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Research Article

A NOTE ON ASYMPTOTIC BEHAVIOR OF FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

The purpose of the study is to present some new criteria for the asymptotic behavior of nonlinear fractional differential equations.
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1. INTRODUCTION

Recently, it has been realized that the fractional calculus has numerous applications in signal processing, geology, dynamics of earthquakes, economics and finance, probability and statistics, chemical engineering, physics, thermodynamics and neural networks and so forth; see [1–3] and the references therein. Due to their widespread applications in the field of engineering, the investigations of fractional differential equations have attracted many researchers during the last decades. A lot of study about the oscillatory behavior for integer order differential equations including the existence of oscillatory and nonoscillatory solutions are presented, see [4–13]. Recently, many articles have discussed the oscillation of fractional differential equations [14–23]. However, we notice that very little attention is paid to asymptotic behavior of nonoscillatory solutions of fractional differential equations. In [14], the authors established some oscillation criteria for the following fractional differential equation

$$\left(D_{0+}^{1+\alpha}y\right)(t)+p(t)\left(D_{0+}^{\alpha}y\right)(t)+q(t)\,f\left(G(t)\right)=0,$$

where $t\geq t_0>0$ and $\alpha\in(0,1)$. In [15], the authors considered the oscillation of the following fractional damped differential equation

$$\left(r(t)\psi\left(x(t)\right)D_{0+}^{\alpha}y(t)\right)'+p(t)\psi\left(x(t)\right)D_{0+}^{\alpha}y(t)+F\left(t,G(t)\right)=0,$$

for $t\geq t_0>0$ and $\alpha\in(0,1)$.

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Motivated by the idea in the above research papers, in this study, we consider the asymptotic behavior of solutions of following equations

$$\left(r(t)\psi\left(G(t)\right)D_{0+}^{\alpha}y(t)\right)' + p(t)D_{0+}^{\alpha}y(t)+q(t)f\left(G(t)\right)=0,\ \ t\ge t_0\ge 0. \tag{1}$$

where D_{0+}^{α} denotes the $\alpha-th$ Riemann-Liouville fractional derivative, $\alpha\in(0,1)$, $r\in C\left([t_0,\infty),\mathrm{R}^+\right)$; $\psi\in C\left(\mathrm{R},\mathrm{R}^+\right)$; $p,q\in C\left([t_0,\infty),\mathrm{R}\right)$; $G(t)=\int_0^t (t-s)^{-\alpha}y(s)ds$; $f\in C^1(\mathrm{R},\mathrm{R})$ and $xf(x)>0$ for $x\neq 0$.

2. PRELIMINARIES

Definition 2.1. *[3,14] The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $y:[0,\infty)\to\mathrm{R}$ is defined by*

$$\left(I_{0+}^{\alpha}y\right)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.2. *[3,14] The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $y:[0,\infty)\to\mathrm{R}$ is defined by*

$$\left(D_{0+}^{\alpha}y\right)(t) := \frac{d^n}{dt^n}\left(I_{0+}^{n-\alpha}y\right)(t),$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $n-1<\alpha\le n$ and $n\ge 1$ is an integer.

Lemma 2.1. *[14] Let $y(t)$ be a solution of (1), and*

$$G(t) := \int_0^t (t-s)^{-\alpha}y(s)ds \text{ for } \alpha\in(0,1),\ t>0.$$

Then

$$G'(t)=\Gamma(1-\alpha)D_{0+}^{\alpha}y(t).$$

3. MAIN RESULTS

Theorem 3.1. *Let $p(t)\equiv 0$, and suppose that*

$$\psi(t)>0 \text{ and } \frac{f'(x)}{\psi(x)}\ge 0,\ \text{for } x\neq 0, \tag{2}$$

$$\int_{t_0}^{\infty}\frac{ds}{r(s)}<\infty, \tag{3}$$

$$\int^{\infty} \frac{\psi(u)}{\Gamma(1-\alpha) f(u)} du < \infty, \tag{4}$$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} q(\tau)\, d\tau ds = \infty. \tag{5}$$

Then for every nonoscillatory solution y of (1), we have $\lim_{t\to\infty} \inf |y(t)| = 0.$

Proof. Let $y(t)$ be a nonoscillatory solution of (1), we may assume that $y(t) \neq 0$ for $t \geq t_0$. Define

$$\omega(t) = \frac{r(t)\psi(G(t)) D_{0+}^{\alpha} y(t)}{f(G(t))}.$$

Then ω is well defined and satisfies

$$\omega'(t) = \frac{\left(r(t)\psi(G(t)) D_{0+}^{\alpha} y(t)\right)' f(G(t)) - r(t)\psi(G(t)) D_{0+}^{\alpha} y(t) f'(G(t)) G'(t)}{f^2(G(t))}$$

$$= -q(t) - \frac{\Gamma(1-\alpha) f'(G(t))}{r(t)\psi(G(t))} \omega^2(t).$$

Using (2), we have

$$\omega'(t) \leq -q(t). \tag{6}$$

Integrating (6) from t_0 to t, we get

$$\frac{r(t)\psi(G(t)) D_{0+}^{\alpha} y(t)}{f(G(t))} \leq \omega(t_0) - \int_{t_0}^{t} q(s) ds,$$

i.e.

$$\frac{r(t)\psi(G(t)) G'(t)}{f(G(t))\Gamma(1-\alpha)} \leq \omega(t_0) - \int_{t_0}^{t} q(s) ds. \tag{7}$$

Dividing (7) by $r(t)$ and then integrating from t_0 to t we obtain

$$\int_{G(t_0)}^{G(t)} \frac{\psi(u)}{\Gamma(1-\alpha) f(u)} du \leq \omega(t_0) \int_{t_0}^{t} \frac{ds}{r(s)} - \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} q(\tau)\, d\tau ds.$$

From (3) and (5), we get

$$\liminf_{t\to\infty} \int_{G(t_0)}^{G(t)} \frac{\psi(u)}{\Gamma(1-\alpha) f(u)} du = -\infty. \tag{8}$$

If $\lim_{t\to\infty} \inf y(t) > 0$, then there exist c_1 and c_2 positive constants such that $y(t) \geq c_1$ and $G(t) \geq c_2$ for all $t \geq t_0$. Consequently, by (4)

$$\left|\int_{G(t_0)}^{G(t)}\frac{\psi(u)}{\Gamma(1-\alpha)f(u)}du\right|\le\int_{c_2}^{G(t)}\frac{\psi(u)}{\Gamma(1-\alpha)f(u)}du<\infty,$$

which contradicts (8). Thus we must have $\lim_{t\to\infty}\inf y(t)>0$. The proof for the case $\lim_{t\to\infty}\inf y(t)<0$ for $t\ge t_0$ is similar and hence is omitted.

Theorem 3.2. *Let (4) holds and*

$$\psi(x)\ge c>0 \text{ and } \frac{f'(x)}{\psi(x)}\ge k>0 \text{ for } x\ne 0. \tag{9}$$

If there exists a positive differentiable function ϕ on $[t_0,\infty)$ such that

$$\phi'(t)p(t)\le 0 \text{ for } t\ge t_0 \tag{10}$$

and

$$\int^{\infty}\frac{ds}{r(s)\phi(s)}<\infty, \tag{11}$$

and

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\frac{1}{r(s)\phi(s)}\int_{t_0}^{s}\left(\phi(\tau)q(\tau)-\frac{r(\tau)\phi(\tau)}{4k\Gamma(1-\alpha)}\left[\frac{\phi'(\tau)}{\phi(\tau)}-\frac{p(\tau)}{cr(\tau)}\right]^2\right)d\tau ds=\infty \tag{12}$$

then for every solution y of (1), we have $\lim_{t\to\infty}\inf|y(t)|=0.$

Proof. Let $y(t)$ be a nonoscillatory solution of (1), we may assume that $y(t)\ne 0$ for $t\ge t_0$. Let

$$\omega(t)=\phi(t)\frac{r(t)\psi(G(t))D_{0+}^{\alpha}y(t)}{f(G(t))}. \tag{13}$$

Differentiating (13), we have

$$\omega'(t)=-q(t)\phi(t)+\left(\frac{\phi'(t)}{\phi(t)}-\frac{p(t)}{r(t)\psi(G(t))}\right)\omega(t)-\frac{\Gamma(1-\alpha)f'(G(t))}{r(t)\phi(t)\psi(G(t))}\omega^2(t)$$

$$=-q(t)\phi(t)+\frac{r(t)\phi(t)}{4\Gamma(1-\alpha)}\frac{\left(\frac{\phi'(t)}{\phi(t)}-\frac{p(t)}{r(t)\psi(G(t))}\right)^2}{\frac{f'(G(t))}{\psi(G(t))}}$$

$$-\left[\sqrt{\frac{\Gamma(1-\alpha)f'(G(t))}{r(t)\phi(t)\psi(G(t))}}\omega(t)-\frac{\left(\frac{\phi'(t)}{\phi(t)}-\frac{p(t)}{r(t)\psi(G(t))}\right)}{2\sqrt{\frac{\Gamma(1-\alpha)f'(G(t))}{r(t)\phi(t)\psi(G(t))}}}\right]^2.$$

Using (9) and (10), we get

$$\omega'(t) \le -q(t)\phi(t) + \frac{r(t)\phi(t)}{4k\Gamma(1-\alpha)}\left(\frac{\phi'(t)}{\phi(t)} - \frac{p(t)}{cr(t)}\right)^2. \tag{14}$$

Integrating (14) from t_0 to t, we obtain

$$\phi(t)\frac{r(t)\psi(G(t))D_{0+}^{\alpha}y(t)}{f(G(t))} \le \omega(t_0) - \int_{t_0}^{t}\left[q(s)\phi(s) - \frac{r(s)\phi(s)}{4k\Gamma(1-\alpha)}\left(\frac{\phi'(s)}{\phi(s)} - \frac{p(s)}{cr(s)}\right)^2\right]ds. \tag{15}$$

Dividing (15) by $\phi(t)r(t)$ and then integrating from t_0 to t, we get

$$\int_{G(t_0)}^{G(t)}\frac{\psi(u)}{\Gamma(1-\alpha)f(u)}du \le \omega(t_0)\int_{t_0}^{t}\frac{ds}{\phi(s)r(s)} - \int_{t_0}^{t}\frac{1}{\phi(s)r(s)}\int_{t_0}^{s}\left[q(\tau)\phi(\tau) - \frac{r(\tau)\phi(\tau)}{4k\Gamma(1-\alpha)}\left(\frac{\phi'(\tau)}{\phi(\tau)} - \frac{p(\tau)}{cr(\tau)}\right)^2\right]d\tau ds.$$

By (11) and (12), we obtain

$$\liminf_{t\to\infty}\int_{G(t_0)}^{G(t)}\frac{\psi(u)}{\Gamma(1-\alpha)f(u)}du = -\infty.$$

The rest of the proof is similar to that of Theorem 3.1., hence is omitted

If we choose $\phi(t) \equiv 1$ in Theorem 3.2., then we obtain the following result.

Corollary 3.1. *Let* (4) *holds and suppose that*

$$\psi(x) \ge c > 0 \text{ and } \frac{f'(x)}{\psi(x)} \ge k > 0 \text{ for } x \ne 0, \tag{16}$$

$$\int^{\infty}\frac{ds}{r(s)} < \infty, \tag{17}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)}\int_{t_0}^{s}\left[q(\tau) - \frac{r(\tau)}{4k}\left(\frac{p(\tau)}{cr(\tau)}\right)^2\right]d\tau ds = \infty. \tag{18}$$

Then for every solution y of (1), we have $\lim_{t\to\infty}\inf|y(t)| = 0.$

4. APPLICATIONS

Example 1. Consider the fractional differential equation for $t \ge 1$,

$$\left(e^t\left(\int_0^t (t-s)^{-\alpha}\, y(s)\,ds\right)^2 D_{0+}^{\alpha} y(t)\right)' + e^{2t}\left(\int_0^t (t-s)^{-\alpha}\, y(s)\,ds\right)^4 = 0, \tag{19}$$

with $\alpha = 1/2$. This corresponds to (1) with $r(t)=e^t, \psi(x)=x^2, q(t)=e^{2t}$ and $f(x)=x^4$. All conditions of Theorem 3.1 are satisfied. Then for every nonoscillatory solutions y of (19), we have $\lim_{t\to\infty}\inf |y(t)| = 0.$

Example 2. Consider the fractional differential equation for $t \geq 0$

$$\left(t^3 \exp\left(\int_0^t (t-s)^{-\alpha}\, y(s)\,ds\right) D_{0+}^{\alpha} y(t)\right)' + D_{0+}^{\alpha} y(t) + t^{3/2}\exp\left(2\int_0^t (t-s)^{-\alpha}\, y(s)\,ds\right) = 0, \tag{20}$$

with $\alpha \in (0,1)$. This corresponds to (1) with $r(t)=t^3, \psi(x)=e^x, p(t)=1, q(t)=t^{3/2}$ and $f(x)=e^{2x}$. All conditions of Corollary 3.1 are satisfied. Then for every solutions $y(t)$ of (20), we have $\lim_{t\to\infty}\inf |y(t)| = 0.$

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