, |x(t) - y(t)|\}$. Since the operator $M(t) \cap K(t, y(t))$ is measurable (see [8, Proposition 4]), there exists a function $u_2(t)$ which is a measurable selection of M. Thus, $u_2(t) \in K(t, y(t))$, and for each $t \in J$, we get $|u_1(t) - u_2(t)| \leq \zeta\, |x(t) - y(t)|$. For each $t \in J$, take

$$\pi_2(t) = I^{p+r}u_2(s)(t) + \frac{1}{\Theta_1}\left[\gamma I^{p+r-\mu}u_2(s)(\xi) - I^{p+r-1}u_2(s)(\tau)\right] \times \left[\frac{\Gamma(p)}{\Gamma(p+r)}t^{p+r-1} - \frac{\Theta_2}{\Theta_3}\right] + \frac{1}{\Theta_3}\left[\beta I^{p+r+q}u_2(s)(\kappa) - I^{p+r}u_2(s)(\delta)\right]. \tag{4.9}$$

From (4.8) and (4.9), we have

$$\begin{aligned} |\pi_1(t)-\pi_2(t)| \leq\ & I^{q+r}|u_2(s)-u_1(s)|(t) \\ & + \frac{\Psi_0}{|\Theta_1|}\Bigg[|\gamma| I^{p+r-\mu}|u_2(s)-u_1(s)|(\xi) \\ & + I^{p+r-1}|u_2(s)-u_1(s)|(\tau)\Bigg] \\ & + \frac{1}{\Theta_3}\Bigg[|\beta| I^{p+r+q}|u_2(s)-u_1(s)|(\kappa) \\ & + I^{p+r}|u_2(s)-u_1(s)|\Bigg] \\ \leq & \left\{\frac{\delta^{p+r}}{\Gamma(p+r)} + \frac{\Psi_0}{|\Theta_1|}\left[|\gamma|\frac{\xi^{p+r-\mu}}{\Gamma(p+r-\mu+1)} + \frac{\tau^{p+r-1}}{\Gamma(p+r)}\right]\right. \\ & \left. + \frac{1}{|\Theta_3|}\left[|\beta|\frac{\kappa^{p+r+q}}{\Gamma(p+r+q+1)} + \frac{\delta^{p+r}}{\Gamma(p+r+1)}\right]\right\}\zeta|x(t)-y(t)|. \end{aligned}$$

From the above inequality, we have $\|\pi_1 - \pi_2\| \leq \Psi\zeta\|x-y\|$. Consequently,

$$H([Fx]_{\alpha(x)}, [Fy]_{\alpha(y)}) \leq \Psi\zeta\sigma(x, y). \tag{4.10}$$

By the assumption (C_3), from (4.10), we have

$$H([Fx]_{\sigma(x)}, [Fy]_{\alpha(y)}) \leq \eta\sigma(x, y). \tag{4.11}$$

for some $\eta \in (0, 1)$. At this point, Corollary 3.7 can be applied to conclude that the BVP (4.2) has at least one solution in X which corresponds to the fuzzy fixed point of F. However, to use Corollary 3.6 to derive the same conclusion, we proceed as follows: Let $E = (0, 1]$ be the universe of parameter set. Then, for all $e \in E$ and $x \in X$, define a soft set-valued map $T_F : X \to [P(X)]^E$ as

$$\begin{aligned} T_F\, x(e) &= \{\mathrm{v} \in X : (Fx)(\mathrm{v}) \geq e\} \\ &= [Fx]_e = [Fx]_{\alpha(x)}. \end{aligned}$$

Therefore, for each $t > 0$, from (4.11) and Lemma 2.15, we have

$$\begin{aligned} H_M((T_F x(e)), (T_F y(e)), \eta t) &= H_M([Fx]_{\alpha(x)}, [Fy]_{\alpha(y)}, \eta t) \\ &= \frac{\eta t}{\eta t + H([Fx]_{\alpha(x)}, [Fy]_{\alpha(y)})} \\ &\geq \frac{\eta t}{\eta t + \eta\sigma(x, y)} \\ &\geq \frac{t}{t + \sigma(x, y)} = M(x, y, t). \end{aligned}$$

Furthermore, for any $j > 0$, it is easy to see that $M(x, y, jt) = 1$ as $t \to 1$. Consequently, all the hypotheses of Corollary 3.6 are satisfied. Hence, the BVP (4.2) has at least one solution in X which coincides with the e-soft fixed points of the soft set-valued map T_F.

In what follows, we supply an example to validate the axioms of Theorem 4.7.

Example 4.8. *Consider the following mixed Riemann-Liouville and Caputo fractional boundary value problem:*

$$\begin{cases} {}^{RL}D^{\frac{6}{7}}({}^{C}D^{\frac{1}{4}}x(t)) \in K(t, x(t)), & t \in (0,5) = J \\ x'(\frac{1}{4}) = \frac{1}{15}\ {}^{C}D^{\frac{1}{4}}x(\frac{1}{6}), & x(5) = 3I^{\frac{4}{5}}x(\frac{1}{4}). \end{cases} \tag{4.12}$$

Comparing the BVPs (4.2) *and* (4.12), *we see that* $p = \frac{6}{7}$, $r = \frac{1}{4}$, $\delta = 5$, $\tau = \frac{1}{4}$, $\gamma = \frac{1}{15}$, $\mu = \frac{1}{4}$, $\xi = \frac{1}{4}$, $\beta = 3$, $q = \frac{4}{5}$, $\kappa = \frac{1}{4}$; *and direct calculation yields* $\Theta_1 \approx 0.416323$, $\Theta_2 \approx 1.584805$, $\Theta_3 \approx -0.0625198$, $\Psi_0 \approx 26.734102$ *and* $\Psi \approx 157.936$. *Now, define the function* $K : J \times \mathbb{R} \longrightarrow P(\mathbb{R})$ *by*

$$K(t, x) = \begin{cases} \left[-1, t^2 + \frac{\sin x}{500}\right], & \text{if } x \in (-\infty, 0) \\ \left[0, \frac{t^2}{2} + \frac{\cos x}{500}\right], & \text{if } x \in [0, \infty). \end{cases}$$

Recalling the fact that every interval and open (closed) subset of $\mathbb{R}$ *is measurable, it follows that* $K(t, x)$ *is measurable. Moreover, it is easy to see that* $K(t, x)$ *is lower semi-continuous and* $H(K(t, x), K(t, y)) \le 1/500\,|x - y| = \zeta|x - y|$, *for almost all* $t \in J$ *and* $x, y \in \mathbb{R}$. *Furthermore, notice that* $\Psi\zeta \le \eta$ *for* $\xi = 1/500$ *and* $\eta = 1/2$. *Hence, all the conditions of Theorem 4.7 are satisfied. Consequently, the BVP* (4.8) *has at least one solution in* $C(J, \mathbb{R})$.

5. CONCLUSION

In this work, the idea of soft set-valued maps in connection with Hausdorff fuzzy metric spaces on compact sets is initiated. To achieve this aim, the existence of e-soft fixed points of soft set-valued map defined on a complete fuzzy metric space is established by using Hausdorff fuzzy distance function. Following the fact that fixed point theorem of soft set-valued map is an extension of fixed point theorems of fuzzy and multivalued mappings, some fixed point results in the framework of single-valued and point-to-set valued mappings are derived as corollaries. Moreover, it is well-known that the existence (and uniqueness) of solution(s) of differential equations is obtained usually by methods which appeal to classical ordinary and partial differential equations. But, more general differential equations require specialized techniques of nonlinear functional analysis. This includes some peculiar notion endemic to the problem and identification of the appropriate solution and function spaces. From the latter point of view, the usability of our results are indicated by an application to mixed fractional differential inclusions comprising of both Riemann-Liouville and Caputo differential coefficients having nonlocal fractional integro-differential boundary conditions. A few nontrivial examples are further provided to support the hypotheses of our results. It is hoped that the presented results herein will motivate further studies of set-valued analysis and related fixed point theorems of multivalued operators as well as their applications in control theory, dynamical systems, game theory, image denoising, optimization theory, Hyers-Ulam-Rassias stability of nonlinear integro-differential equations, collocation-type method for integral inclusions, and so on. It is worthy of note that the ideas of this paper can be improved upon when further presented in the setting of fuzzy b-metric spaces, rectangular fuzzy metric spaces, intuitionistic fuzzy metric spaces, neutrosophic metric spaces, tripled fuzzy metric spaces and other quasi or pseudo-metric spaces.

Competing Interests

The authors declare that they have no competing interests.

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