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Research Article

β-SELECTION PROPERTIES IN DITOPOLOGICAL TEXTURE SPACES

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ABSTRACT

In the present paper, we continue previous investigations on the selection properties in ditopological texture spaces. We introduce the β-Menger and β-Rothberger properties and examine some their properties in ditopological texture spaces. The subject of our investigation is also the preservation of β-Menger property under subspaces, products, and β-continuous mappings.
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1. INTRODUCTION

In general topological spaces, by introducing the properties of Menger and Rothberger were carried out in the classical text by Menger [21] and also in [22]. A general form of two classical selection principles is as follows: A topological space Y has the Menger (resp. Rothberger) property if for each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of Y there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ (resp. $(U_n)_{n\in\mathbb{N}}$) such that each $n\in\mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ (resp. $U_n\in\mathcal{U}_n$) and $Y=\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{V}_n$ (resp. $Y=\bigcup_{n\in\mathbb{N}}U_n$). Each compact space Y has obviously the Menger property, and every Menger space is Lindelöf. Menger's property is hereditary for closed subsets and preserved by continuous mappings. By Todorčević [27], there are Menger spaces Y and Z whose product space $Y\times Z$ is not Menger.

On the other hand, generalized open sets and its counterpart generalized closed sets are one of the significant areas of research in general topology. Many authors studied these notions with their various properties. Semi-open sets and semi-continuity which are early notions related to this area were introduced by Levine [19], and the other one was pre-open sets introduced by Mashhour et al. [20]. Every pre-open set is semi-open in a topological space [2]. The role of generalized open sets in selection principles theory in topological spaces was discussed in a number of papers (see, for example, [26], [4], [15], and the survey paper [14] and references therein).

Apart from introducing β-open set in topology, Abd El Monsef et al. [1], and the equivalent notion of semi-pre-open set was given by Andrijevic in [3], and examined by Ganster and

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Andrijevic [13]. Af ter that many researchers studied these concepts and related notions. For more detailed studies of β-open sets, we refer the reader to Caldas and Jafari [9].

The fundamental concept of a texture space was initiated by Brown and the primary motivation ditopological texture spaces are to offer a new extension of classical fuzzy sets. Since then various aspects of general topology were investigated and carried out in ditopological texture space sense by several authors of this field. In recent papers on textures indicate that these structures are a good tool for the theory of rough sets [10], [11], and semi-separation axioms in soft fuzzy topological spaces [18]. For a study of selection principles in ditopological texture spaces see [16], [17], [23], [24], [25].

In this paper, we define and extend the idea of β-Menger and β-Rothberger spaces in the setting of ditopological texture spaces.

These are the motivations for our paper. The paper is organized as follows. The next section contains a review of the well-known properties of ditopological texture spaces. In Section 3, we define β-Menger and β-Rothberger property in ditopological texture spaces and give some examples. In Section 4, we investigate the behavior of β-Menger and co-β-Menger properties with respect to subspaces, products and β-continuous mappings.

2. PRELIMINARIES

In this chapter, we will give some elementary definitions and basic properties of the theory of ditopological texture spaces.

Texture Space: [5] Given a set S, a texturing of S is a subset $\mathcal{S}$ of $\mathcal{P}(S)$. The pair $(S,\mathcal{S})$ is said to be a *texture space* if $\mathcal{S}$ is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, such that arbitrary meets coincide with intersections and finite joins coincide with unions. Given a texture $(S,\mathcal{S})$, the sets $P_s = \bigcap\{A \in \mathcal{S} | s \in A\}$ is called the *p-sets* and $Q_s = \bigvee\{A \in \mathcal{S} | s \notin A\}$ is called the *q-sets*.

Complementation: [5] Since a texturing of S need not be closed under the operation of taking the set-complement, but there exists a mapping $\sigma: \mathcal{S} \to \mathcal{S}$ verifying the condition $\sigma(\sigma(A)) = A$ for all $A \in \mathcal{S}$ and for all $A, B \in \mathcal{S}$, $A \subseteq B \Longrightarrow \sigma(B) \subseteq \sigma(A)$. Thus, a texture $(S,\mathcal{S})$ with a complementation σ is called a complemented texture $(S,\mathcal{S},\sigma)$.

Examples 2.1 [5] *Now, we give well-known reference examples:*

1. *Textures of the form $(X,\mathcal{P}(X),\pi)$, where X is a set and $\pi(Y) = X\backslash Y$ for $Y \subseteq X$, are called complemented discrete textures. Obviously, for all $x \in X$ we get $Q_x = X\backslash\{x\}$, $P_x = \{x\}$.*
2. *For $\mathbb{I} = [0,1]$ define $\mathcal{J} = \{[0,\rho) | \rho \in \mathbb{I}\} \cup \{[0,\rho] | \rho \in \mathbb{I}\}$ with a complementation ι may be defined as follows*

$$\iota([0,\rho)) = [0,1-\rho] \text{ and } \iota([0,\rho]) = [0,1-\rho), \rho \in \mathbb{I}.$$

$(\mathbb{I},\mathcal{J},\iota)$ gives the unit interval texture, where $Q_\rho = [0,\rho)$ and $P_\rho = [0,\rho]$ for all $\rho \in \mathbb{I}$.

Definition 2.2 [5] *Let $(S,\mathcal{S})$ be a texture space and (τ,κ) be a pair of subsets of $\mathcal{S}$, where τ is the open sets family and κ the closed sets family. If the followings are hold, then the pair (τ,κ) is called a ditopology on $(S,\mathcal{S})$.*

1. $S, \emptyset \in \tau$, $\qquad S, \emptyset \in \kappa$,
2. $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$, $\qquad K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$,
3. $G_i \in \tau, i \in I \Rightarrow \bigvee G_i \in \tau$. $\qquad K_i \in \kappa, i \in I \Rightarrow \bigcap K_i \in \kappa$.

Therefore a ditopology is essentially a "topology" for which there is no *a priori* relation between τ and κ.

The *closure* $cl(A)$ and the *interior* $int(A)$ of $A \in \mathcal{S}$ are defined, respectively:

$$cl(A) = \bigcap \{K \in \kappa | A \subseteq K\} \text{ and } int(A) = \bigvee\{G \in \tau | G \subseteq A\}.$$

On the other hand, if (τ,κ) is a ditopology on a complemented texture $(S,\mathcal{S},\sigma)$ we say (τ,κ) is complemented where $\kappa=\sigma(\tau)$. Thus, we get $\sigma(cl\,(A))=\ int(\sigma(A))$ and $\sigma(int(A))=cl(\sigma(A))$.

We begin by recalling [12] that A is β-open if and only if $A\subseteq cl\left(int\left(cl(A)\right)\right)$ and A is β-closed if and only if $int\left(cl\left(int(A)\right)\right)\subseteq A$. The collection of all β-open subsets of S is denoted by $\beta O(S)$, or when there can be no confusion by βO. Likewise, $\beta C(S)$ or βC denote the set of β-closed sets.

Examples 2.3

1. *Let $\mathcal{T}$ be a topology on X. Then $(X,\mathcal{P}(X),\ \pi_X,\ \mathcal{T},\mathcal{T}')$ is a complemented ditopological texture space where $\pi_X(Y)=X\backslash Y$ for $Y\subseteq X$. Obviously the β-open sets and the β-closed sets in $(X,\mathcal{T})$ are equivalent to the β-open, β-closed sets, respectively, in this space.*
2. *For the unit interval texture $(\mathbb{I},\mathcal{I},\iota)$ of Examples 2.1(2), $(\tau_{\mathbb{I}},\kappa_{\mathbb{I}})$ the standard complemented ditopology given as follows:*

$$\tau_{\mathbb{I}}=\{[0,\rho)|\rho\in\mathbb{I}\}\cup\{\mathbb{I}\},\kappa_{\mathbb{I}}=\{[0,\rho]|\rho\in\mathbb{I}\}\cup\emptyset.$$

For this space we obviously get $\beta O(\mathbb{I})=\beta C(\mathbb{I})=\mathcal{I}$.

Selection properties in ditopological texture space: [16] A ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ has the Menger (resp. Rothberger) property if for each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of S there exists a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ (resp. $(U_n)_{n\in\mathbb{N}}$) such that every $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ (resp. $U_n\in\mathcal{U}_n$) and $S=\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{V}_n$ (resp. $S=\bigcup_{n\in\mathbb{N}}U_n$). Dually, a ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ has the co-Menger (resp. co-Rothberger) property if for each sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of closed cocovers of $\emptyset$ there exists a sequence $(\mathcal{K}_n)_{n\in\mathbb{N}}$ (resp. $(F_n)_{n\in\mathbb{N}}$) such that every $\mathcal{K}_n$ is a finite subset of $\mathcal{F}_n$ (resp. $F_n\in\mathcal{F}_n$) and $\emptyset=\bigcap_{n\in\mathbb{N}}\bigcap\mathcal{K}_n$ (resp. $\emptyset=\bigcap_{n\in\mathbb{N}}F_n$).

Here we recall from **[5]** that the family $\{G_i\in\mathcal{S}|i\in I\}$ is a cover (resp. cocover) of A if $A\subseteq\bigvee_{i\in I}G_i$ (resp. $\bigcap_{i\in I}G_i\subseteq A$).

Definition 2.4 [5] *The ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is said to be:*

1. *Compact if there exists a finite subcover of every cover S which is set by elements of τ.*
2. *Cocompact if there exists a finite subcover of every cocover of $\emptyset$ which is set by elements of κ.*
3. *Stable if all closed subset $K\in\mathcal{S}\backslash\{S\}$ is compact.*
4. *Costable if all open subset $G\in\mathcal{S}\backslash\{\emptyset\}$ is co-compact.*

When $(S,\mathcal{S},\tau,\kappa)$ satisfies the above four properties, then it is said to be a dicompact space. Now we present a similar definition of β-dicompactness in ditopological texture spaces.

Definition 2.5 [12]*The ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is said to be:*

1. *β-compact if given $\bigvee_{i\in I}G_i=S$, $G_i\in\beta O(S)$, $i\in I$ there exists a finite subset J of I with $\bigcup_{j\in J}G_j=S$.*
2. *β-cocompact if given $\bigcap_{i\in I}F_i=\emptyset$, $F_i\in\beta C(S)$, $i\in I$ there exists a finite subset J of I with $\bigcap_{j\in J}F_j=\emptyset$.*
3. *β-stable if for every $F\in\beta C(S)$ with $S\neq F$, whenever G_i, $i\in I$, are β-open sets in $\mathcal{S}$ verifying $F\subseteq\bigvee_{i\in I}G_i$, there exists a finite subset J of I for which $F\subseteq\bigcup_{j\in J}G_j$.*
4. *β-costable if for every $G\in\beta O(S)$ with $\emptyset\neq G$, whenever F_i, $i\in I$, are β-closed sets in $\mathcal{S}$ verifying $\bigcap_{i\in I}F_i\subseteq G$, there exists a finite subset J of I for which $\bigcap_{j\in J}F_j\subseteq G$.*

A ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is said to be β-dicompact when it satisfies all of the above four properties.

Now, we refer the notion of difunction which has an important role in ditopological texture spaces.

Difunction: [8] A *difunction from* $(S,\mathcal{S})$ *to* $(T,\mathcal{T})$ verifies the following conditions.

1. For $u,u' \in S$, $P_u \nsubseteq Q_{u'} \Rightarrow$ there exist $v \in T$ with $f \nsubseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v)} \nsubseteq F$.
2. For $v,v' \in T$ and $u \in S$, $f \nsubseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u,v')} \nsubseteq F \Rightarrow P_{v'} \nsubseteq Q_v$.

Definition 2.6 *Let* $(f,F)\colon (S,\mathcal{S}) \to (T,\mathcal{T})$ *be a difunction. For* $A \in \mathcal{S}$*, the A-sections with respect to* (f,F) *is given as follows:*

$$f^{\rightarrow}A = \bigcap \{Q_v \mid \text{for each } u, f \nsubseteq \overline{Q}_{(u,v)} \Longrightarrow A \subseteq Q_u\},$$

$$F^{\rightarrow}A = \bigvee \{P_v \mid \text{for each } u, \overline{P}_{(u,v)} \nsubseteq F \Longrightarrow P_u \subseteq A\}.$$

Also, for $B \in \mathcal{T}$*, the B-presections with respect to* (f,F) *is given as follows*:

$$f^{\leftarrow}B = \bigvee \{P_u \mid \text{for each } s, f \nsubseteq \overline{Q}_{(u,v)} \Longrightarrow P_v \subseteq B\},$$

$$F^{\leftarrow}B = \bigcap \{Q_u \mid \text{for each } v, \overline{P}_{(u,v)} \nsubseteq F \Longrightarrow B \subseteq Q_v\},$$

respectively.

The inverse image and the inverse co-image are equal; and the image and co-image are usually not, for a given difunction.

We note that $((f^{\leftarrow})^{\leftarrow})(A) = f^{\rightarrow}(A)$ and $((F^{\leftarrow})^{\leftarrow})(A) = F^{\rightarrow}(A)$.

Definition 2.7 [12] *The difunction* $(f,F)\colon (S_1,\mathcal{S}_1,\tau_1,\kappa_1) \to (S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ *will be called:*

1. β*-continuous if for each* $B \in \tau_2$ *we have* $f^{\leftarrow}B \in \beta O(S_1)$.
2. β*-cocontinuous if for each* $B \in \kappa_2$ *we have* $F^{\leftarrow}B \in \beta C(S_1)$.
3. β*-bicontinuous if the difunction satisfies* (1) *and* (2) *together*.

Definition 2.8 [12] *The difunction* $(f,F)\colon (S_1,\mathcal{S}_1,\tau_1,\kappa_1) \to (S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ *will be called:*

1. β*-irresolute if for each* $B \in \beta O(S_2)$ *we have* $f^{\leftarrow}B \in \beta O(S_1)$.
2. β*-co-irresolute if for each* $B \in \beta C(S_2)$ *we have* $F^{\leftarrow}B \in \beta C(S_1)$.
3. β*-bi-irresolute if the difunction satisfies* (1) *and* (2) *together*.

3. β-MENGER AND β-ROTHBERGER DITOPOLOGICAL TEXTURE SPACES

In this chapter, we present $\beta-$Menger and $\beta-$Rothberger selection properties in ditopological texture spaces.

Definition 3.1 *Consider the ditopological texture space* $(S,\mathcal{S},\tau,\kappa)$ *and let* A *be an element of* $\mathcal{S}$*. The collection* $\{G_\alpha : \alpha \in \Delta\}$ *will be called* β*-open cover of* A *if for each* $\alpha \in \Delta$ *we get* $G_\alpha \in \beta O(S)$ *so that* $A \subseteq \bigvee_{\alpha\in\Delta} G_\alpha$*. Dually, the collection* $\{F_\alpha : \alpha \in \Delta\}$ *is a* β*-closed cocover of* A *when* $\bigcap_{\alpha\in\Delta} F_\alpha \subseteq A$ *such that* $F_\alpha \in \beta C(S)$ *for each* $\alpha \in \Delta$*.*

We will use $\beta\mathcal{O}$ (resp. $\beta\mathcal{C}$) instead of the collection of all β-open (resp. β-closed) covers of a ditopological texture space $(S,\mathcal{S},\tau,\kappa)$.

Definition 3.2 *Consider the ditopological texture space* $(S,\mathcal{S},\tau,\kappa)$ *and let* $A \subset S$.

1. *A will said to have the* β*-Menger property for the ditopological texture space if whenever each sequence* $(\mathcal{U}_n)_{n\in\mathbb{N}}$ *of* β*-open covers of A there exists a sequence* $(\mathcal{V}_n)_{n\in\mathbb{N}}$ *so that* $\mathcal{V}_n$ *is a finite subset of* $\mathcal{U}_n$ *and* $A \subseteq \bigvee_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$*, for each* $n \in \mathbb{N}$*. In particular* $(S,\mathcal{S},\tau,\kappa)$ *will be called* β*-Menger if S is* β*-Menger.*

2. *A will said to have the co-*β*-Menger property for the ditopological texture space if whenever each sequence* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *of* β*-closed cocovers of A, there exists a sequence* $(\mathcal{K}_n)_{n\in\mathbb{N}}$*, so that* $\mathcal{K}_n$ *is a finite subset of* $\mathcal{F}_n$ *and* $\bigcap_{n\in\mathbb{N}} \bigcap \mathcal{K}_n \subseteq A$*, for each* $n \in \mathbb{N}$*. In particular* $(S,\mathcal{S},\tau,\kappa)$ *will be called co-*β*-Menger if* $\emptyset$ *is co-*β*-Menger.*

Since the class of β-open (resp. β-closed) covers (resp. cocovers) contains the class of open (closed) covers of S, every ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ which has β-Menger (resp. co-β-Menger) property has Menger (resp. co-Menger) property. The following example shows that the inverse does not hold.

Example 3.3 *There is a Menger space which is not β-Menger.*

Consider the product of the texture $(\{a,b\},\{\emptyset,\{a\},\{a,b\}\})$ and the principle subtexture [7] of the unit interval texture $(\mathbb{I},\mathcal{J},\iota)$ on the set $[0,1)$. The resulting plain texturing of $S=\{a,b\}\times[0,1)$ is easily seen to consist of unions of sets of the form

$$\{a,b\}\times[0,r],0\le r<1,\quad \{a\}\times[0,s],0\le s<1,$$
$$\{a,b\}\times[0,r),0<r<1,\quad \{a\}\times[0,s),0<s<1,$$

together with $\emptyset$ and S. We define a ditopology on this texture by setting

$$\tau=\{\emptyset\}\cup\big\{\{a\}\times[0,s)|0<s<1\big\}\cup\{S\},$$
$$\kappa=\{\emptyset\}\cup\big\{\{a,b\}\times[0,r]|0\le r<1\big\}\cup\{S\}.$$

It is easy to see that the elements of κ are β-open, whence $\mathcal{U}_n=\left\{\{a,b\}\times\left[0,1-\frac{1}{n}\right]|n=2,3,\dots\right\}$, is a β-open cover of S which has no finite subset $\mathcal{V}_n$ such that $S=\bigvee_{n\in\mathbb{N}}\cup\mathcal{V}_n$, for each $n\in\mathbb{N}$. Hence this space is not β-Menger. On the other hand it is clear that the only way of obtaining a covering of S by open sets is to include S, whence $\{S\}$ is a finite subset and the space is Menger.

It is left to the interested reader to produce a modification of this example that is co-Menger but not co-β-Menger.

Definition 3.4 *Consider the ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ and let $A\subset S$.*

1. *A will said to have the β-Rothberger property for $(S,\mathcal{S},\tau,\kappa)$ if whenever each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of β-open covers of A, there exists a sequence $(U_n)_{n\in\mathbb{N}}$ so that $U_n\in\mathcal{U}_n$ and $A\subseteq\bigvee_{n\in\mathbb{N}}U_n$, for each $n\in\mathbb{N}$. In particular $(S,\mathcal{S},\tau,\kappa)$ will be called β-Rothberger if S is β-Rothberger.*
2. *A will said to have the co-β-Rothberger property for $(S,\mathcal{S},\tau,\kappa)$ if whenever each sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of β-closed cocovers of A, there exists a sequence $(F_n)_{n\in\mathbb{N}}$, so that $F_n\in\mathcal{F}_n$ and $\bigcap_{n\in\mathbb{N}}F_n\subseteq A$, for each $n\in\mathbb{N}$. In particular $(S,\mathcal{S},\tau,\kappa)$ will be called co-β-Rothberger if $\emptyset$ is co-β-Rothberger.*

Since the class of β-open (resp. β-closed) covers (resp. cocovers) contains the class of open (closed) covers of S, every ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ which has β-Rothberger (resp. co-β-Rothberger) property has Rothberger (resp. co-Rothberger) property. Also it is obvious that every β-Rothberger (resp. co-β-Rothberger) ditopological texture space is β-Menger (resp. co-β-Menger).

Definition 3.5 *A ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is called σ-β-compact (resp. σ-β-cocompact) if there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of β-compact (resp. β-cocompact) subsets of S such that $\bigvee_{n\in\mathbb{N}}A_n=S$ (resp. $\bigcap_{n\in\mathbb{N}}A_n=\emptyset$).*

Theorem 3.6 *Consider the ditopological texture space $(S,\mathcal{S},\tau,\kappa)$.*

1. *$(S,\mathcal{S},\tau,\kappa)$ has β-Menger property, when $(S,\mathcal{S},\tau,\kappa)$ is σ-β-compact.*
2. *$(S,\mathcal{S},\tau,\kappa)$ has co-β-Menger property, when $(S,\mathcal{S},\tau,\kappa)$ is σ-β-cocompact.*

Proof.

1. Let S be σ-β-compact. Suppose that $S = \vee_{t\in\mathbb{N}} A_t$, where each A_t is β-compact. Take a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of β-open covers of S. For each $t \in \mathbb{N}$ there exists a finite subset $\mathcal{V}_t \subseteq \mathcal{U}_t$ such that $A_t \subseteq \bigcup \mathcal{V}_t$. It follows that the sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ indicates that S is β-Menger.

2. Let S be σ-β-cocompact. We get that $\emptyset = \bigcap_{t\in\mathbb{N}} A_t$, where each A_t is β-cocompact. Take a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of β-closed covers of $\emptyset$. For each $t \in \mathbb{N}$, choose a finite subset $\mathcal{K}_t \subseteq \mathcal{F}_t$ such that $\bigcap \mathcal{K}_t \subseteq A_t$. Then $\bigcap_{n\in\mathbb{N}} \bigcap \mathcal{K}_n \subseteq \bigcap_{n\in\mathbb{N}} A_n = \emptyset$, so S is co-β-Menger.

Evidently, we have

$$\begin{array}{lllll} \sigma-\beta-compact & \Rightarrow & \beta-Menger & \Leftarrow & \beta-Rothberger \\ \sigma-\beta-cocompact & \Rightarrow & co-\beta-Menger & \Leftarrow & co-\beta-Rothberger \end{array}$$

Example 3.7 *There exists a ditopological texture space which is β-Rothberger (thus β-Menger) (resp. co-β-Rothberger (thus co-β-Menger)) but not σ-β-compact (resp. σ-β-cocompact).*

Take into consideration the real line $(\mathbb{R},\mathfrak{R})$ where $\mathbb{R}$ is the set of real numbers and $\mathfrak{R} = \{(-\infty,r]: r \in \mathbb{R}\} \cup \{(-\infty,r): r \in \mathbb{R}\} \cup \{\mathbb{R},\emptyset\}$. The ditopology $(\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ on $(\mathbb{R},\mathfrak{R})$ is defined by topology $\tau_{\mathbb{R}} = \{(-\infty,r): r \in \mathbb{R}\} \cup \{\mathbb{R},\emptyset\}$ and cotopology $\kappa_{\mathbb{R}} = \{(-\infty,r]: r \in \mathbb{R}\} \cup \{\mathbb{R},\emptyset\}$. Since the β-open cover $\mathcal{U} = \mathfrak{R}$ does not contain a finite subcover we see that $(\mathbb{R},\mathfrak{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ is not β-compact. Also $(\mathbb{R},\mathfrak{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ is not β-cocompact because its β-closed cocover $\mathfrak{R}$ does not contain a finite cocover.

But $(\mathbb{R},\mathfrak{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ is β-Rothberger. Let us prove that $\mathcal{U}_n = \{(-\infty,n): n \in \mathbb{N}\}$ is a sequence of β-open covers of $\mathbb{R}$. Write $\mathbb{R} = \bigcup\{(-\infty,n): n \in \mathbb{N}\}$. For each n, $\mathcal{U}_n$ is a β-open cover of $\mathbb{R}$, hence there is some $r_n \in \mathbb{R}$ such that $(-\infty,n) \subseteq (-\infty,r_n) \in \mathcal{U}_n$. Then the collection $\{(-\infty,r_n): n \in \mathbb{N}\}$ shows that $(\mathbb{R},\mathfrak{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ is β-Rothberger.

Also $(\mathbb{R},\mathfrak{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ is co-β-Rothberger. Let us prove that $\mathcal{F}_n = \{(-\infty,n]: n \in \mathbb{N}\}$ is a sequence of β-closed covers of $\mathbb{R}$. Write $\mathbb{R} = \bigcup\{(-\infty,n): n \in \mathbb{N}\}$. For each n, $\mathcal{U}_n$ is a β-closed cover of $\mathbb{R}$, hence there is some $r_n \in \mathbb{R}$ such that $(-\infty,n] \supseteq (-\infty,r_n] \in \mathcal{F}_n$. Then the collection $\{(-\infty,r_n]: n \in \mathbb{N}\}$ shows that $(\mathbb{R},\mathfrak{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ is co-β-Rothberger.

Theorem 3.8 *Consider a texture $(S,\mathcal{S},\sigma)$ with the complementation σ. Take a complemented ditopology (τ,κ) on the texture $(S,\mathcal{S},\sigma)$. Then S is β-Menger iff $\emptyset$ is co-β-Menger.*

Proof. Let S be β-Menger and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of β-closed cocovers of $\emptyset$. Hence $\left(\sigma(\mathcal{F}_n)\right)_{n\in\mathbb{N}}$ is a sequence of β-open covers of S. For each $n \in \mathbb{N}$, there exists a sequence $\mathcal{V}_n$ of finite sets such that $\mathcal{V}_n \subseteq \sigma(\mathcal{F}_n)$ and $\vee_{n\in\mathbb{N}} \bigvee \mathcal{V}_n = S$ since S is β-Menger. We observe that

$$\emptyset = \sigma(S) = \sigma\left(\bigvee_{n\in\mathbb{N}} \bigvee \mathcal{V}_n\right) = \bigcap_{n\in\mathbb{N}} \bigcap \sigma(\mathcal{V}_n),$$

where $n \in \mathbb{N}$, $\sigma(\mathcal{V}_n)$ is a sequence of finite sets. Then, $\emptyset$ is co-β-Menger.

Using dual arguments, it can be obtained the other direction.

Theorem 3.9 *Consider a complemented ditopology (τ,κ) on a texture $(S,\mathcal{S},\sigma)$. Then for $K \in \kappa$ with $K \neq S$, K is β-Menger iff G is co-β-Menger, for some $G \in \tau$ and $G \neq \emptyset$.*

Proof. Take $K \in \kappa$ such that $K \neq S$. Suppose that $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a sequence of β-open covers of K. Because $K = \sigma(G)$ but G is co-β-Menger, so for the sequence $\left(\sigma(\mathcal{U}_n)\right)_{n\in\mathbb{N}}$ of β-closed cocovers of G there exists a sequence $(\mathcal{K}_n)_{n\in\mathbb{N}}$ such that finite $\mathcal{K}_n \subseteq \sigma(\mathcal{U}_n)$ with $\bigcap_{n\in\mathbb{N}} \bigcap \mathcal{K}_n \subseteq G$ for each $n \in \mathbb{N}$. Hence $\sigma((\mathcal{K}_n)_{n\in\mathbb{N}})$ is a sequence such that for each $n \in \mathbb{N}$, $\sigma(\mathcal{K}_n)$ is a finite subset of $\mathcal{U}_n$. Hence $K = \sigma(G) \subseteq \sigma(\bigcap_{n\in\mathbb{N}} \bigcap \mathcal{K}_n) = \vee_{n\in\mathbb{N}} \bigcup \sigma(\mathcal{K}_n)$, and we see that K is β-Menger.

The proof of $K \in \kappa$ is β-Menger with $S \neq K$ implies G is co-β-Menger with $G \in \tau$ and $G \neq \emptyset$ is obtained by using dual arguments.

4. RESULTS

In this section, we investigate various properties of β-Menger and co-β-Menger ditopological texture space. Principally, we examine the preservation of these properties under subspaces, products, and mappings.

4.1. Subspaces

Take into consideration a texture space $(S,\mathcal{S})$ and let $A\in\mathcal{S}$. The texturing $\mathcal{S}_A=\{A\cap K: K\in\mathcal{S}\}$ is said to be the induced structure on A and $(A,\mathcal{S}_A)$ is said to be a principal subtexture of $(S,\mathcal{S})$. Also (τ_A,κ_A) is said to be the induced ditopology on A where $\kappa_A=\{A\cap K: K\in\kappa\}$ and $\tau_A=\{A\cap G: G\in\tau\}$. The principle ditopological subtexture of $(S,\mathcal{S},\tau,\kappa)$ is denoted by $(A,\mathcal{S}_A,\tau_A,\kappa_A)$.

Note that A is called open subspace if $A\in\tau$ and A is closed subspace if $A\in\kappa$.

Theorem 4.1 *Consider a complemented ditopology (τ,κ) on a texture $(S,\mathcal{S},\sigma)$ and $A\in\mathcal{S}$. Then $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ is β-Menger, if S is β-Menger and $A\in\beta C$.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of A by β-open sets. Considering that $\mathcal{V}_n=\mathcal{U}_n\cup\{\sigma(A)\}$ is a β-open cover of S for each n. Since S is β-Menger, for $n\in\mathbb{N}$, there exists finite families $\mathcal{W}_n\subseteq\mathcal{V}_n$ such that $\vee_{n\in\mathbb{N}}\cup\mathcal{W}_n$ covers S. It follows that $\mathcal{Z}_n=\mathcal{W}_n\backslash\{\sigma(A)\}$, for $n\in\mathbb{N}$. So, $\mathcal{Z}_n$ is a finite subset of $\mathcal{U}_n$ such that $A\subseteq\vee_{n\in\mathbb{N}}\cup\mathcal{Z}_n$, for each $n\in\mathbb{N}$. In this case, the principle ditopological subtexture $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ has the β-Menger property.

Theorem 4.2 *Consider a complemented ditopology (τ,κ) on a texture $(S,\mathcal{S},\sigma)$ and $A\in\mathcal{S}$. Then $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ is co-β-Menger, if $\emptyset$ is co-β-Menger and $A\in\beta O$.*

Proof. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of cocovers of A by β-closed sets. Considering that $\mathcal{K}_n=\mathcal{F}_n\cup\{\sigma(A)\}$ is a β-closed cocover of S for each n. Since $\emptyset$ is co-β-Menger, for $n\in\mathbb{N}$, there exists finite families $\mathcal{L}_n\subseteq\mathcal{K}_n$ such that $\bigcap_{n\in\mathbb{N}}\cap\mathcal{L}_n\subseteq S$. It follows that $\mathcal{G}_n=\mathcal{L}_n\backslash\{\sigma(A)\}$, for $n\in\mathbb{N}$. So, $\mathcal{G}_n$ is a finite subset of $\mathcal{F}_n$ such that $\bigcap_{n\in\mathbb{N}}\cap\mathcal{G}_n\subseteq A$ for each $n\in\mathbb{N}$. In this case, the principle ditopological subtexture $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ has the co-β-Menger property.

Example 4.3 *Take a complemented ditopology (τ,κ) on a texture $(S,\mathcal{S},\sigma)$ and $A\in\mathcal{S}$. When this space is β-Menger and $A\in\beta C(S)$, then the principle ditopological subtexture $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ has the β-Menger property.*

Consider again the real line textured $\mathbb{R}$ by $\mathfrak{R}=\{(-\infty,r]: r\in\mathbb{R}\}\cup\{(-\infty,r): r\in\mathbb{R}\}\cup\{\mathbb{R},\emptyset\}$, with complemented ditopology $(\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ such that $\beta C(S)=\mathfrak{R}$. This space is β-Menger and co-β-Menger by Example 3.7. Take $A=\{(-\infty,r]: r\in\mathbb{R}\}$, then $(A,\mathfrak{R}_A)$ is an induced subtexture of $(\mathbb{R},\mathfrak{R})$ with texturing $\mathfrak{R}_A=\{A\cap K: K\in\mathfrak{R}\}=\{(-\infty,a]: a\in A\}\cup\{(-\infty,a): a\in A\}\cup\{A,\emptyset\}$, topology $\tau_A=\{A\cap G: G\in\tau_{\mathbb{R}}\}=\{(-\infty,a): a\in A\}\cup\{A,\emptyset\}$, cotopology $\kappa_A=\{A\cap K: K\in\kappa_{\mathbb{R}}\}=\{(-\infty,a]: a\in A\}\cup\{A,\emptyset\}$ and $\beta C(A)=\mathfrak{R}_A$. Thus $(A,\mathfrak{R}_A,\tau_A,\kappa_A)$ is the induced ditopological texture space.

Now, we will show that $(A,\mathfrak{R}_A,\tau_A,\kappa_A)$ is β-Menger. The proof that $(A,\mathfrak{R}_A,\tau_A,\kappa_A)$ is co-β-Menger is similar. Let a sequence of β-open covers of $\mathbb{R}$ be $\mathcal{U}_n=\{(-\infty,n): n\in\mathbb{N}\}$. Take $A=\bigcup\{(-\infty,n]: n\in\mathbb{N}\}$. Since $\mathcal{U}_n$ is a β-open cover of A, for each n, then there exists some $r_n\in\mathbb{R}$ so that $(-\infty,n]\subseteq(-\infty,r_n)\in\mathcal{U}_n$. Therefore the family $\{(-\infty,r_n): n\in\mathbb{N}\}$ shows that $(A,\mathfrak{R}_A,\tau_A,\kappa_A)$ is β-Menger.

4.2. Products

When we consider a Menger space and a compact space, the product of them is Menger. Now we examine the above information for β-Menger in a ditopological context. First, we recall [6] the product of ditopological spaces. For textures $(S,\mathcal{S})$ and $(T,\mathcal{T})$, we will denote by $\mathcal{S}\otimes\mathcal{T}$ the *product texturing* of $S \times T$. Thus, $\mathcal{S} \otimes \mathcal{T}$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$ where $A \in \mathcal{S}$ and $B \in \mathcal{T}$. $(S \times T, \mathcal{S} \otimes \mathcal{T})$ is said to be the *product* of $(S, \mathcal{S})$ and $(T, \mathcal{T})$. We obviously get $P_{(s,t)} = P_s \times P_t$ and $Q_{(s,t)} = (Q_s \times T) \cup (S \times Q_t)$, for $s \in S$, $t \in T$.

Theorem 4.4 *Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be ditopological spaces. Then we have*

1. *If $(S, \mathcal{S})$ is β-*Menger*, and $(T, \mathcal{T})$ is β-compact, then $(S \times T, \mathcal{S} \otimes \mathcal{T})$ is β-Menger*;
2. *If $(S, \mathcal{S})$ is co-β-Menger, and $(T, \mathcal{T})$ is β-cocompact, then $(S \times T, \mathcal{S} \otimes \mathcal{T})$ is co-β-Menger*.

Proof. We concentrate on (1), leaving the essentially dual proof of (2) to the interested reader.
Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of $S \times T$ by β-open sets in $S \times T$. Hence for each $n \in \mathbb{N}$ there exist β-open covers $\mathcal{V}_n$ and $\mathcal{W}_n$ of S and T, respectively, such that $\mathcal{U}_n = \mathcal{V}_n \times \mathcal{W}_n$. By β-Mengerness of S there are finite subsets $\mathcal{V}_n' \subseteq \mathcal{V}_n$, $n \in \mathbb{N}$, so that $S = \vee_{n\in\mathbb{N}} \cup \mathcal{V}_n'$. Since T is β-compact, choose a finite subset A_n of $(\mathcal{W}_n)_{n\in\mathbb{N}}$ which is a β-open cover of T. Now let us consider the family

$$\mathcal{X}_n = \mathcal{V}_n' \times A_n.$$

Hence for each $n \in \mathbb{N}$, $\mathcal{X}_n$ is a finite subset of $\mathcal{U}_n$. That implies

$$S \times T = \bigvee_{n\in\mathbb{N}} \cup \mathcal{V}_n' \times \cup A_n = \bigvee_{n\in\mathbb{N}} \cup (\mathcal{V}_n' \times A_n) = \bigvee_{n\in\mathbb{N}} \cup \mathcal{X}_n$$

which concludes the proof.

4.3. Mappings

In this subsection, we consider the behaviour of the β-Menger property under β-difunction.

Theorem 4.5 *Let $(S_t, \mathcal{S}_t, \tau_t, \kappa_t)$, $t = 1,2$, be ditopological texture spaces. Suppose that (f, F) is a β-continuous difunction from $(S_1, \mathcal{S}_1)$ to $(S_2, \mathcal{S}_2)$. Then $f^{\to}(A) \in \mathcal{S}_2$ is β-Menger, if $A \in \mathcal{S}_1$ is β-Menger.*

Proof. Suppose that $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a sequence of covers of $f^{\to}A$ by open sets in S_2. Then, by [8, Theorem 2.24(2a); Corollary 2.12(2)] and β-continuity of (f, F), for each n we have

$$A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}(\bigvee \mathcal{U}_n) = \bigvee F^{\leftarrow}\mathcal{U}_n,$$

so that each $F^{\leftarrow}\mathcal{U}_n$ is a β-open cover of A. So A is β-Menger, there exist finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $A \subseteq \vee_{n\in\mathbb{N}} \cup F^{\leftarrow}\mathcal{V}_n$ for each n. In this case, by [8, Theorem 2.24(2b); Corollary 2.12(2)], we get that

$$f^{\to}A \subseteq f^{\to}\left(\bigvee_{n\in\mathbb{N}} \cup F^{\leftarrow}\mathcal{V}_n\right) = \bigvee_{n\in\mathbb{N}} \cup \left(f^{\to}(F^{\leftarrow}\mathcal{V}_n)\right) \subseteq \bigvee_{n\in\mathbb{N}} \cup \mathcal{V}_n,$$

i.e. $f^{\to}A$ is β-Menger.

Theorem 4.6 *Let $(S_t, \mathcal{S}_t, \tau_t, \kappa_t)$, $t = 1,2$, be ditopological texture spaces. Suppose that (f, F) is a β-cocontinuous difunction from $(S_1, \mathcal{S}_1)$ to $(S_2, \mathcal{S}_2)$. Then $F^{\to}(A) \in \mathcal{S}_2$ has the co-β-Menger property, if $A \in \mathcal{S}_1$ has the co-β-Menger property.*

Proof. Suppose that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a sequence of cocovers of $F^{\to}A$ by closed sets in S_2. Hence, by [8, Theorem 2.24(2a); Corollary 2.12(2)] and β-cocontinuity of (f, F), for each n we get

$$\cap\, \mathcal{F}_n \subseteq F^{\to}(A)$$

$$f^{\leftarrow}(\cap\, \mathcal{F}_n) \subseteq f^{\leftarrow}\left(F^{\to}(A)\right) \subseteq A$$

$$\cap\, f^{\leftarrow}(\mathcal{F}_n) \subseteq A$$

so that each $f^{\leftarrow}(\mathcal{F}_n)$ is a β-closed cocover of A. So A is co-β-Menger, there are for each n finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ so that $\cap_{n\in\mathbb{N}} \cap f^{\leftarrow}(\mathcal{K}_n) \subseteq A$. So, by [8, Theorem 2.24(2b); Corollary 2.12(2)], we get

$$F^{\to}\left(\bigcap_{n\in\mathbb{N}} \cap f^{\leftarrow}(\mathcal{K}_n)\right) \subseteq F^{\to}(A)$$
$$\bigcap_{n\in\mathbb{N}} \cap F^{\to}\left(f^{\leftarrow}(\mathcal{K}_n)\right) \subseteq F^{\to}(A)$$
$$\bigcap_{n\in\mathbb{N}} \cap \mathcal{K}_n \subseteq F^{\to}(A),$$

i.e. $F^{\to}(A)$ is co-β-Menger.

Theorem 4.7 *Let $(S_t, \mathcal{S}_t, \tau_t, \kappa_t)$, $t = 1,2$, be ditopological texture spaces. Suppose that (f, F) is a β-irresolute difunction from $(S_1, \mathcal{S}_1)$ to $(S_2, \mathcal{S}_2)$. If $A \in \mathcal{S}_1$ is β-Menger, then $f^{\to}(A) \in \mathcal{S}_2$ is also β-Menger.*

Proof. Suppose that $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a sequence of covers of $f^{\to}A$ by β-open sets in S_2. Then, by [8, Theorem 2.24(2a); Corollary 2.12(2)] and β-irresolute of (f, F), for each n we may write

$$A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}(\bigvee \mathcal{U}_n) = \bigvee F^{\leftarrow}\mathcal{U}_n,$$

so that each $F^{\leftarrow}\mathcal{U}_n$ is a β-open cover of A. As A is β-Menger, there exists for each n a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ so that $A \subseteq \bigvee_{n\in\mathbb{N}} \cup F^{\leftarrow}\mathcal{V}_n$. In this case, by [8, Theorem 2.24(2b); Corollary 2.12(2)], we get that

$$f^{\to}A \subseteq f^{\to}\left(\bigvee_{n\in\mathbb{N}} \cup F^{\leftarrow}\mathcal{V}_n\right) = \bigvee_{n\in\mathbb{N}} \cup (f^{\to}F^{\leftarrow}\mathcal{V}_n) \subseteq \bigvee_{n\in\mathbb{N}} \cup \mathcal{V}_n.$$

Hence, $f^{\to}A$ is β-Menger.

Theorem 4.8 *Let $(S_t, \mathcal{S}_t, \tau_t, \kappa_t)$, $t = 1,2$, be ditopological texture spaces. Suppose that (f, F) is a β-coirresolute difunction from $(S_1, \mathcal{S}_1)$ to $(S_2, \mathcal{S}_2)$. If $A \in \mathcal{S}_1$ is co-β-Menger, then $F^{\to}(A) \in \mathcal{S}_2$ is also co-β-Menger.*

Proof. Suppose that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a sequence of cocovers of $F^{\to}A$ by β-closed sets in S_2. Then, by [8, Theorem 2.24(2a); Corollary 2.12(2)] and β-coirresolute of (f, F), for each n we may write

$$\cap \mathcal{F}_n \subseteq F^{\to}(A)$$
$$f^{\leftarrow}(\cap \mathcal{F}_n) \subseteq f^{\leftarrow}\left(F^{\to}(A)\right) \subseteq A$$
$$\cap f^{\leftarrow}(\mathcal{F}_n) \subseteq A$$

so that each $f^{\leftarrow}(\mathcal{F}_n)$ is a β-closed cocover of A. As A is co-β-Menger, there exists for each n finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ such that $\bigcap_{n\in\mathbb{N}} \cap f^{\leftarrow}(\mathcal{K}_n) \subseteq A$. Hence, by [8, Theorem 2.24(2b); Corollary 2.12(2)], we get

$$F^{\to}\left(\bigcap_{n\in\mathbb{N}} \cap f^{\leftarrow}(\mathcal{K}_n)\right) \subseteq F^{\to}(A)$$
$$\bigcap_{n\in\mathbb{N}} \cap F^{\to}f^{\leftarrow}(\mathcal{K}_n) \subseteq F^{\to}(A)$$
$$\bigcap_{n\in\mathbb{N}} \cap \mathcal{K}_n \subseteq F^{\to}(A).$$

Thus, $F^{\to}(A)$ is co-β-Menger.

A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called βM-stable if for all $K \in \beta C$ with $K \neq S$ is β-Menger, and βM-co-stable if for every $G \in \beta O$ with $G \neq \emptyset$ is co-β-Menger.

Theorem 4.9 *Let $(S_t, \mathcal{S}_t, \tau_t, \kappa_t)$, $t = 1,2$, be ditopological texture spaces with a β-bicontinuous surjective difunction (f, F) between them. If S_1 is βM-stable, then S_2 is also βM-stable.*

Proof. Take $K \in \kappa_2$ with $K \neq S_2$. We get that $f^{\leftarrow}(K) \in \beta C(S_1)$ because (f, F) is β-cocontinuous. We prove that $f^{\leftarrow}(K) \neq S_1$. Suppose that $f^{\leftarrow}(K) = S_1$. By [8, Lemma 2.28(1c)], we get $f^{\leftarrow}(S_2) = S_1$ which satisfies that $f^{\leftarrow}(S_2) \subseteq f^{\leftarrow}(K)$ because (f, F) is surjective. According to [8, Corollary 2.33(1 ii)] and surjectivity of (f, F), we find $S_2 \subseteq K$ which is a contradiction. Thus $f^{\leftarrow}(K) \neq S_1$.

Then $f^{\leftarrow}(K)$ is a β-Menger set in S_1 by βM-stability. By Theorem 4.5 and β-continuity of (f, F), we get $F^{\to}\left(f^{\leftarrow}(K)\right)$ is a β-Menger set in S_2. By [8, Corollary 2.33(1)] this set is equal to K. This gives that S_2 is βM-stable.

As expected, we have dual result for βM-co-stable. We omit the proofs.

Theorem 4.10 *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces with a β-bicontinuous surjective (f, F) difunction between them. If S_1 is βM-co-stable, then S_2 is also βM-co-stable.*

Note that a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called β-di-Menger if it is β-Menger, co-β-Menger, βM-stable and βM-costable.

Theorem 4.11 *Let $(f, F): (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a surjective β-bicontinuous difunction from a β-di-Menger ditopological texture space to a ditopological texture space. If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is β-di-Menger, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also β-di-Menger.*

Proof. Clear by Theorems 4.5, 4.6, 4.9 and 4.10.

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