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Research Article

THE LOCAL GENERALIZED DERIVATIVE AND MITTAG-LEFFLER FUNCTION

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ABSTRACT

In this paper, we present a general definition of a generalized derivative of local type using the well known Mittag-Leffler function. Some methodological remarks on the local fractional derivatives are also presented.
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1. PRELIMINARIES

We know that the ”birth” of the fractional calculation is practically the same as the ordinary calculation, in fact this date is known: September 30, 1695, however only until very recently, they were considered global fractional derivatives, the ones they use in their definition a certain integral, with which the locality of the classical derivative was not present. In addition to this “detail”, we know that these derivatives D^{α} have a set of insufficiencies which can be summarized in the following:

1. Almost all of these derivatives, except those of the Caputo type, they do not satisfy that the derivative of a constant is zero, if $\alpha \notin \mathbb{N}$.
2. All fractional derivatives do not satisfy the familiar Product Rule to calculate the derivative of product of two functions $D^{\alpha}(fg) = gD^{\alpha}(f) + fD^{\alpha}(g)$.
3. For these global derivatives, the rule of the derivative of a quotient for two functions is not satisfied, i.e. $D^{\alpha}\frac{f}{g} = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$ with $g \neq 0$.

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4. The know n Rule of the Chain of ordinary calculus, to calculate the derivative of a compound function, is not fulfilled for the global fractional derivatives, i.e. $D^{\alpha}(f \circ g)(t) = D^{\alpha}(f(g))D\alpha g(t)$.

5. A theoretical body, a "calculus" for these global derivatives has not been developed.

6. These global fractional derivatives do not always satisfy the law of exponents (semigroup property) $D^{\alpha}D^{\beta}(f) = D^{\alpha+\beta}(f)$.

7. It is known of the different existence and uniqueness theorems for systems of classical differential equations, that if these requirements are satisfied, two different solutions cannot be intercepted in a finite time, however, there are very simple (even linear) examples of fractional systems that exhibit up to self-interceptions.

The fractional calculation has attracted many investigations and researchers to this day. This fractional calculation has clearly impacted, both from a theoretical and practical point of view in many areas of science and technology (cf. [13], [14], [17] and [18]). Although derivatives began to appear from the 60s that we now call local fractionals, it is with Khalil's work that new derivatives are formalized using the limit of a certain incremental quotient. These derivatives have an additional value: they overcome almost all the difficulties mentioned above for the global ones (see [1], [8], [12], [16] and references cited therein). In this way, a new direction in fractional calculus was opened, which has shown to be interesting from a theoretical viewpoint and useful in the applications. In this paper we present a generalized local derivative, defined in terms of the Mittag-Leffler Function, which adds a generality that did not have the definitions precedent.

2. A NEW LOCAL GENERALIZED DERIVATIVE

Definition 1 *Be the function* $f : [0,+\infty)\to\mathbb{R}$. *The N-derivative of function f of order* α *is defined by*

$$N_F^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon F(t,\alpha)) - f(t)}{\varepsilon} \tag{1}$$

for all $t > 0$, $\alpha \in (0,1)$ *being* $F(\alpha, t)$ *is some function. Here we will use some cases of F defined in function of* $E_{a,b}(.)$ *the classic definition of Mittag-Leffler function with* $Re(a);Re(b) > 0$. *Also we consider* $E_{a,b}(.)_k$ *is the k-nth term of* $E_{a,b}(.)$.

If f is α*–differentiable in some* $(0,\alpha)$, *and* $\lim_{t\to 0^+} N_F^{(\alpha)} F f(t)$ *exists, then define* $N^{(\alpha)}{}_F f(0) = \lim_{t\to 0^+} N_F^{(\alpha)} f(t)$.

The function $E_{a,b}(z)$ was defined and studied by Mittag-Leffler in the year 1903. This Mittag-Leffler function is a generalization of the well-known exponential function, Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953 established definitely this generalization. Examples. Let's see some particular cases that provide us with new non-conforming derivatives.

1. Mellin-Ross Function. In this case we have

$$E_t(\alpha,a) = t^{\alpha} E_{1,\alpha+1}(at) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\alpha+k+1)}$$

with $E_{1,\alpha+1}(.)$ the Mittag-Leffler two-parameter function. So, we obtain $\lim_{\alpha\to 1} N^{\alpha}_{E_t(\alpha,a)} f(_t) = f'(t)tE_{1,2}(at)$, i.e.,

$$N^1_{E_t(1,a)} f(t) = f'(t)t \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+2)}.$$

2. Robotov's Functio n. That is to say

$$R_\alpha(\beta,t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma(1+\alpha)(k+1)} = t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1})$$

like before, $E_{\alpha+1,\alpha+1}(.)$ is the Mittag-Leffler two-parameter function. Now, we obtain $\lim_{\alpha\to 1} N^\alpha_{R_\alpha(\beta,t)} f(t)$ $= f'(t)tE_{2,2}(\beta t^2)$ and

$$N^1_{R_1(\beta,t)} f(t) = \frac{f'(t)t}{\Gamma(2)} \sum_{k=0}^{\infty} \frac{\beta^k t^{2k}}{(k+1)}.$$

3. Let $F(t,\alpha) = E_{1,1}(t-\alpha)$. In this case we obtain, from Definition 1, the derivative $N^\alpha_1 f(t)$ defined in [8] (and [16]).

4. Be now $F(t,\alpha) = E_{1,1}(t^{-\alpha})1$, in this case we have $F(t,\alpha) = 1t_\alpha$, a new derivative with a remarkable propertie $\lim_{t\to\infty} N^\alpha_1 f(t)$ $= 0$, i.e., the derived N is annulled to infinity.

5. If we now take the development of function E to order 1, we have $E_{a,b}(t^{-\alpha}) = 1 + 1 / t^\alpha$. Then $\lim_{t\to\infty} N^\alpha_F f(t) = \lim_{t\to\infty} N^\alpha_1 f(t) = f'(t)$, in this case we have a derivative.

Remark 2 *It is easy to check but tedious, following for example, that the general derivative fulfills properties very similar to those known from the classical calculus. As well as its most important consequences, among them the Chain Rule, of vital importance in many applications, among them the Second Method of Liapunov.*

Following the ideas of the work [8] (see Theorem 3) we can easily prove the following result.

Theorem 3 *Let f and g be N-differentiable at a point* $t > 0$ *and* $\alpha \in (0,1]$. *Then*

a) $N^\alpha_F(af+bg)(t) = aN^\alpha_F(f)(t)+bN^\alpha_F(g)(t)$.

b) $N^\alpha_F(t^p) = e^{t-\alpha} pt^{t-1}$, $p \in \mathbb{R}$.

c) $N^\alpha_F(\lambda) = 0$, $\lambda \in \mathbb{R}$.

d) $N^\alpha_F(fg)(t) = fN^\alpha_F(g)(t)+gN^\alpha_F(f)(t)$.

e) $N^\alpha_F(f/g)(t) = \frac{gN^\alpha_F(f)(t) - fN^\alpha_F(g)(t)}{g^2(t)}$.

f) If, in addition, f is differentiable then $N^\alpha_F(f) = F(t,\alpha) f'(t)$.

3. ON THE TARASOV'S AFFIRMATIONS

Regarding the works of Tarasov (see [20], [21] and [22]), I would like to make some reflections on the concept of fractional derivative, for this I would like to go back to the genesis of calculus.

Many historians claim that the genesis of Calculus can be traced back to Greek mathematics and its first logicians, probably Zeno, around 450 BC, when posing the Achilles paradox and the Turtle, however that infinite, convergent sum was out of reach of Greek mathematicians with the use of potential and non-actual infinity. Later, about 370 BC Eudoxio, when formulating the method of exhaustion quite solidly, allowed Archimedes (287-212 BC) to use it in determining the areas and volumes of many geometric figures such as circle, sphere, cone, etc. These works configured what is known today as integration: the whole from the infinitesimal. We have to wait several centuries until in the seventeenth century there are several original approaches to develop and give a proper path to these advances, the rediscovery of Greek mathematics reached its peak at the dawn of this century. Translations and reissues multiplied; In addition, the numerous attempts to reconstitute lost or altered works reveal the new scope that mathematical culture reached at that time. Kepler (1615), Galileo, Cavalieri in 1635 (with his method of indivisibles), in 1637 Descartes and his Analytical Geometry, Fermat and his method to determine maximums and minimums and tangents to curves, which was succeeded by Isaac Barrow's in 1669 and the wave of creativity finally found its peak in Leibniz and Newton, who independently consolidated what has been called today as Calculus.

In various resources, physical and virtual, there is a lot of information about this whole process, however in order to achieve a clearer exposition of our position regarding Tarasov's claims, we would like to present some outstanding milestones of this development (see [10]):

1. Fermat investigated the problem of maximums and minimums, in the modern conception, taking the tangent to the curve at these points with zero slope, that is, parallel to the x-axis, which made Lagrange consider Fermat as the true creator of the calculus.

2. Similar ideas about the conception of the derivative, are found in the works of Hudde and Barrow, the latter considered an interesting problem, that of variable speed and obtaining the derivative of distance. Barrow as such had been on the trail of the fundamental relation between integration and differentiation when Newton arrived on the scene. Finding the tangent to a curve is a known problem that has been studied by many mathematicians since Archimedes, however the first method that can be called ”modern” is Gilles Personne de Roberval during the 30s and 40s of the 17th century. Pierre of Fermat, almost simultaneously, used his own ideas to find the tangent to a curve, although it was not until Leibniz and Newton that they rigorously defined their tangent method.

3. Both Leibniz and Newton developed the process in geometric terms, rather than analytical. Newton considered that the variables are variable over time and limited himself to the creation of a geometric technique to present his own physical discoveries while Leibniz thinks of the variables as infinitely small increments dx and dy and chose to develop it as an analytical tool with appropriate notations.

4. Leibniz is the one who uses the notation for the first time $f(x)dx$, the 21 November 1675. Newton had been using his method of fluxions to deal with change and motion since1665 but he published the ideas only in 1687 – three years after the publication of seminal Leibniz's paper, **"A new method for maxima and minima as well as tangents. . . and a curious type of calculus for it"** of Acta Eruditorium.

5. Newton combined the ideas of the Greeks and the analytic Geometry of Descartes and conceived geometrical figures as 'fluents' evolving from the continuous motion of a point or line and the velocity of the moving point or line became the fluxion of the fluent.Newton developed the ideas of Fermat and Barrow and culminated in the definition of the derivative in terms of a limit (the rate of change). This limit is what we know today as the derivative or the instantaneous slope of the curve at the precise place where the two points merge: the derivative and with respect to ax, while the integration process considered the inverse process and Newton used it to raise and solve their known laws of movement and gravitation.

6. Apart from the deductions obtained by differentiating the Keplers 2nd law, the new operator yielded a great insight to Newton that the constant factor in various processes of nature is the reason according to which the rate of change changes. As for example if we consider the equation of free fall $y = f(t)$, $\acute{y} = d f(t) / dt$ and $\ddot{y} = d^2 f(t) / dt^2$. i.e in the case of $f(t) = at^2 + bt + c$ the rate of increase in the speed of a falling body is $(2at + b)$ m/s.

7. For the creators of Calculus, the problem of the integration of differential equations, at the beginning, was presented as part of a more general problem: the inverse problem of infinitesimal analysis (the integration). Naturally, attention was initially focused on the different first-order equations. Its solution was sought in the form of algebraic or elementary transcendent functions,

with the help of more or less successfully chosen methods. To reduce this problem to the search operation of primitive functions, the creators of the analysis and their disciples tended in each differential equation to separate the variables. This method, with which the systematic texts of the theory of differential equations currently begin, was, apparently, historically the first. Finally, it is convenient to highlight some of the most important characteristics with which we abandoned the historical moment of Newton-Leibniz: at this time the problems were still tackled with a geometric-Euclidean vision. In this sense, obviously the concept of tangent was the Euclidean. In Leibniz there is a different but ambiguous element of conceiving the tangent line as one that joins two infinitely close points in any case, the notion that was handled on a tangent line was clearly intuitive.

Let's go back to point 2.

Seeing the historical development of the calculation of the tangent to a curve at a point, we have arrived at the current geometric interpretation of the notion of derivative: the tangent to a given curve at an arbitrary P point is nothing other than the secant limit through P and another point Q of the curve, when Q converges to P. It is clear that this definition, rests on the notion of limit, and interprets our idea of what is the line tangent to a curve. Leibniz's original publication in 1684 encountered the difficulty if a definite value could be assigned to this ratio of indefinitely small increments. This difficulty persisted until 1696, the year in which L'Hopital began writing *d/dx*: giving the conclusion sought, an indefinitely small increase divided by a similar one may have a definite value, that of a boundary position towards which it moves closer and closer. In the eighteenth century there were several objections to Newton's method of fluxions, basically because the elements used by him, first and last reasons, limits, derivatives and differentials were not clearly defined, we can consider that the reasons used are very similar to those used for Zeon more than 20 centuries before they rested, in turn, on the use of infinity and continuity.

Consider the following classical formula for $x = f(t)$:

$$\frac{dx}{dt} = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon}.$$

We know that this is the classic definition of the notion of derivative used in classical calculus. This notion measures the reason at which a certain variable quantity changes. For example, if we consider a mobile that travels at a certain speed, this is only that derived from the position with respect to time, so if the mobile travels 45 kilometers per hour, we say that the mobile has changed its position 45 kilometers. Of course, in many processes or phenomena studied by science, she is interested in determining how they change, obviously the derivative and the integral are the central tools in these studies.

It is clear that the first derivative presented above, leads us to the notion of derivative of order n in the following way [†]:

$$\frac{d^n f(t)}{dt^n} = \frac{d}{dt}\left(\frac{d^{n-1} f(t)}{dt^{n-1}}\right).$$

Fractional calculus was introduced over 300 years ago. When Leibniz wrote a letter to L'Hopital, raising the possibility of generalizing the meaning of derivatives from integer order to noninteger order derivatives. L'Hopital wanted to know the result for the derivative of order n = 1/2. Leibniz replied that *"one day, useful consequences will be drawn"* and, in fact, his vision became a reality[‡].

[†]A general definition of the derivative of order n of a function in a point, originates discussions even today, the diverse approximations to the subject are insufficient. We do not recommend [15] for readers interested in going deeper into the subject.

[‡]Consult [19]

Therefore, from its very origins, the notion of derivative is a “local” notion, opposed to the globality of the integral, hence they are not inverse operators in the strict sense. It has always been referred to instants, points, specific magnitudes and not at intervals. The classical notions of fractional derivatives ”forgot” this fact and built an operator that is not local, therefore, from its conception, the classical fractional derivatives are “not derivatives”, it is an operator of another nature. As we have said, it is impossible to compare them, so Tarasov’s statements should be reformulated as follows: *“No nonlocality. No derivative operator”*.

However, a new local derivative that violates Leibniz’s Rule can be constructed, so the violation of this rule cannot be a necessary condition for a given operator to be a fractional derivative, let’s go back to (1). It is clear that the violation of this rule does not depend (at least not only) on the incremental quotient, but on a factor that we can add to the increased function, from which the non-symmetry of the product rule would be obtained.

Taking into account [23] we can write from (1) the following derivative ($\alpha+\beta=1$):

$$DH_{\beta}^{\alpha} f(t) := \lim_{\varepsilon\to 0}\frac{H(\varepsilon,\beta)f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon} \tag{2}$$

with $H(\varepsilon,\beta)\to k$ if $\varepsilon\to 0$. In the case that $k\equiv 1$, we can consider two simple cases:

I) $H(\varepsilon,\beta)=1+\varepsilon\beta$ as in [23] and so

$$DL_{\beta}^{\alpha} f(t) := \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon}.$$

If $F(t,\alpha)=e^{t-\alpha}$, that is, a generalization of the local fractional derivative presented in example 4 above. In this case we have:

$$NL_{2}^{\alpha} f(t) := \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}. \tag{3}$$

II) $H(\varepsilon,\beta)=1+\varepsilon\beta_r$, $r>0$, in this way we obtain

$$DP_{\beta}^{\alpha} f(t) := \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta^{r})f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon}.$$

Refer to our N-derivative of [8] we have:

$$NP_{2}^{\alpha} f(t) := \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta^{r})f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}. \tag{4}$$

If $k\neq 1$, as $e^{x}=1+x+x^{2}/2!+\ldots$ we can take (as a first possibility):

III) $H(\varepsilon,\beta)=E_{1,1}(\varepsilon\beta)$ and so we have

$$DE_{\beta}^{\alpha} f(t) := \lim_{\varepsilon\to 0}\frac{E_{1,1}(\varepsilon\beta)f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon},$$

and regarding our N-derivative of [8] it becomes:

$$NE_{\beta}^{\alpha} f(t) := \lim_{\varepsilon\to 0}\frac{E_{1,1}(\varepsilon\beta)f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}. \tag{5}$$

From (2) we can easily obtain the following conclusions:

1. Is a derivative local operator, that is, defined at a point.

2. They are derivative in the strict sense of the word.

3. It does not comply with Leibniz's rule, so for (3) we have (the calculations are similar for (4) and (5)):

$$NL_2^{\alpha}\left[f(t)g(t)\right] = \left(N_2^{\alpha} f(t)\right) g(t) + f(t)\left(N_F^{\alpha} g(t)\right),$$

Also for (3) we have (again the calculations for (4) and (5) are very similar):

4. If $\alpha = 0$, $\beta = 1$ then $N^{\alpha}_{2} f(t) = N^{0}_{F} f(t) + f(t) = (1+e) f(t)$.

5. If $\alpha = 1$, $\beta = 0$ then

$$N_2^1 f(t) = N^1_{e^{t^{-1}}} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon e^{t^{-1}}) - f(t)}{\varepsilon} = e^{t^{-1}} \left[\lim_{\varepsilon \to 0} \frac{f(t+\varepsilon e^{t^{-1}}) - f(t)}{\varepsilon e^{t^{-1}}} \right] = e^{t^{-1}} f'(t),$$

if f is derivable.

6. If the limit exists in (5) then we have

$$NL_{\beta}^{\alpha} f(t) = N_F^{\alpha} f(t) + \beta f'(t). \tag{6}$$

7. Unfortunately, "we lose" the Chain Rule that was valid for our N-derivative (see [8]), so for NL^{α}_{β} we obtain:

$$NL_{\beta}^{\alpha}\left[f(g(t))\right] = N_F^{\alpha} f(g(t)) + \beta f(g(t)).$$

8. From (6) we derive that

$$\lim_{t\to\infty} NL_{\beta}^{\alpha} f(t) = \lim_{t\to\infty} N_F^{\alpha} f(t) + \lim_{t\to\infty} \beta f'(t) = f'(t) + \beta f(\infty).$$

Where we can draw the following: if the term $\beta f(\infty)$ exists, then the derivative $N^{\alpha}_{\beta} f(t)$ is only a "translation" of the derivative of the function when $t \to \infty$, so it does not affect the qualitative behavior of the ordinary derivative, this is of vital importance in the study of asymptotics properties of solutions of fractional differential equations with NL^{α}_{β}. Unfortunately, the non-existence of the limit of the function to infinity makes the qualitative study of these fractional differential equations impossible.

9. Let's go back to the equation (2), it is clear that the function $H(\varepsilon,\beta)$ can be generalized although that would complicate the calculations extraordinarily. Of course this does not close the discussion on what terms can be "added" to the increased function that give local fractional derivatives that violate the Leibniz Rule, which would maintain the locality, as a historical inheritance of the derivative, and would default Leibniz's Rule, as a "necessary" condition so that there is a fractional derivative.

4. ON SOLUTIONS OF LOCAL FRACTIONAL EQUATIONS AND REMARKS ON THE INTERSECTIONS OF TRAJECTORIES OF SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

As we pointed out in insufficiency 7), in fractional systems, it may happen that two different solutions intersect in finite time, something that in the case of the integer order is not possible, under the conditions of existence and uniqueness. Different papers (cf. [2], [3], [5], [6], [7], and [11]) have studied this question, in [5] a separation theorem is proved, for one-dimensional fractional systems, which states that two different trajectories can not intersect in finite time, however in the general case this theorem is not valid and this is due to the non-local nature of the global fractional derivatives. Even in the relatively simple case of linear fractional systems of order n, this situation persists (see in particular [6] for what follows). We consider the following linear fractional autonomous system, with $0 < \alpha < 1$ and Caputo fractional derivative:

$$D^{\alpha}x(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n}, \quad x \in \mathbb{R}^{n}, \quad t \geq 0, \quad x(0) = x_0. \tag{7}$$

Recall that for global fractional derivatives, particularly for Caputo, there is no Chain Rule (insufficiency 4) and therefore, are not translation invariant (see [18]). Let's specify this a little more. For $x_0, y_0 \in \mathbb{R}^n$, $t \geq 0$, trajectories $u(t; x_0)$ and $u(t; y_0)$ of (7) are said to intersect at the point $p \in R^n$ if there exist T, $\check{T} \geq 0$, such that $p=u(T; x_0)=u(\check{T};y_0)$. As highlighted before, the fractional derivative is not translation invariant and thus $u(T+t; x_0)$ and $u(T+t; y_0)$ are not solutions of (7) with $x(0) = p$. Therefore, unlike $\alpha = 1$ case, the existence of intersections does not contradict uniqueness.

It is clear that if we consider a local fractional system, in the sense of Definition 1, this situation is excluded under the conditions of existence and uniqueness and which is consistent with the known classic results they pose "A trajectory which passes through at least one point that is not a critical point cannot cross itself unless it is a closed curve. In this case the trajectory corresponds to a periodic solution of the system" ([4], p.379-380). For example, if we consider the ordinary damped linear spring, which is described by a second order equation of the type $mu = -kx - nv$, or the equivalent system:

$$\dot{x} = v, \dot{v} = -\frac{k}{m}x - \frac{n}{m}v, \tag{8}$$

the cause is that this damped spring is a second order system in which different complexity versions can be distinguished (from linear to nonlinear):

a) linear damped spring $mu = -kx - cv$,
b) monotonic spring force $mu = -f(x) - cv$,
c) monotonic damping $mu = -kx - g(v)$,
d) general damped spring $u = -g(x) - f(v)$,

with f and g functions that satisfy the conditions of the existence and uniqueness theorem. This model is relevant since similar equations are presented in different contexts (e.g. circuits, biological systems and control). Therefore, it is an example that is worth studying in the context of local derivatives, for this, we will use the following kernel and different resources, in particular, we will consider the following fractional version of the system:

$$N_F^{\alpha}x = v, N_F^{\alpha}v = -\frac{k}{m}x - \frac{c}{m}v. \tag{9}$$

The system (9) is equivalent to the fractional equation:

$$N_F^{\alpha}(N_F^{\alpha}x) + \frac{c}{m}N_F^{\alpha}x + \frac{k}{m}x = 0. \tag{10}$$

We present below, the definition of generalized integral:

Definition 4 *Let* $\alpha \in (0,1]$ *and* $0 \leq u \leq v$. *We say that a function* $h : [u,v] \to \mathbb{R}$ *is* α*-generalized integrable on* $[u,v]$*, if the integral*

$$_F J_u^{\alpha} f(x) = \int_u^x h(\zeta)d_{\alpha}\zeta = \int_u^x \frac{h(\zeta)}{F(\zeta,\alpha)}d\zeta \tag{11}$$

exists and is finite.

The result we present below is similar to that known from the classical calculus.

Theorem 5 *Let f be N-differentiable function in* (t_0,∞) *with* $\alpha \in (0,1]$. *Then for all* $t > t_0$ *we have*

a) If f is differentiable, then $F\,J^{\alpha}_{t0}(N^{\alpha}_F f(t)) = f(t) - f(t_0)$.
b) $N^{\alpha}_F (F\,J^{\alpha}_{t0} f(t)) = f(t)$.

Proof. a) Since f' is a locally integrable function on I, from [16] we have

$$J^{\alpha}_{F,t_0}(N^{\alpha}_F(f))(t) = \int_{t_0}^{t} \frac{N^{\alpha}_F(f)(s)}{F(s,\alpha)}ds = \int_{t_0}^{t} f'(s)ds = f(t) - f(t_0),$$

which is the desired result.

b) Let f be *a* continuous function f on I. Taking into account the property Theorem 3 f) gives for every $t_0,\, t \in I$

$$\left(J^{\alpha}_{F,t_0}(f)(t)\right)' = \left(\int_{t_0}^{t} \frac{f(s)}{F(s,\alpha)}\,ds\right)' = \frac{f(t)}{F(t,\alpha)},$$

so

$$N^{\alpha}_F J^{\alpha}_{F,t_0}(f)(t) = F(t,\alpha)\left(J^{\alpha}_{F,t_0}(f)(t)\right)' = F(t,\alpha)\,\frac{f(t)}{F(t,\alpha)} = f(t).$$

■

We are now in a position to establish some conclusions about the asymptotic behavior of the fractional equation (10), using classical tools.

Definition 6 *The generalized exponential function is defined for every* $t \geq 0$ *by*:

$$E_{1,1}(c\mathscr{F}(x)) \tag{12}$$

where $c \in \mathbb{R}$, $0 < \alpha < 1$ *and* $\mathscr{F}(f) = F\,J^{\alpha}_{u}(1)(x) = \int^{x}_{u} d\alpha\xi = \int_u^x \frac{1}{F(\zeta,\alpha)}\,d\xi$ *and* $u \in \mathbb{R}^{+}$.

Using Theorem 3 f) we have the simple identity

$$N^{\alpha}_F(E_{1,1}(c\mathscr{F}(t)) = F(t,\alpha)(E_{1,1}(c\mathscr{F}(t))' = F(t,\alpha)(E_{1,1}(c\mathscr{F}(t))\,(c\mathscr{F}(t))' =$$
$$= F(t,\alpha)(E_{1,1}(cF(t))\frac{c}{F(t,\alpha)} = c(E_{1,1}(cF(t)).$$

From the above we obtain the following result for the non conformable version of system (7):

$$N^{\alpha}_F x(t) = Ax(t), \quad A \in \mathbb{R}^{nxn}, \quad x \in \mathbb{R}^{n}, \quad t \geq 0, \quad x(0) = x_0, \tag{13}$$

Theorem 7 *(Existence and Uniqueness). The solution x(t) of Cauchy Problem* (13) *exists and it is unique for all* $t \geq t_0 \geq 0$.

Further we can write the general solution of the form equation (10) of the following form:

$$x(t) = C_1 E_{1,1}\left(a\mathscr{F}(t)\right) + C_2 E_{1,1}\left(b\mathscr{F}(t)\right), \tag{14}$$

where α and b are the roots of the equation $\lambda^2 + c/m\,\lambda + k/m = 0$ and C_1 and C_2 are arbitrary constants. It is easy to verify that the known behavior of the ordinary case is maintained in this frame (overdamped, critical damping and underdamped).

On the other hand it is clear, given the local nature of the derivative N^{α}_F, that autointersections or intersections between different solutions can not occur for the equation (13).

5. FINAL REMARKS

In the Second Lyapunov Method, the Chain Rule is vital to calculate the total derivative of the Lyapunov Function. Using classical fractional derivatives, this is a problem that is not solved (see [9]), however, using our Theorem 8 (of [16]) it is easy to verify that difficulty is overcome, in a future work we will present concrete results in this direction. Nevertheless, we want to advance something in this direction, be it the Generalized Lienard System:

$$N_F^{\alpha}x(t) = y(t) - H(x(t)), N_F^{\alpha}y(t) = -g(x(t)), \tag{15}$$

as a natural generalization of the classical Lienard system, with $H(x) = {}_FJ^{\alpha}_0\,(h)(x)$, and f and g are continuous functions such that $h : \mathbb{R} \to \mathbb{R}_{+}$, and $g : \mathbb{R} \to \mathbb{R}$ with $xg(x) > 0$ for $x \neq 0$. The system (15) is equivalent to the equation $(N^{\alpha}_F\,(N^{\alpha}_F))x(t) + N^{\alpha}_F\,[H(x(t))] + g(x(t)) = 0$. We consider the following Lyapunov Function

$$V(x, y) = G(x) + \frac{y^2}{2}. \tag{16}$$

With $G(x) = {}_FJ^{\alpha}_0\,(g)(x)$. We calculate the generalized derivative of (16) along the system (15):

$$N_F^{\alpha}V(x(t), y(t)) = N_F^{\alpha}[G(x(t))] + N_F^{\alpha}\left[\frac{y^2(t)}{2}\right].$$

$$N_F^{\alpha}V(x(t), y(t)) = g(x(t))N_F^{\alpha}x(t) + y(t)N_F^{\alpha}y(t).$$

From this we have

$$N_F^{\alpha}V(x(t), y(t)) = -g(x(t))H(x(t)).$$

Under conditions previously imposed on f and g, we have that V is a positive definite function and its derivative throughout the system (5) is non-positive, from this we have the stability according to Lyapunov of the trivial solution of the system (15).

Finally, we would like to point out that a limitation of our definition is that it assumes that the variable $t > 0$. Thus, the following open problem arises naturally: *if this condition can be overcome for some kinds of functions and if so, what are these functions?*

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