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Research Article

CHARACTERIZATIONS OF HELICES BY USING THEIR DARBOUX VECTORS

Mustafa DÜLDÜL[*1], Bahar UYAR DÜLDÜL[2]

[1] *Yildiz Technical University, Department of Mathematics, ISTANBUL*; ORCID: 0000-0002-7306-6006
[2] *Yildiz Technical University, Department of Mathematics and Science Education, ISTANBUL*;
ORCID: 0000-0003-3281-8918

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ABSTRACT

In this study, firstly, new proofs of the theorems which characterize the generalized helices, slant helices and relatively-normal slant helices are presented by using the Darboux vectors of these helices. Also, a characterization of a generalized helix lying on an oriented surface is given in terms of its geodesic curvature, geodesic torsion and normal curvature. Secondly, by defining $\mathbf{B}_1$-slant helix in Euclidean 4-space, we give its characterization.
Keywords: generalized helix, slant helix, Darboux vector, curvatures.

1. INTRODUCTION

This paper includes characterizations of some helical curves in Euclidean 3-space $\mathbb{E}^3$ and Euclidean 4-space $\mathbb{E}^4$. Helical curves have been defined by their Frenet vector fields which make constant angles with fixed directions. Such curves have always been of interest and have been studied and characterized not only in Euclidean spaces but also in non-Euclidean spaces either in 3-space or in higher dimensional spaces. The most encountered helical curve is a generalized helix. Generalized helices which have an important place not only in differential geometry but also in CAGD can also be seen in nature such as a tool of designing highways [15]. One can see such helices also in fractal geometry and in the structure of DNA [2, 13]. Another most attractive helical curve is a slant helix [5]. These curves are the geodesics of helix surfaces [6] and are applied in some applications of quaternion algebra which plays an important role in several areas of physics, such as simulation of particle motion [16]. A slant helix is also related with a special surface curve called isophotic curve. An isophotic curve which plays an important role in visual psychophysics and vision theory is a geodesic if and only if it is a slant helix [4, 7]. Another helical curve which have been recently defined as a relatively-normal slant helix [8] is also related with generalized and slant helices.

Generalized helix [10] and slant helix [5] have been defined by the unit tangent vector and principal normal vector of a regular curve in $\mathbb{E}^3$, respectively. Besides, a relatively-normal slant helix [8] has been defined as a surface curve by the tangent normal vector field of the surface curve. The characterizations of generalized helices, slant helices and relatively-normal slant helices in $\mathbb{E}^3$ are well-known. The characterizations of slant helices and relatively-normal slant helices in $\mathbb{E}^3$ have been proved by using their principal normal and relatively-normal indicatrices, respectively. However, a proof for the sufficiency

*Corresponding Author: e-mail: mduldul@yildiz.edu.tr, tel: (212) 383 43 45

part of the characterization of generalized helices given by [14] identifies a vector field which is linearly dependent with Darboux vector field of the curve. This easy proof led us to look for similar proofs for slant helices and relatively-normal slant helices in $\mathbb{E}^3$.

In addition, generalized (or cylindrical) helix [9, 12], slant helix [1], and $\mathbf{B}_2$-slant helix [11] have been defined by the unit tangent vector, principal normal vector, and second binormal vector of a regular curve in $\mathbb{E}^4$, respectively. However, similar to such helices in $\mathbb{E}^4$, the notion $\mathbf{B}_1$-slant helix in $\mathbb{E}^4$ is missing.

The aim of this paper is, firstly, to give the characterizations of generalized helix, slant helix, and relatively-normal slant helix in $\mathbb{E}^3$ by using their Darboux vectors. It is also aimed to present a characterization of a generalized helix lying on an oriented surface in terms of its geodesic curvature, geodesic torsion and normal curvature. The second aim is to define $\mathbf{B}_1$-slant helix in $\mathbb{E}^4$ and to give its characterization.

2. PRELIMINARIES

2.1. Curves in $\mathbb{E}^3$

Let us consider a curve $\alpha : I \to \mathbb{E}^3$, where $I \subset \mathbb{R}$ is an open interval. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ denotes the Frenet frame of α. The Frenet formulas are then given by

$$\mathbf{t}' = \kappa\mathbf{n}, \quad \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}' = -\tau\mathbf{n}, \tag{1}$$

where κ and τ are curvature and torsion of α, respectively.

It is well-known that $\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$ is called as the Darboux vector field of the curve. The Darboux vector field enables us to rewrite the Frenet formulas as [10]

$$\mathbf{t}' = \mathbf{d} \times \mathbf{t}, \quad \mathbf{n}' = \mathbf{d} \times \mathbf{n}, \quad \mathbf{b}' = \mathbf{d} \times \mathbf{b}. \tag{2}$$

Let us now consider a regular surface $\mathbb{S}$ in $\mathbb{E}^3$. Let $\beta : J \to \mathbf{S}$ be a unit speed curve on $\mathbb{S}$, where J denote an open real interval, and $\mathbf{U}$ denote the unit surface normal along β. Let $\mathbf{T} = \beta'$ and $\mathbf{V} = \mathbf{U} \times \mathbf{T}$ (the vector $\mathbf{V}$ is called as tangent normal vector of β). The frame field $\{\mathbf{T}, \mathbf{V}, \mathbf{U}\}$ is called as Darboux frame. The Darboux formulas are given by

$$\begin{aligned} \mathbf{T}' &= \kappa_g\mathbf{V} + \kappa_n\mathbf{U}, \\ \mathbf{V}' &= -\kappa_g\mathbf{T} + \tau_g\mathbf{U}, \\ \mathbf{U}' &= -\kappa_n\mathbf{T} - \tau_g\mathbf{V}, \end{aligned} \tag{3}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of β [10]. Similarly, the Darboux vector field of β is defined by $\mathbf{D} = \tau_g\mathbf{T} - \kappa_n\mathbf{V} + \kappa_g\mathbf{U}$, and it enables us to rewrite the Darboux formulas as

$$\mathbf{T}' = \mathbf{D} \times \mathbf{T}, \quad \mathbf{V}' = \mathbf{D} \times \mathbf{V}, \quad \mathbf{U}' = \mathbf{D} \times \mathbf{U}. \tag{4}$$

The vector fields

$$\mathbf{D}_n = -\kappa_n\mathbf{V} + \kappa_g\mathbf{U}, \quad \mathbf{D}_r = \tau_g\mathbf{T} + \kappa_g\mathbf{U}, \quad \mathbf{D}_o = \tau_g\mathbf{T} - \kappa_n\mathbf{V}$$

along β are called the normal Darboux vector field, the rectifying Darboux vector field and the osculating Darboux vector field, respectively [4].

Definition 1 (Generalized helix)**.** *Let a regular curve be given in $\mathbb{E}^3$. If its tangential direction makes a constant angle with a fixed direction, then the curve is called a generalized helix.*

Generalized helices are characterized as follows (see [10, 14] for the proof):

Theorem 1. *A curve with $\kappa > 0$ in $\mathbb{E}^3$ is a generalized helix if and only if the ratio of its torsion to the curvature is a non-zero constant, i.e.*

$$\frac{\tau}{\kappa} = constant.$$

Definition 2 (Slant helix)**.** *Let a regular curve be given in $\mathbb{E}^3$. If its principal normal direction makes a constant angle with a fixed direction, then the curve is called a slant helix [5].*

Slant helices are characterized as follows (see [3, 5] for the proof):

Theorem 2. *A space curve with $\kappa > 0$ is a slant helix if and only if*

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = constant.$$

Definition 3 (Relatively-normal slant helix)**.** *A regular curve in $\mathbb{E}^3$ lying on a regular surface $\mathbb{S}$ is called a relatively-normal slant helix if its tangent normal vector makes a constant angle with a fixed unit vector [8].*

Relatively-normal slant helices are characterized as follows (see [8] for the proof):

Theorem 3. *A unit speed curve on a surface $\mathbb{S}$ with $(\tau_g, \kappa_g) \neq (0,0)$ is a relatively normal-slant helix if and only if*

$$\frac{1}{\left(\kappa_g^2+\tau_g^2\right)^{\frac{3}{2}}}\left(\tau_g'\kappa_g - \kappa_g'\tau_g - \kappa_n\left(\kappa_g^2+\tau_g^2\right)\right) = constant.$$

2.2. Curves in $\mathbb{E}^4$

Let γ be a regular curve in $\mathbb{E}^4$ and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ denote its Frenet frame. Then, Frenet formulas along γ is given by

$$\mathbf{T}' = \kappa_1\mathbf{N}, \quad \mathbf{N}' = -\kappa_1\mathbf{T} + \kappa_2\mathbf{B}_1, \quad \mathbf{B}_1' = -\kappa_2\mathbf{N} + \kappa_3\mathbf{B}_2, \quad \mathbf{B}_2' = -\kappa_3\mathbf{B}_1,$$

where $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$, and $\mathbf{B}_2$ denote the tangent, principal normal, first binormal and second binormal vector fields, respectively, and κ_i, $i = 1, 2, 3$ denotes the i-th curvature function of the curve.

Definition 4. *Let γ be a regular curve in $\mathbb{E}^4$, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ denote its Frenet frame and $\kappa_i, i = 1, 2, 3$ denote the i-th curvature. If $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}_2$ make a constant angle with a fixed direction, then the curve γ is called a generalized helix, slant helix, and $\mathbf{B}_2$-slant helix, respectively [1, 9, 11, 12].*

Such curves have been characterized as follows:

A regular curve in $\mathbb{E}^4$ with non-vanishing curvatures is a

- generalized helix if and only if

$$\frac{\kappa_1^2}{\kappa_2^2} + \left[\frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'\right]^2 = \text{constant, [9]}, \tag{5}$$

- slant helix if and only if

$$\left\{\frac{\kappa_2}{\kappa_3} + \frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\int \kappa_1 ds\right)'\right\}^2 + \left(\frac{\kappa_1}{\kappa_2}\int \kappa_1 ds\right)^2 + \left(\int \kappa_1 ds\right)^2 = \text{constant, [1]}, \tag{6}$$

- $\mathbf{B}_2$-slant helix if and only if

$$\frac{\kappa_3^2}{\kappa_2^2} + \left[\frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\right)'\right]^2 = \text{constant, [11]}. \tag{7}$$

3. CHARACTERIZATIONS OF HELICES BY USING THEIR DARBOUX VECTORS

In this section, we reobtain the above given characterizations by using the Darboux vectors of the mentioned helices. The following new proofs show again how important the Darboux vector of a curve is.

3.1. New proof of Theorem 1

($\Rightarrow$:) Let α be a unit speed generalized helix in $\mathbb{E}^3$. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ denote the Frenet frame, and $\mathbf{d}=\tau\mathbf{t}+\kappa\mathbf{b}$ denote the Darboux vector of α. Then, by the definition, there exists a constant angle θ and a fixed unit vector $\mathbf{g}$ such that $\langle \mathbf{t},\mathbf{g}\rangle=\cos\theta$. If we differentiate this equation according to arc-length, we get $\langle \mathbf{n},\mathbf{g}\rangle=0$, i.e. $\mathbf{n}\perp\mathbf{g}$. Since we also have $\mathbf{n}\perp\mathbf{t}$, we obtain $\mathbf{n}$ is parallel to $\mathbf{g}\times\mathbf{t}$. This means

$$\mathbf{n}=\pm\frac{\mathbf{g}\times\mathbf{t}}{||\mathbf{g}\times\mathbf{t}||}=\pm\frac{1}{\sin\theta}\Big(\mathbf{g}\times\mathbf{t}\Big), \tag{8}$$

where $||\cdot||$ denotes the norm of a vector. Since $\mathbf{g}$ and θ are constant, differentiating (8) with respect to arc-length and using Frenet formulas, we get

$$\mathbf{n}'=\pm\frac{1}{\sin\theta}\Big(\mathbf{g}\times\mathbf{t}'\Big)=\pm\frac{\kappa}{\sin\theta}\Big(\mathbf{g}\times\mathbf{n}\Big). \tag{9}$$

Combining $\mathbf{n}'=\mathbf{d}\times\mathbf{n}$ with (9) yields

$$\mathbf{d}=\pm\frac{\kappa}{\sin\theta}\mathbf{g}=\tau\mathbf{t}+\kappa\mathbf{b}.$$

By taking the inner product of both hand sides of the last equation with $\mathbf{t}$ implies that

$$\frac{\tau}{\kappa}=\pm\cot\theta=\text{constant}.$$

($\Leftarrow$:) See [14].

Slant helices are characterized by using the geodesic curvature function of the principal normal indicatrix of the curve. Now, let us give a new method for finding the characterization of a slant helix.

3.2. New proof of Theorem 2

($\Rightarrow$:) Let α be a unit speed slant helix with $\kappa>0$ in $\mathbb{E}^3$. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ denote the Frenet frame and $\mathbf{d}=\tau\mathbf{t}+\kappa\mathbf{b}$ denote the Darboux vector of α. Then, by the definition of slant helix, there exists a constant angle φ and a fixed unit vector $\mathbf{h}$ such that $\langle\mathbf{n},\mathbf{h}\rangle=\cos\varphi$. Differentiating this equation according to arc-length of α yields $\langle\mathbf{n}',\mathbf{h}\rangle=\mathbf{0}$, i.e. $\mathbf{n}'\perp\mathbf{h}$. Since we also have $\mathbf{n}'\perp\mathbf{d}$, we obtain $\mathbf{n}'=\mathbf{d}\times\mathbf{n}$ is parallel to $\mathbf{d}\times\mathbf{h}$. This means $\mathbf{d},\mathbf{n}$, and $\mathbf{h}$ are planar. Since $\langle\mathbf{d},\mathbf{n}\rangle=\mathbf{0}$, we may write

$$\mathbf{h}=\cos\varphi\,\mathbf{n}\pm\sin\varphi\frac{\mathbf{d}}{||\mathbf{d}||}. \tag{10}$$

Differentiating (10) with respect to arc-length and using Frenet formulas gives

$$\mathbf{0}=\left(-\kappa\cos\varphi\pm\frac{\kappa\big(\kappa\tau'-\kappa'\tau\big)}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}\sin\varphi\right)\mathbf{t}+\left(\tau\cos\varphi\pm\frac{\tau\big(\kappa'\tau-\kappa\tau'\big)}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}\sin\varphi\right)\mathbf{b}.$$

Thus, we obtain

$$-\kappa\cos\varphi\pm\frac{\kappa\big(\kappa\tau'-\kappa'\tau\big)}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}\sin\varphi=0,\quad \tau\cos\varphi\pm\frac{\tau\big(\kappa'\tau-\kappa\tau'\big)}{\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}}\sin\varphi=0$$

which yields $\dfrac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\dfrac{\tau}{\kappa}\right)' = \pm\cot\varphi = \text{constant}.$

($\Leftarrow$:) Let's assume that $\sigma = \dfrac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\dfrac{\tau}{\kappa}\right)'$ is constant. Let

$$\mathbf{H} = \sigma\,\mathbf{n} + \frac{\mathbf{d}}{||\mathbf{d}||}.$$

Thus, $||\mathbf{H}|| = 1+\sigma^2$ is constant. If we differentiate $\mathbf{H}$ according to arc-length and employ Frenet formulas, we obtain $\mathbf{H}' = \mathbf{0}$. Then, $\mathbf{H}$ is a nonzero constant vector, and $\langle \mathbf{n}, \mathbf{H}\rangle = \sigma = $ constant which means that $\mathbf{n}$ makes a constant angle with $\mathbf{H}$, i.e. α is a slant helix.

Similar to slant helices, relatively-normal slant helices are characterized by using the constancy of the geodesic curvature function of the spherical image of tangent normal vector. Now, let us give a new method for finding the characterization of a relatively-normal slant helix lying on a regular surface in $\mathbb{E}^3$.

3.3. New proof of Theorem 3

($\Rightarrow$:) Let β be a unit speed relatively-normal slant helix with $(\tau_g, \kappa_g) \neq (0,0)$ on a regular surface $\mathbb{S} \subset \mathbb{E}^3$. Let $\{\mathbf{T}, \mathbf{V}, \mathbf{U}\}$ denote the Darboux frame and $\mathbf{D} = \tau_g\mathbf{T} - \kappa_n\mathbf{V} + \kappa_g\mathbf{U}$ denote the Darboux vector of β. Then, by the definition of relatively-normal slant helix, there exists a constant angle ψ and a fixed unit vector $\mathbf{r}$ such that $\langle \mathbf{V}, \mathbf{r}\rangle = \cos\psi$. If we differentiate this equation according to arc-length of β, we get $\langle \mathbf{V}', \mathbf{r}\rangle = 0$, i.e. $\mathbf{V}' \perp \mathbf{r}$. Since we also have $\mathbf{V}' \perp \mathbf{D}_r$, we obtain $\mathbf{V}' = \mathbf{D}_r \times \mathbf{V}$ is parallel to $\mathbf{r} \times \mathbf{D}_r$. This means $\mathbf{r}, \mathbf{D}_r$, and $\mathbf{V}$ are planar. Since $\langle \mathbf{D}_r, \mathbf{V}\rangle = \mathbf{0}$, we may write

$$\mathbf{r} = \cos\psi\,\mathbf{V} \pm \sin\psi\frac{\mathbf{D}_r}{||\mathbf{D}_r||}. \tag{11}$$

Differentiating (11) with respect to arc-length of β and using Darboux formulas gives

$$\begin{aligned}\mathbf{0} = &\left(-\kappa_g\cos\psi \pm \frac{\kappa_g\left(\kappa_g\tau_g' - \kappa_g'\tau_g - \kappa_n(\kappa_g^2+\tau_g^2)\right)}{\left(\kappa_g^2+\tau_g^2\right)^{\frac{3}{2}}}\sin\psi\right)\mathbf{T}\\ &+\left(\tau_g\cos\psi \pm \frac{\tau_g\left(\kappa_g'\tau_g - \kappa_g\tau_g' + \kappa_n(\kappa_g^2+\tau_g^2)\right)}{\left(\kappa_g^2+\tau_g^2\right)^{\frac{3}{2}}}\sin\psi\right)\mathbf{U}\end{aligned}$$

from which we have

$$-\kappa_g\cos\psi \pm \frac{\kappa_g\left(\kappa_g\tau_g' - \kappa_g'\tau_g - \kappa_n(\kappa_g^2+\tau_g^2)\right)}{\left(\kappa_g^2+\tau_g^2\right)^{\frac{3}{2}}}\sin\psi = 0,$$

$$\tau_g\cos\psi \pm \frac{\tau_g\left(\kappa_g'\tau_g - \kappa_g\tau_g' + \kappa_n(\kappa_g^2+\tau_g^2)\right)}{\left(\kappa_g^2+\tau_g^2\right)^{\frac{3}{2}}}\sin\psi = 0.$$

The assumption $(\tau_g, \kappa_g) \neq (0,0)$ yields

$$\frac{1}{\left(\kappa_g^2+\tau_g^2\right)^{\frac{3}{2}}}\left(\tau_g'\kappa_g - \kappa_g'\tau_g - \kappa_n\left(\kappa_g^2+\tau_g^2\right)\right) = \pm\cot\psi = \text{constant}$$

which completes the proof.

($\Leftarrow$:) Let's assume that $\rho = \dfrac{1}{\left(\kappa_g^2+\tau_g^2\right)^{\frac{3}{2}}}\left(\tau_g'\kappa_g - \kappa_g'\tau_g - \kappa_n\left(\kappa_g^2+\tau_g^2\right)\right)$ is constant. Let

$$\mathbf{R} = \rho\,\mathbf{V} + \frac{\mathbf{D}_r}{||\mathbf{D}_r||}.$$

Thus, $||\mathbf{R}|| = 1+\rho^2$ is constant. Differentiating $\mathbf{R}$ according to arc-length and employing Frenet formulas yields $\mathbf{R}' = \mathbf{0}$. Then, $\mathbf{R}$ is a nonzero constant vector, and $\langle \mathbf{V}, \mathbf{R}\rangle = \rho =$ constant which means that $\mathbf{V}$ makes a constant angle with $\mathbf{R}$, i.e. β is a relatively-normal slant helix.

Now, by using the Darboux vector, let us give a characterization of a generalized helix lying on an oriented surface in terms of its geodesic curvature, geodesic torsion and normal curvature.

Theorem 4. *A unit speed curve on a surface $\mathbb{S}$ with $(\kappa_n, \kappa_g) \neq (0,0)$ is a generalized helix if and only if*

$$\frac{1}{\left(\kappa_n^2+\kappa_g^2\right)^{\frac{3}{2}}}\Big(\kappa_g'\kappa_n - \kappa_g\kappa_n' - \tau_g\big(\kappa_n^2+\kappa_g^2\big)\Big) = \textit{constant}.$$

Proof. ($\Rightarrow$:) Let β be a unit speed generalized helix with $(\kappa_n, \kappa_g) \neq (0,0)$ on a regular surface $\mathbb{S} \subset \mathbb{E}^3$. Let $\{\mathbf{T}, \mathbf{V}, \mathbf{U}\}$ denote the Darboux frame of β. Then, by definition of generalized helix, there exists a constant angle ϕ and a fixed unit vector $\mathbf{c}$ such that $\langle \mathbf{T}, \mathbf{c}\rangle = \cos\phi$. If we differentiate this equation according to arc-length of β, we get $\langle \mathbf{T}', \mathbf{c}\rangle = 0$, i.e. $\mathbf{T}' \perp \mathbf{c}$. Since we also have $\mathbf{T}' \perp \mathbf{D}_n$, we obtain $\mathbf{T}' = \mathbf{D}_n \times \mathbf{T}$ is parallel to $\mathbf{c} \times \mathbf{D}_n$. This means $\mathbf{c}, \mathbf{D}_n$, and $\mathbf{T}$ are planar. Since $\langle \mathbf{D}_n, T\rangle = 0$, we may write

$$\mathbf{c} = \cos\phi\,\mathbf{T} \pm \sin\phi\frac{\mathbf{D}_n}{||\mathbf{D}_n||}. \tag{12}$$

Differentiating (12) with respect to arc-length and using Darboux formulas gives

$$\begin{aligned}\mathbf{0} = &\left(\kappa_g\cos\phi \pm \frac{\kappa_g\big(\kappa_n\kappa_g' - \kappa_n'\kappa_g - \tau_g(\kappa_g^2+\kappa_n^2)\big)}{\left(\kappa_g^2+\kappa_n^2\right)^{\frac{3}{2}}}\sin\phi\right)\mathbf{V}\\ &+\left(\kappa_n\cos\phi \pm \frac{\kappa_n\big(\kappa_n\kappa_g' - \kappa_n'\kappa_g - \tau_g(\kappa_g^2+\kappa_n^2)\big)}{\left(\kappa_g^2+\kappa_n^2\right)^{\frac{3}{2}}}\sin\phi\right)\mathbf{U}.\end{aligned}$$

Thus, the assumption $(\kappa_n, \kappa_g) \neq (0,0)$ yields the desired result.
($\Leftarrow$:) Let's assume that $\xi = \dfrac{1}{\left(\kappa_g^2+\kappa_n^2\right)^{\frac{3}{2}}}\Big(\kappa_g'\kappa_n - \kappa_g\kappa_n' - \tau_g\big(\kappa_g^2+\kappa_n^2\big)\Big)$ is constant. Let

$$\mathbf{C} = \xi\,\mathbf{T} - \frac{\mathbf{D}_n}{||\mathbf{D}_n||}.$$

Thus, the rest of the proof follows as previously. □

Remark 1. *The constant function given in Theorem 4 is equal to $\dfrac{-\delta_n}{\sqrt{\kappa_g^2+\kappa_n^2}}$, where δ_n is defined by [4].*

4. $\mathbf{B}_1$-SLANT HELIX IN $\mathbb{E}^4$

In Euclidean 4-space, generalized helix, slant helix, and $\mathbf{B}_2$-slant helix have been defined by the unit tangent vector, principal normal vector, and second binormal vector of a regular curve, respectively. However, similar to these curves, a new type of curve which is missing in the literature can be defined by using the first binormal vector of a regular curve.

Definition 5. *Let γ be a regular curve with nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$ in $\mathbb{E}^4$. γ is called a $\mathbf{B}_1$-slant helix if its first binormal vector $\mathbf{B}_1$ makes a constant angle with a fixed direction.*

Theorem 5. *Let γ be a regular curve with nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$ in $\mathbb{E}^4$. γ is a $\mathbf{B}_1$-slant helix if and only if*

$$\left\{\frac{\kappa_2}{\kappa_1} + \frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)'\right\}^2 + \left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)^2 + \left(\int \kappa_3 ds\right)^2 \tag{13}$$

is a constant function.

Proof. ($\Rightarrow$:) Let γ be a $\mathbf{B}_1$-slant helix and s denote its arc-length. Then, there exists a fixed unit vector $\mathbf{w}$ and a constant angle ξ such that $\langle \mathbf{B}_1, \mathbf{w}\rangle = \cos\xi$. Differentiating this equation according to s and using Frenet formulas in $\mathbb{E}^4$ gives $\langle -\kappa_2\mathbf{N} + \kappa_3\mathbf{B}_2, \mathbf{w}\rangle = 0$ or

$$\langle \mathbf{N}, \mathbf{w}\rangle = \frac{\kappa_3}{\kappa_2}\langle \mathbf{B}_2, \mathbf{w}\rangle. \tag{14}$$

On the other hand, by using $\langle \mathbf{B}_1, \mathbf{w}\rangle = \cos\xi$, we may write

$$\langle -\kappa_3\mathbf{B}_1, \mathbf{w}\rangle = -\kappa_3\cos\xi \quad \text{or} \quad \langle \mathbf{B}_2', \mathbf{w}\rangle = -\kappa_3\cos\xi$$

which yields

$$\langle \mathbf{B}_2, \mathbf{w}\rangle = -\cos\xi\int \kappa_3 ds. \tag{15}$$

Then, from (14) and (15), we obtain

$$\langle \mathbf{N}, \mathbf{w}\rangle = -\cos\xi\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds. \tag{16}$$

Furthermore, by using $\langle \mathbf{B}_1, \mathbf{w}\rangle = \cos\xi$, we may also write $\langle \kappa_2\mathbf{B}_1, \mathbf{w}\rangle = \kappa_2\cos\xi$ or $\langle \mathbf{N}' + \kappa_1\mathbf{T}, \mathbf{w}\rangle = \kappa_2\cos\xi$. This equation yields

$$\langle \mathbf{T}, \mathbf{w}\rangle = \frac{\kappa_2}{\kappa_1}\cos\xi - \frac{1}{\kappa_1}\langle \mathbf{N}', \mathbf{w}\rangle. \tag{17}$$

If we differentiate (16) with respect to s and substitute the result into (17), we get

$$\langle \mathbf{T}, \mathbf{w}\rangle = \left\{\frac{\kappa_2}{\kappa_1} + \frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)'\right\}\cos\xi. \tag{18}$$

If we differentiate (18) according to s and consider (16), we also have

$$\frac{d}{ds}\left\{\frac{\kappa_2}{\kappa_1} + \frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)'\right\} = -\frac{\kappa_1\kappa_3}{\kappa_2}\int \kappa_3 ds. \tag{19}$$

Thus, since $\mathbf{w}$ has unit length, by using

$$\langle \mathbf{T}, \mathbf{w}\rangle^2 + \langle \mathbf{N}, \mathbf{w}\rangle^2 + \langle \mathbf{B}_1, \mathbf{w}\rangle^2 + \langle \mathbf{B}_2, \mathbf{w}\rangle^2 = 1,$$

we may write

$$\left\{\frac{\kappa_2}{\kappa_1} + \frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)'\right\}^2 + \left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)^2 + \left(\int \kappa_3 ds\right)^2 = \tan^2\xi = \text{constant}.$$

($\Leftarrow$:) We assume that the function given in (13) is constant. We denote this constant with $\tan^2\xi$. Let us consider the unit vector

$$\cos\xi\left[\left\{\frac{\kappa_2}{\kappa_1} + \frac{1}{\kappa_1}\left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)'\right\}\mathbf{T} + \left(\frac{\kappa_3}{\kappa_2}\int \kappa_3 ds\right)\mathbf{N} + \mathbf{B}_1 + \left(\int \kappa_3 ds\right)\mathbf{B}_2\right]$$

defined along the curve γ. Differentiating this vector with respect to s and using (19) shows that it is a fixed vector which makes the constant angle ξ with the first binormal vector $\mathbf{B}_1$ of γ. Then γ is a $\mathbf{B}_1$-slant helix. □

Remark 2. *By using the same method presented above, we can obtain the characterizations of generalized helix, slant helix and* $\mathbf{B}_2$*-slant helix in* $\mathbb{E}^4$ *easier than the earlier given proofs in [1, 9, 11].*

Theorem 6. *Let* γ *be a space curve in* $\mathbb{E}^4$ *with* $\kappa_1(s) = \kappa_3(s)$*. Then* γ *is a* $\mathbf{B}_1$*-slant helix if and only if* γ *is a slant helix.*

Proof. The proof can be seen easily by using the characterizations of these curves given by (6) and (13). □

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