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**Research Article**

# ON THE ASYMPTOTIC BEHAVIOR OF A GENERALIZED NONLINEAR EQUATION

**Juan E. NÁPOLES VALDÉS*[1], María Nubia QUEVEDO CUBILLOS[2], Adrian R. GÓMEZ PLATA[3]**

[1]*Unne, FaCENA. Av. Libertad 5450, Corrientes 3400; UTN-FRRE, FRENCH 414, Resistencia, Chaco 3500. ARGENTINA;* ORCID: 0000-0003-2470-1090
[2]*Universidad Militar Nueva Granada, Bógota D.C., COLOMBIA;* ORCID: 0000-0002-4137-7408
[3]*Universidad Militar Nueva Granada, Bógota D.C., COLOMBIA;* ORCID: 0000-0001-8047-5388

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**ABSTRACT**

In this article we present a second-order differential equation in the framework of the derivative N, and various qualitative properties of the solutions are studied, firstly conditions are obtained under which the equation under study has a non-continuity solution at infinity. Later we study the conditions for the prolongation of the solutions and their oscillation.
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## 1. PRELIMINARIES

Consider the following non-autonomous linear differential equation of the second order

$$\ddot{x}(t)+a(t)x(t) = 0, \tag{L}$$

where $a : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function with first continuous derivative. This equation describes the motion of a material point of unit mass under the action of restoring force with changing elasticity coefficient. Many qualitative properties of equation (L) have been studied for a long time and are classic problems of the theory of stability systems and ordinary nonautonomous differential equations. In particular, the following results are of great importance.

**Theorem 1** *Let a be a nondecreasing continuous function such that $a(0) \neq 0$ and $a(t)\to\infty$ as $t\to\infty$, Then all solutions of (L) are bounded.*

**Theorem 2** *Let $x = x(t)$ be a solution of (L) and let $\int_0^\infty t\,|a(t)|dt < \infty$. Then $\lim_{t\to\infty}\dot{x}(t)$ exists and further the general solution of (L) is asymptotic to $a_0+a_1t$, where $a_0$ and $a_1$ are constants simultaneously not equal to zero.*

Let us now consider a more general case than equation (L), i.e., consider the second order nonlinear differential equation

* Corresponding Author: e-mail: jnapoles@exa.unne.edu.ar, tel: +54-379-4265637

$$x^{\cdot\cdot}(t)+a(t)g(x(t)) = 0, \tag{N}$$

where $a(t) \in C[0,\infty)$, $g(t) \in C^1(-\infty,\infty)$, $g'(t) \geq 0$, and $xg(t) > 0$ for $x \neq 0$. The prototype of equation (N) is the so-called Emden-Fowler equation (see [8] and [10]):

$$x^{\cdot\cdot}(t)+a(t)|x(t)|^{\gamma}\,\mathrm{sgn}x(t) = 0,\ \gamma > 0,$$

which is used in mathematical physics, theoretical physics, and chemical physics. The above equation has interesting ma thematical and physical properties, and it has been investigated from various points of view, in particular, the solutions of this equation represent the Newton–Poisson gravitational potential of stars, such as the Sun, considered as spheres filled with polytropic gas. Other interesting results can be found in [29], [30], [31], [32] and the references cited in these.

The concept of rate of change in any function versus change in the independent variables was defined as derivative, first of an integer order, and this concept attracted many scientists and mathematicians such as Newton, L'Hospital, Leibniz, Abel, Euler, Riemann, etc. Later, several types of fractional derivatives, what will we denote $D^{\alpha}$, have been introduced to date Euler, Riemann–Liouville, Abel, Fourier, Caputo, Hadamard, Grunwald–Letnikov, Miller–Ross, Riesz among others, extended the derivative concept to fractional order derivative (see [18], [24] and [25]). Most of these derivatives are defined on the basis of the corresponding fractional integral in the Riemann–Liouville sense and, based on this, they are called global fractional derivatives or with memory.

To date, the study of this area has attracted the attention of many researchers, not only in Pure Mathematics, but in multiple fields of applied science. Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in recent years, in such a way that a single definition of "fractional derivative or integral" does not exist, or at least is not unanimously accepted, in [3] suggests and justifies the idea of a fairly complete classification of the known operators of the Fractional Calculus, on the other hand, in the work [2] the authors study this phenomenon and support the appearance of various operators, both in theoretical and practical research. Let us point out that these developments have been obtained in different contexts, and not with a single starting point, that is, they are taken as a basis, from the Riemann-Liouville integral, to that of Katugampola, through other formulations such as Weyl's, Hadamard, or Erdelyi-Kober, in this way various authors have defined different integral operators, even from different notions of generalized local derivatives, this last point of view is the one present in our work.

We believe that it is convenient to take into account the historical route that is presented in Chapter 1 of [1] where a historical route of differential operators, whether local or global, is made, starting from Newton's classical formulation and arriving at Caputo's Definition, which serves as the basis for presenting a differential operator, with a new parameter, and providing a great variety of applications, taking into account the difference between both types of differential operators, global and local. A seminal question is addressed in 1.5.1 (p.24), where sentence "We can therefore conclude that both the Riemann – Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [33] that, the local fractional operator is not a fractional derivative" (p.24). For all the above, we can affirm that there is a great variety of integral operators, which have proven their usefulness in solving a great variety of applications and in successive theoretical developments.

As an attempt to overcome these difficulties, Khalil et al. [16], came up with an interesting idea that extends the familiar limit definition of the derivative and allows to introduce successfully a conformable local fractional derivative, more recently, a non-conformable local derivative is introduced in [11] (see also [21]). In this way, a new direction in fractional calculus was opened, which has shown to be interesting from a theoretical viewpoint and useful in the applications. In the spirit of completeness, we remember the definition of the derivative N because it will be basic in our work.

In [22] (see also [35]) a generalized derivative was defined in the following way.

**Definition 3** *Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order $\alpha$ is defined by*

$$N_F^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon F(t,\alpha)) - f(t)}{\varepsilon} \tag{1}$$

*for all $t > 0$, $\alpha \in (0,1)$ being $F(\alpha, t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a);Re(b) > 0$. Also we consider $E_{a,b}(t^{-\alpha})k$ is the k-th term of $E_{a,b}(.)$.*

*If f is $\alpha$-differentiable in some $0 < \alpha \le 1$, and $\lim_{t \to 0^+} N_F^{(\alpha)} f(t)$ exists, then define $N_F^{(\alpha)} f(0) = \lim_{t \to 0^+} N_F^{(\alpha)} f(t)$.*

The function $E_{a,b}(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential function. This generalization was studied by Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953, and others.

Examples. Let's see some particular cases that provide us with new local derivatives.

1) Let $F(t, \alpha) = E_{1,1}(t^{-\alpha})$. In this case we obtain, from Definition 3, the non-conformable derivative $N^{\alpha}_1 f(t)$ defined in [11] (see also [21]).

2) Be now $F(t, \alpha) = E_{1,1}(t^{-\alpha})_1$, in this case we have $F(t, \alpha) = 1/t^{\alpha}$ , a new non-conformable derivative with a remarkable property
$\lim_{t\to\infty} N^{\alpha}_2 f(t) = 0$, i.e., the derived N is annulled to infinity (see [23, 19]).

3) If we now take $F(t, \alpha) = E_{1,1}((1 - \alpha)t) = e(^{1-\alpha})_t$ , we have the conformable derivative used in [9].

4) $F(t, \alpha) = E_{1,1}(t^{1-\alpha})_1 = t^{1-\alpha}$ with this kernel we have $F(t, a) \to 0$ as $\alpha \to 1$ (see [16]), a conformable derivative.

5) $F(t, \alpha) = E_{1,1}(t^{\alpha})_1 = t^{\alpha}$ with this kernel we have $F(t, \alpha) \to x$ as $\alpha \to 1$ (see [23]). It is clear that since it is a non-conformable derivative, the results will differ from those obtained previously, which enhances the study of these cases.

6) Using the Robotov's Function, that is to say

$$F(t,\alpha) = R_{\alpha}(\beta,t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma(1+\alpha)(k+1)} = t^{\alpha} E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1})$$

like before, $E_{\alpha+1,\alpha+1}(.)$ is the Mittag-Leffler two-parameter function, we can obtain a non-conformable derivative (see [34]).

If the N-derivative of the function $x(t)$ of order $\alpha$ exists and is finite in $(t_0,\infty)$, we will say that $x(t)$ is N-differentiable in $I = (t_0,\infty)$.

**Remark 4** *The N-derivative solves almost all the insufficiencies that are indicated to the classical fractional derivatives. In particular, if $F(t, \alpha) = E_{1,1}(t^{-\alpha}) = e^{t-\alpha}$, we have the following result.*

**Theorem 5** *(See [11]) Let f and g be N-differentiable at a point $t > 0$ and $\alpha \in (0,1]$. Then*

*a) $N^{\alpha}_1 (af + bg)(t) = a N^{\alpha}_1 (f)(t) + b N^{\alpha}_1 (g)(t)$.*

*b) $N^{\alpha}_1 (t^p) = e^{t-\alpha} pt^{p-1}$, $p \in \mathbb{R}$.*

*c) $N^{\alpha}_1 (\lambda) = 0$, $\lambda \in \mathbb{R}$.*

*d) $N^{\alpha}_1 (fg)(t) = f N^{\alpha}_1 (g)(t) + g N^{\alpha}_1 (f)(t)$.*

*e)* $$N_1^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{gN_1^{\alpha}(f)(t) - fN_1^{\alpha}(g)(t)}{g^2(t)}.$$

*f) If, in addition, f is differentiable then $N^{\alpha}_1 (f) = e^{t-\alpha} f'(t)$:*

*g) Being f differentiable and* $\alpha = n$ *integer, we have* $N^{\alpha}_1(f)(t) = e^{t-n} f'(t)$.

**Remark 6** *The relations a), c), d) and (e) are similar to the classical results mathematical analysis, these relationships are not established (or do not occur) for fractional derivatives of global character (see [17] and [25] and bibliography there). The relation c) is maintained for the fractional derivative of Caputo. Cases c), f) and g) are typical of this non conformable local fractional derivative.*

These results for the derivative $N^{\alpha}_1$ can be extended without difficulty to the case of the generalized derivative $N^{\alpha}_F$. The next result will be used later (see[12]).

**Theorem 7** *Let* $a > 0$ *and* $f : [a,b] \to \mathbb{R}$ *be a given function that satisfies:*

*i) f is continuous on* $[a,b]$,
*ii) f is N-differentiable for some* $\alpha \in (0;1)$.

*Then, we have that if* $N^{\alpha}_1 f(t) \geq 0$ $(\leq 0)$ *then f is a non-decreasing (increasing) function.*

Now we will present the equivalent result, for $N^{\alpha}_1$, of the well-known chain rule of classic calculus and that is basic in the Second Method of Lyapunov, for the study of stability of perturbed motion (see [11]).

**Theorem 8** *Let* $\alpha \in (0,1]$, *g N-differentiable at* $t > 0$ *and f differentiable at g(t) then* $N^{\alpha}_1(f \circ g)(t) = f'(g(t))\, N^{\alpha}_1 g(t)$.

Now, we give the definition of a general fractional integral (in [14] was studied in detail, see also [35]). Throughout the work we will consider that the integral operator kernel *F* defined below is an absolutely continuous function.

Let *I* be an interval $I \subseteq \mathbb{R}$, $a, t \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J^{\alpha}_{F,a}$, right and left, is defined for every locally integrable function *f* on *I* as

$$J^{\alpha}_{F,a+}(f)(t) = \int_a^t \frac{f(s)}{F(s,\alpha)} ds, t > a. \tag{2}$$

$$J^{\alpha}_{F,b-}(f)(t) = \int_t^b \frac{f(s)}{F(s,\alpha)} ds, b < t. \tag{3}$$

**Remark 9** *It is easy to see that the case of the* $J^{\alpha}_F$ *operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases listed at the beginning of the work. So, we have*

*1) If* $G(t,\alpha) = t1^{-\alpha}$, $F(t,\alpha) = \Gamma(\alpha)G(x-t,\alpha)$, *from* (2) *we have the right side Riemann-Liouville fractional integrals* $(R^{\alpha}_{\alpha+} f)(t)$, *similarly from (3) we obtain the left derivative of Riemann-Liouville. Then its corresponding right differential operator is*

$$({}^{RL}D^{\alpha}_{a^+} f)(t) = \frac{d}{dt}(R^{1-\alpha}_{a+} f)(t),$$

*analogously we obtain the left.*

*2) With* $G(t,\alpha) = t^{1-\alpha}$, $F(t-x,\alpha) = \Gamma(\alpha)G(\ln t - \ln x, \alpha)t$, *we obtain the right Hadamard integral from (2), the left Hadamard integral is obtained similarly from* (3). *The right derivative is*

$$({}^{H}D^{\alpha}_{a^+} f)(t) = t\frac{d}{dt}(H^{1-\alpha}_{a+} f)(t),$$

*in a similar way we can obtain the left.*

*3) The right Katugampola integral is obtained from* (2) *making*

$$G(t,\alpha)=t^{1-\alpha}, \quad e(t)=t^{\rho}, \quad F(t,\alpha)=\frac{\Gamma(\alpha)}{G(\rho,\alpha)}\frac{G(e(t)-e(x),\alpha)}{e'(t)},$$

*analogously for the integral left fractional. In this case, the right derivative is*

$$({}^{K}D_{a^+}^{\alpha,\rho}f)(t)=t^{1-\rho}\frac{d}{dt}K_{a^+}^{1-\alpha,\rho}f(t)=F(t,\rho)\frac{d}{dt}K_{a^+}^{1-\alpha,\rho}f(t),$$

*and we can obtain the left derivative in the same way.*

*4) The solution of equation* $(-\Delta)^{-\alpha/2}\phi(u)=-f(u)$ *called Riesz potential, is given by the expression* $\phi=C_n^{\alpha}\int_{R^n}\frac{f(v)}{|u-v|^{n-\alpha}}dv,$ *where* $C^{\alpha}_n$ *is a constant (see [5, 15, 20]). Obviously, this solution can be expressed in terms of the operator* (2) *very easily.*

*5) Obviously, we can define the lateral derivative operators (right and left) in the case of our generalized derivative, for this it is sufficient to consider them from the corresponding integral operator. To do this, just make use of the fact that if f is differentiable, then* $N^{\alpha}_{F}f(t)=F(t,\alpha)f'(t)$ *where* $f'(t)$ *is the ordinary derivative. For the right derivative we have* $\left(N_{F,a+}^{\alpha}f\right)(t)=N_F^{\alpha}\left[J_{F,a+}^{\alpha}(f)(t)\right]=\frac{d}{dx}\left[J_{F,a+}^{\alpha}(f)(t)\right]F(t,\alpha),$ *similarly to the left.*

*6) It is clear then, that from our definition, new extensions and generalizations of known integral operators can be defined.*

*7) We can define the function space* $L^{p}_{\alpha}[a,b]$ *as the set of functions over* $[a,b]$ *such that* $(J^{\alpha}_{F,a+}[f(t)]^{p}(b))<+\infty$.

An important detail that we want to point out is the fact that generalized integral operators, corresponding to local derivatives, can also be obtained from the (2) and (3) operators. For example, the non-conformable integral operator can be defined this way (see [12]).

**Definition 10** *The non-conformable fractional integral of order a is defined by the expression* ${}_{N}J_{t_0}^{\alpha}f(t)=\int_{t_0}^{t}\frac{f(s)}{e^{s^{-\alpha}}}ds.$

The following statement is analogous to the one known from the Ordinary Calculus (see [12]).

**Theorem 11** L*et f be N-differentiable function in* $(t_0,\infty)$ *with* $\alpha\in(0,1]$. *Then for all* $t>t_0$ *we have*

*a) If f is differentiable, then* ${}_{N}J^{\alpha}_{t0}(N^{\alpha}_{1}f(t)=f(t)-f(t_0)$.

b) $N^{\alpha}_{1}(f\,{}_{N}J^{\alpha}_{t0}f(t))=f(t)$.

Proof.

a) From definition we have

$${}_{N}J_{t_0}^{\alpha}\left(N_1^{\alpha}f(t)\right)=\int_{t_0}^{t}\frac{N_1^{\alpha}f(s)}{e^{s^{-\alpha}}}ds=\int_{t_0}^{t}\frac{f'(s)e^{s^{-\alpha}}}{e^{s^{-\alpha}}}ds=f(t)-f(t_0).$$

b) Analogously we have

$$N_1^{\alpha}\left({}_{N}J_{t_0}^{\alpha}f(t)\right)=e^{t^{-\alpha}}\frac{d}{dt}\left[\int_{t_0}^{t}\frac{f(s)}{e^{s^{-\alpha}}}ds\right]=f(t).$$

■

**Remark 12** *From the definition of the integral operator J, it is easy to deduce that if f is differentiable* ${}_NJ^{\alpha}_a f(b) = {}_NJ^{\alpha}_a f(t) - {}_NJ^{\alpha}_b f(t)$.

It is clear that many "classical" properties of integration theory can be proved without much difficulty. For example, we can prove the well-known Mean Value Theorems for Integral Calculus (see [14]).

**Theorem 13** *If f is continuous on* $[a,b]$, $0 < a < b$, *there exists a value c on the interval (a,b) such that* ${}_NJ^{\alpha}_a f(b) = f(c)(b-a)$.

**Theorem 14** *If f is continuous on* $[a,b]$, *and g is an integrable function that does not change sign on* $[a,b]$, *then there exists a value c on the interval (a,b) such that* ${}_NJ^{\alpha}_a f g(b) = f(c)\,{}_NJ^{\alpha}_a g(b)$.

In our work, we are interested in studying the continuability and oscillation of the solutions of the following equation, under suitable assumptions on functions $a$ and $g$:

$$N^{\alpha}_F (N^{a}_F x(t)) + a(t)g(x(t)) = 0. \tag{4}$$

The coefficient $a(t)$ is allowed to be negative for arbitrarily large values of $t$. Under this premise, in general not every solution to the second order nonlinear differential equation (N) is continuable throughout the entire half real axis. For this reason, and as a natural generalization of the ordinary case, we confine ourselves to those solutions of (4) that exist and can be continued on some interval of the form $[t_0, +\infty)$, where $t_0 > 0$ may depend on the particular solution. A solution $x(t)$ is said to be oscillatory if it has arbitrarily large zeros, the equation (4) is called oscillatory if all continuable solutions are oscillatory. Here we are concerned with sufficient conditions on a(t) so that all solutions of (4) are oscillatory.

## 2. RESULTS

Next to equation (4), we will consider the following equivalent system:

$$N^{a}_F x(t) = y(t),$$
$$N^{a}_F y(t) = a(t)g(x(t)), \tag{5}$$

with $a \in C[0,+\infty)$, $g \in C(R)$, $xg(x) > 0$ if $x \neq 0$ and $G(t) = {}_NJ^{\alpha}_0\, g(s)ds$. Thus we have the following results, the first concerning the non-prolongability of the solutions, of vital importance in the qualitative theory. Later the following functions will be used.

$$b(t) = e^{-{}_NJ^{\alpha}_0\left[\frac{N^{\alpha}_F a(s)_-}{a(s)}\right](t)}, \tag{6}$$

$$c(t) = a(0)e^{-{}_NJ^{\alpha}_0\left[\frac{N^{\alpha}_F a(s)_+}{a(s)}\right](t)}, \tag{7}$$

so that

$$a(t) = b(t)c(t), \tag{8}$$

where $b(t)$ is non-increasing and $c(t)$ is non-decreasing function with $N^{\alpha}_F\, a(t)_+ = \max(N^{\alpha}_F\, a(t), 0)$ and $N^{\alpha}_F\, a(t)\_ = \max(-N^{\alpha}_F\, a(t), 0)$, so that $N^{\alpha}_F\, a(t) = N^{\alpha}_F\, a(t)_+ - N^{\alpha}_F\, a(t)_-$.

### 2.1. On the continuability of solutions

**Theorem 15** *Suppose in (4)* $a(t_1) < 0$ *for some* $t_1 > 0$. *If either*

a) ${}_NJ^{\alpha}{}_0\,(1+G(u))^{-1/2}(+\infty) < \infty$, *or*
b) ${}_NJ^{\alpha}{}_0\,(1+G(u))^{-1/2}(-\infty) > -\infty$,

*Then (4) has solution x(t) which is not continuable to* $+\infty$*:*

Proof. As a consequence of the continuity of a and of the fact that $a(t1) < 0$, there are positive numbers $\delta$, $M$ and $m$ such that $-M \le a(t) \le -m$, if $t_1 \le t \le t_1 + \delta$. Assume that condition a) holds and let $(t(t), y(t))$ be a solution of (5) satisfying $x(t_1) = 0$ with $y(t_1)$ large and to be determined. From (5) we have both $x(t)$ and $y(t)$ are monotonically increasing while $(x(t), y(t))$ is defined on $t_1 \le t \le t_1 + \delta$. Integrating the second equation of (5) and taken into account the Theorem 14 we have

$$y(t)N_F^{\alpha}y(t) = a(t)g(x(t))N_F^{\alpha}x(t), \quad t_1 \le t \le t_1 + \delta$$

$$y(t) = \sqrt{y^2(t_1) - 2a(\xi)G(x(t))}$$

and so

$$\sqrt{y^2(t_1) + 2mG(x(t))} \le y(t) \le \sqrt{y^2(t_1) + 2MG(x(t))}.$$

Since $N^{\alpha}_F\, x(t) = y(t)$ we have, from the left hand side of the above

$${}_NJ_0^{\alpha}(y^2(t_1) + 2mG(s))^{-1/2}(+\infty) \ge k(t - t_1), \quad k \quad \textit{being} \quad \textit{some} \quad \textit{constant}.$$

Since a) holds, we may choose $y^2(t_1)$ so large that the integral is smaller than $\delta$. It the follows that $x(t) \to \infty$ before $t$ reaches $t_1 + \delta$. Let $\varepsilon > 0$ be given, by a) there exist $T > 0$ such that

$${}_NJ_F^{\alpha}(1 + G(s))^{-1/2}(+\infty) < \varepsilon.$$

Write $NF = \left(\frac{y^2(t_1)}{2m}\right) - 1$ and agree that $y(t_1)$ will be taken so large that $N_F > 0$, then

$${}_NJ_0^{\alpha}(y^2(t_1) + 2mG(s))^{-1/2}(+\infty) = \frac{1}{\sqrt{2m}}\left\{{}_NJ_0^{\alpha}((N_F + 1) + 2mG(s))^{-1/2}\right\} =$$

$$= \quad \frac{1}{\sqrt{2m}}\left\{{}_NJ_0^{\alpha}[N_F + 1 + G(s)]^{-1/2}(T) + {}_NJ_X^{\alpha}([N_F + 1 + G(s)]^{-1/2}(+\infty)\right\} <$$

$$< \quad \frac{1}{\sqrt{2m}}\left\{{}_NJ_0^{\alpha}[N_F + 1 + G(s)]^{-1/2}(T) + \varepsilon\right\}.$$

Since $T$ is fixed, we may take $y(t_1)$ so large that the integral is smaller than $\varepsilon$. From here we have the desired result, choosing $\varepsilon$ such that $\frac{2\varepsilon}{\sqrt{2m}} < \delta$. This completes the proof in case a), the case b) is proved in a similar way working in the quadrant III of the $xy$-plane. ■

**Remark 16** *It is easy to check the scope of the previous result, considering the equation* $N^{\alpha}_F\,(N^{\alpha}_F\, x(t)) - \alpha t^{-(1+\alpha)}e^{2t-a}\, x(t) = 0$, *does not satisfy conditions a) and b) of the Theorem and which has the non-oscillatory solution* $x(t) = e^{2t}$.

It is known that the only way in which a solution $(x(t), y(t))$ of (5) can fail to be defined past some $T$ is if $\lim_{t \to T^-} (x^2(t) + y^2(t)) = +\infty$ (cf. [6], p.61). It is easily shown that if $a(t)$ is continuous and non negative for all $t \geq 0$, then there is no $T$ for any solution of (5) satisfying $\lim_{t\to T^-} x^2(t) = +\infty$. Thus, the only way in which it is possible for solutions to behave as in the proof of Theorem 1 is for $a(t_1) < 0$ for some $t_1$.

**Theorem 17** *Let $a(t)$ be continuous and satisfy $a(t) < 0$ on an interval $t_1 \leq t < t_2$ with $a(t_2) \leq 0$. Then (5) has a solution $(x(t), y(t))$ defined for $t = t_1$ and satisfying $\lim_{t\to T^-} |x(t)| = +\infty$ for some $T$ satisfying $t_1 \leq T < t_2$ if and only if either a) or b) holds.*

**Proof.** Sufficiency. From Theorem 15 we have that such solution exist if $a(t_1) < 0$ and if a) or b) holds.

Necessity. We assume that such solution exist, in this way $(x(t); y(t))$ is defined on $[t_1,T)$, show that a) is true. Using the same considerations as in the above Theorem we have

$$\sqrt{y^2(t*) + 2m\{G(x(t)) - G(x(t*))\}} \leq y(t) \leq \sqrt{y^2(t*) + 2M\{G(x(t)) - G(x(t*))\}}.$$

for $t^* \leq t < T$. As $N^{\alpha}_F x(t) = y(t)$ we have

$$N^{\alpha}_F x(t) \leq \sqrt{y^2(t*) + 2M\{G(x(t)) - G(x(t*))\}},$$

from this we have

$$_N J^{\alpha}_{x(t*)} \left(y^2(t*) + 2M\{G(s) - G(x(t*))\}\right)^{-1/2}(x(t)) \leq (t - t*).$$

Now $G(x)$ is an increasing function for $x > 0$ and we know that $x(t)$ is increasing. Thus since $y^2(t^*) > 0$ the integrand is defined. Since $x(t) \to \infty$ as $t \to T$, we see that

$$_N J^{\alpha}_{x(t*)} \left(y^2(t*) + 2M\{G(s) - G(x(t*))\}\right)^{-1/2}(x(t)) < \infty.$$

Making $w(t^*) = \frac{y^2(x*)}{2M} - G(x(t*))$ we obtain

$$_N J^{\alpha}_{x(t*)} \left(w(t*) + G(s)\right)^{-1/2}(+\infty) < \infty.$$

If $w(t^*) \leq 1$, a) holds. Now suppose that $w(t^*) > 1$, then

$$\frac{1}{\sqrt{w(t*)}}\,{}_N J^{\alpha}_{x(t*)} \left(1 + \frac{G(s)}{w(t*)}\right)^{-1/2}(+\infty) > \frac{1}{\sqrt{w(t*)}}\,{}_N J^{\alpha}_{x(t*)} \left(1 + G(s)\right)^{-1/2}(+\infty).$$

Since the first integral converges, so does the second and hence a) holds.

If $x(t) \to -\infty$ as $t \to T^-$ then a similar proof may be carried out in quadrant III of the $xy$-plane showing that b) holds. This completes the proof. ■

Now we present a result equivalent to Theorem 1 for equation (4). This problem has received a considerable amount of attention during the past century, particularly when (L) is a nonlinear ordinary differential equation of type (N).

**Theorem 18** *Under assumption on function $g$ of Theorem 15 let a a continuous and positive function on* $[0, +\infty)$ *satisfying*

$$a(t) \to \infty, \text{ as } t \to \infty. \tag{9}$$

*Then all solutions of (5) can be defined for all $t \geq t_0 > 0$.*

**Proof.** We will develop an extension of Liapunov's Second Method in this proof. For this, we define the following functions.

$W(t,x(t), y(t)) = b(t)V(t,x(t), y(t))$ (10)

where $b(t)$ is defined by (6) and $V$ is given by

$$V(t,x(t),y(t)) = \frac{y^2}{2a(t)} + G(x), \tag{11}$$

where $G$ is as before. Then along solutions of system (5), we have

$$N_F^\alpha W(t,x(t),y(t)) = N_F^\alpha b(t)V(t,x(t),y(t)) + b(t)N_F^\alpha V(t,x(t),y(t))$$

and

$$N_F^\alpha V(t,x(t),y(t)) = -\frac{y^2}{2}\frac{N_F^\alpha a(t)}{a^2(t)}.$$

Using (6), (7) and (8) we obtain

$$N_F^\alpha W(t,x(t),y(t)) \leq 0, \tag{12}$$

so $W$ is non-increasing function. As we pointed out before, suppose such a $T$ exists for some solution of system (5), i.e. satisfying $\lim_{t\to T^-} |x(t)| = +\infty.$ . Now

$$b(T)\left[G(x(t)) + \frac{y^2(t)}{2M}\right] \leq W(t,x(t),y(t)) \leq W(t_0,x_0,y_0),$$

being $M = \max_{t\in[t0,T]} a(t)$. From this we have $|y(t)|$ is uniformly bounded, say $|y(t)| \leq K$ for $t_0 \leq t < T$. But $N^\alpha_F\, x(t) = y(t)$ so $|x(t)| \leq u_0+K(t - t_0) \leq x_0 + K(T - t_0)$. This completes the proof. ■

## 2.2. Oscillation

**Theorem 19** *Under assumptions $a(t) \in C[0, \infty)$, $g(x) \in C^1(-\infty, \infty)$, $g'(x) \geq 0$, and $xg(x) > 0$ for $x \neq 0$, suppose that*

$$_N J^\alpha_{x(t)}\left(\frac{1}{g(s)}\right)(+\infty) < \infty; \quad _N J^\alpha_{-x(t)}\left(\frac{1}{g(s)}\right)(-\infty) < \infty \tag{13}$$

*for every $x(t) > 0$, then any solution of equation (4) is either oscillatory or tends monotonically to zero as $t \to +\infty$.*

**Proof.** Suppose that $x(t)$ is a non-oscillatory solution of (4), that is, it is of constant sign from a certain value of $t$, $x(t) > 0$ for all $t \in [t_0,+\infty)$ being $t_0 > 0$. From (4) we have

$$\frac{N_F^\alpha\left(N_F^\alpha x(t)\right)}{g(x(t))}t = -a(t)t,$$

integrating by parts and taking into account that $g'(t) \geq 0$ we obtain

$$\frac{t\left(N_F^\alpha x(t)\right)}{g(x(t))} \leq_N J^\alpha_{x(t_0)}\left[\frac{1}{g(z)}\right]x(t) -_N J^\alpha_{t_0}(sa(s))(t) + c_0,$$

with $c_0 = \frac{t_0(N_F^\alpha x(t_0))}{g(t(t_0))}$. By (13) we have $\frac{t(N_F^\alpha x(t))}{g(t(t))} \to -\infty$, *as* $t \to +\infty$ this means that we can find some constant $k > 0$ such that

$$\frac{(N_F^\alpha x(t))}{g(x(t))} \leq -\frac{k}{t}$$

and from this we have

$$_N J_{x(t_0)}^{\alpha} \left(\frac{1}{g(s)}\right)(x(t)) \leq k ln\left(\frac{t_0}{t}\right).$$

The negativity of right hand side implies lim $\lim_{t \to +\infty} x(t) = 0.$ Thus, every solution of (4) is oscillatory or tends to zero monotonically.

■

## 3. CONCLUSIONS

In this paper, we study the oscillatory character of a generalized nonlinear equation of order $\alpha + \alpha$ using the analysis of the phase plane, in this way, extending methods used in the integer case, to the non-conformable equation (4). It is clear that the methods used can be used for other kernels of the Definition 3.

On the other hand, one might think that this study is designed for generalized differential equations only, however we want to conclude this work with a methodological observation.

Consider the following second order nonlinear ordinary differential equation (see [4])

$$(p(t)x'(t))' + a(t)f(x(t)) = 0, \tag{14}$$

where $p,a : [t_0, +\infty) \to (0, +\infty)$ are continuous and $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the known signum condition $f(x)x > 0$, for $x \neq 0$.

One of the most valued asymptotic properties is that of oscillation, two issues are central in this case: the existence of nonoscillatory solutions and the oscillation of all solutions. Taking into account the Definition 3 we can define the following generalized derivative, using the following function. Let $F(t, \alpha) = p(t, \alpha)$ such that $F(t, 1) = p(t)$ where $p(t)$ is the function involved in the equation (14). From this equation we easily obtain the following generalized equation

$$N_p^\alpha(N_p^\alpha x(t)) + q(t)g(x(t)) = 0, \tag{15}$$

with $q(t, \alpha) = a(t)p(t, \alpha)$. In this way, we can study the equation (14) with the help of the equation (15) and with the same techniques of this work (or following to [13]).

Of course, if we use other of the kernels indicated at the beginning of the work, we can study in a similar way, a great variety of problems that arise from various applications, for example, equations of type p-Laplacian as studied in [7].

We would like to point out finally, two important details. First, one of the criticisms of local generalized derivatives is based on various works (see mainly [26, 27]) because they satisfies the well-known Leibniz Rule of the product, at [22] we build local derivatives that violate this rule, which means this is no longer an essential condition for fractional derivatives. For example, taking $H(\varepsilon, \beta) = E_{1,1}(\varepsilon\beta)$ and so we have, from Definition 3

$$DE_{\beta}^{\alpha} f(t) := \lim_{\varepsilon \to 0} \frac{E_{1,1}(\varepsilon\beta) f(t + \varepsilon F(t,\alpha)) - f(t)}{\varepsilon},$$

and regarding our $N_1$-derivative of [11] it becomes:

$$NE_{\beta}^{\alpha} f(t) := \lim_{\varepsilon \to 0} \frac{E_{1,1}(\varepsilon\beta) f(t + \varepsilon e^{t^{-\alpha}}) - f(t)}{\varepsilon}. \tag{16}$$

From (16) we can easily obtain the following conclusions:

a) Is a local operator.
b) Is a no conformable fractional derivative.
c) It does not comply with Leibniz's rule.
d) If $\alpha = 0$, $\beta = 1$ then $NE^0_1 f(t) = N^0_1 f(t) + f(t) = (1+e) f(t)$.
e) If $\alpha = 1$, $\beta = 0$ then $NE^1_0 f(t) = N^1_1 f(t) = e^{t-1} f'(t)$, if f is derivable.
f) If we linearize the exponential in (16) and if the limit exists, then we have (writing $NL^{\alpha}_{\beta}$ instead of $NE^{\alpha}_{\beta}$)

$$NL_{\beta}^{\alpha} f(t) = N_1^{\alpha} f(t) + \beta f'(t). \tag{17}$$

g) Unfortunately, "we lose" the Chain Rule that was valid for our N-derivative (see [11]), so for $NL^{\alpha}_{\beta}$ we obtain:

$$NL_{\beta}^{\alpha}\left[f(g(t))\right] = N_1^{\alpha} f(g(t)) + \beta f(g(t)).$$

h) From (17) we derive that

$$\lim_{t\to\infty} NL_{\beta}^{\alpha} f(t) = \lim_{t\to\infty} N_1^{\alpha} f(t) + \lim_{t\to\infty} \beta f'(t) = f'(t) + \beta f(\infty).$$

Where we can draw the following: if the term $\beta f(\infty)$ exists, then the derivative $N^{\alpha}_{\beta} f(t)$ is only a "translation" of the derivative of the function when $t \to \infty$, so it does not affect the qualitative behavior of the ordinary derivative, this is of vital importance in the study of asymptotics properties of solutions of fractional differential equations with $NL^{\alpha}_{\beta}$. Unfortunately, the non-existence of the limit of the function to infinity makes the qualitative study of these fractional differential equations impossible.

i) Let's go back to the definition (2), it is clear that the function $H(\varepsilon, \beta)$ can be generalized although that would complicate the calculations extraordinarily. Of course this does not close the discussion on what terms can be "added" to the increased function that give local fractional derivatives that violate the Leibniz Rule, which would maintain the locality, as a historical inheritance of the derivative, and would default Leibniz's Rule, as a "necessary" condition so that there is a fractional derivative.

The second criticism is related to the local character of the generalized derivatives of Definition 3 (see [28]). From its very origins, the notion of derivative is a "local" notion, opposed to the globality of the integral, hence they are not inverse operators in the strict sense. It has always been referred to instants, points, specific magnitudes and not at intervals. The classical notions of fractional derivatives "forgot" this fact and built an operator that is not local, therefore, from its conception, the global fractional derivatives are not derivative, it is an operator of another nature. As we have said, it is impossible to compare them, so Tarasov's statements should be reformulated as follows: *"No locality. No differential operator"*.

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