Sigma J Eng & Nat Sci 38 (3), 2020, 1307-1319

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

**Research Article**

# A NUMERICAL ALGORITHM BASED ON ULTRASPHERICAL WAVELETS FOR SOLUTION OF LINEAR AND NONLINEAR KLEIN-GORDON EQUATIONS

**Neslihan OZDEMIR*[1], Aydin SECER[2]**

[1]*Yildiz Technical University, Dept. of Mathematical Engineering, ISTANBUL;* ORCID: 0000-0003-1649-0625
[2]*Yildiz Technical University, Dept. of Mathematical Engineering, ISTANBUL;* ORCID: 0000-0002-8372-2441

**Received: 08.04.2020 Accepted: 20.08.2020**

**ABSTRACT**

In this paper, Galerkin method based on the Ultraspherical wavelets expansion together with operational matrix of integration is developed to solve linear and nonlinear Klein Gordon (KG) equations with the given initial and boundary conditions. Firstly, we present the ultraspherical wavelets, then the corresponding operational matrix of integration is presented. To transform the given PDE into a system of linear-nonlinear algebraic equations which can be efficiently solved by suitable solvers, we utilize the operational matrix of integration and both properties of Ultraspherical wavelets. The applicability of the method is shown by two test problems and acquired results show that the method is good accuracy and efficiency.
**Keywords:** Ultraspherical wavelets, Klein-Gordon equation, Galerkin method, operational matrix of integration.

## 1. INTRODUCTION

The goal of this paper is to present a numerical solution by means of the Ultraspherical wavelet Galerkin method for the following Klein Gordon (KG) equation which has the nonlinear term as [1]:

$$\frac{\partial^2 u(\eta,t)}{\partial t^2}+\gamma_1\frac{\partial^2 u(\eta,t)}{\partial \eta^2}+\gamma_2 u(\eta,t)+\gamma_3 u^2(\eta,t)= f(\eta,t), \qquad \eta\in[0,1],\ t\in[0,1]$$

subject to initial and boundary conditions

$$u(\eta,0)=0, \quad u_t(\eta,0)=0 \qquad \eta\in\left[0,1\right]$$

and

$$u(0,t)=h_1(t), \quad u(1,t)=h_2(t), \quad t\in[0,1]$$

where $h_1(t)$ and $h_2(t)$ are known functions, $\gamma_1,\gamma_2$ and $\gamma_3$ are constants . The Klein-Gordon (KG) equation is basically a relativistic wave equation version of the Schrödinger equation. It has

* Corresponding Author: e-mail: ozdemirn@yildiz.edu.tr, tel: (212) 479 07 91

wide applications in many scientific fields, such as nonlinear optics, fluid dynamics, solid state physics, and quantum mechanics [2].

As a kind of essential n onlinear PDEs, the KG type equations have been studied to get both analytical and numerical solutions in different studies. Analytical and numerical solutions of KG equations were presented by using the taylor matrix method [3], the the Adomian's decomposition and variational iterative methods [4], the lattice boltzmann method, the exponential cubic b-spline collocation method [5], the perturbation iteration technique and optimal perturbation iteration method [6], the variational iteration method [7], the finite-difference method [8], the decomposition method [9], the transformation and Exp-function method and comparison with Adomian’s method [10], the variational iteration method combined with the Exp-function method [11].

Methods based on wavelets have been used to obtain numerical solutions of differential equations over the past 30 years. Up to now, a large number of papers focus on this topic. Some of methods used these paper are the Gegenbauer wavelets based on methods [12,13], the Legendre Wavelet Operational Matrix Method [14], the Cheybshev wavelet collocation method [15], the Legendre wavelet method [16], wavelets Galerkin method [17], the modified Laguerre wavelet based Galerkin method [18], Genocchi wavelet method [19] and the discontinuous Legendre wavelet Galerkin method [20]. In this study, Galerkin method based on ultraspherical wavelets was used to obtain the numerical solution of linear and nonlinear KG equations. The proposed method presents an understandable algorithm to reduces KG equations and transforms such equations to a system of algebraic equations, which is the most important advantage of the proposed method.

## 2. ULTRASPHERICAL POLYNOMIALS AND ULTRASPHERICAL WAVELETS

Ultraspherical polynomials $C_n^{\beta}(\eta)$ is defined on the interval $\left[-1,1\right]$ and Ultraspherical polynomials can be obtained by the following recurrence relations [21]:

$$C_0^{\beta}(\eta)=1, C_1^{\beta}(\eta)=2\beta\eta,$$
$$C_{n+1}^{\beta}(\eta)=\frac{1}{n+1}\left(2\left(n+\beta\right)\eta C_n^{\beta}(\eta)-(n+2\beta-1)C_{n-1}^{\beta}(\eta)\right),\ n\geq 1, \beta>-\frac{1}{2}.$$

Some properties of Ultraspherical polynomials are

$$\frac{d}{d\eta}\left(C_n^{\beta}\left(\eta\right)\right)=2\beta C_{n-1}^{\beta+1}\left(\eta\right),\ \frac{d^k}{d\eta^k}\left(C_n^{\beta}\left(\eta\right)\right)=2^k\beta^k C_{n-k}^{\beta+k}\left(\eta\right),\ n\geq 1$$

$$\left(n+\beta\right)C_n^{\beta}\left(\eta\right)=\beta\left(C_n^{\beta+1}\left(\eta\right)-C_{n-2}^{\beta+1}\left(\eta\right)\right),\ n\geq 2$$

$$\frac{d}{d\eta}\left(C_{n+1}^{\beta}\left(\eta\right)-C_{n-1}^{\beta}\left(\eta\right)\right)=2\beta\left(C_n^{\beta+1}\left(\eta\right)-C_{n-2}^{\beta+1}\left(\eta\right)\right)=2\left(n+\beta\right)C_n^{\beta}\left(\eta\right).$$

The equation given as

$$\int\left(1-\eta^2\right)^{\beta-1/2}C_n^{\beta}\left(\eta\right)d\eta=-\frac{2\beta\left(1-\eta^2\right)^{\beta+1/2}}{n\left(n+2\beta\right)}C_{n-1}^{\beta+1}\left(\eta\right),\ \ n\geq 1.$$

is obtained from Rodrigues formula [21].

Ultraspherical polynomials are orthogonal with respect to the weight function $\omega(\eta)=\left(1-\eta^2\right)^{\beta-\frac{1}{2}}$; that is,

$$\int_{-1}^{1}\left(1-\eta^2\right)^{\beta-\frac{1}{2}} C_m^\beta(\eta) C_n^\beta(\eta)\,d\eta = K_n^\beta \delta_{nm}, \beta > -\frac{1}{2}$$

in which $K_n^\beta = \dfrac{\pi 2^{1-2\beta}\Gamma(n+2\beta)}{n!(n+\beta)\left(\Gamma(\beta)\right)^2}$ is called the norm alizing factor, and $\delta_{nm}$ is the Kronecker delta function.

Legendre polynomials and Chebyshev polynomials are special types of Ultraspherical polynomials. For $\beta = 0$, $\beta = 1/2$ and $\beta = 1$, we obtain the first- kind Chebyshev polynomials, Legendre polynomials, the second- kind Chebyshev polynomials, respectively.

The basic wavelet (Mother wavelet) is given on the basis of scaling and translation parameters as:

$$\psi_{p,q}(\eta) = \frac{1}{\sqrt{|p|}}\psi\left(\frac{\eta - q}{p}\right), p, q \in R, p \neq 0,$$

in which $p$ and $q$ are the scaling and translation parameters, respectively. By restricting $p, q$ to discrete values as: $p = p_0^{-k}, q = nq_0 p_0^{-k}$, where $p_0 > 1, q_0 > 0$ and $k, n \in \mathbb{N}$, the following discrete wavelets are obtained:

$$\psi_{k,n}(\eta) = \left(p_0\right)^{\frac{k}{2}}\psi\left(p_0^k \eta - nq_0\right)$$

in which an orthogonal basis of $L_2(R)$ is formed. If $p_0 = 2$ and $q_0 = 1$, then $\psi_{k,n}$ forms an orthonormal basis.

The discrete wavelet transform is defined as

$$\psi_{k,n}(\eta) = (2)^{\frac{k}{2}}\psi\left(2^k \eta - n\right).$$

Ultraspherical wavelets are defined on the interval $[0,1]$ by

$$\psi_{n,m}^\beta(\eta) = \begin{cases} \dfrac{1}{\sqrt{K_m^\beta}} 2^{\frac{k}{2}} C_m^\beta\left(2^k \eta - \hat{n}\right), & \dfrac{\hat{n}-1}{2^k} \le \eta \le \dfrac{\hat{n}+1}{2^k} \\ 0, & elsewhere \end{cases},$$

in which $C_m^\beta\left(2^k \eta - \hat{n}\right)$ is Ultraspherical polynomials of degree $m$, $k = 1,2,3,...,$ is the level of resolution, $n = 1,2,3,...,2^{k-1}, \hat{n} = 2n-1,$ is the translation parameter, and $m = 0,1,2,...,M-1$ is the order of the *Ultraspherical* polynomials, $M > 0$. Corresponding to each $\beta > -\frac{1}{2}$, a different wavelet family is obtained, i.e., when $\beta = \frac{1}{2}$, Ultraspherical wavelets are identical with Legendre wavelets. For $\beta = 0$ and $\beta = 1$, we get the Chebyshev wavelets of the first kind and the Chebyshev wavelets of the second kind, respectively. In this study, we use the Ultraspherical wavelets at the values $\beta = \frac{1}{2}$ and $\beta = \frac{3}{2}$.

Ultraspherical wavelets' the weight function is given as follows:

$$\omega_n(\eta)=\begin{cases}\omega\left(2^k\eta-2n+1\right)=\left(1-\left(2^k\eta-2n+1\right)^2\right)^{\beta-\frac{1}{2}}, & \eta\in\left[\dfrac{n-1}{2^{k-1}},\dfrac{n}{2^{k-1}}\right].\\ 0, & otherwise\end{cases}$$

## 3. FUNCTION APPROXIMATION

$u(\eta)\in L^2[0,1]$ can be expanded in terms of Ultraspherical wavelets as:

$$u(\eta)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}^{\beta}\psi_{n,m}^{\beta}(\eta) \tag{1}$$

where $c_{n,m}^{\beta}$ values are wavelet coefficients, and $c_{n,m}^{\beta}$ wavelet coefficients are calculated by

$$c_{n,m}^{\beta}=\left\langle u(\eta),\psi_{n,m}^{\beta}(\eta)\right\rangle_{\omega_n}.$$

We approximate infinite series expansion in equation (1) by truncated series as:

$$u(\eta)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}^{\beta}\psi_{n,m}^{\beta}(\eta)=\left(C^{\beta}\right)^T\Psi^{\beta}(\eta) \tag{2}$$

in which the matrices $\Psi^{\beta}(\eta)$ and C are of order $2^{k-1}M\times 1$.

Equation (2) can be also expressed as:

$$u(\eta)=\sum_{i=1}^{\tilde{m}}c_i^{\beta}\psi_i^{\beta}(\eta) \tag{3}$$

where $\tilde{m}=\left(2^{k-1}M\right)$, $C^{\beta}\triangleq\left[c_1,c_2,...,c_{\tilde{m}}\right]^T$,

$$\Psi^{\beta}(\eta)\triangleq\left[\psi_1^{\beta}(\eta),...,\psi_{\tilde{m}}^{\beta}(\eta)\right]^T \tag{4}$$

and we use the relation $i=M(n-1)+m+1$ to find the index $i$.

Similarly, $u(\eta,t)\in L^2\left([0,1]\times[0,1]\right)$ can be approximated in terms of Ultraspherical wavelet as:

$$u(\eta,t)=\sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{m}}u_{i,j}^{\beta}\psi_i^{\beta}(\eta)\psi_j^{\beta}(t)=\left(\Psi^{\beta}\right)^T(\eta)U\Psi^{\beta}(t) \tag{5}$$

in which $u_{i,j}$ wavelets coefficients can be calculated by

$$u_{i,j}^{\beta}=\left\langle\psi_i^{\beta}(\eta),\left\langle u(\eta,t),\psi_j^{\beta}(t)\right\rangle_{\omega_n}\right\rangle_{\omega_n}$$

By substituting the collocation points $\eta_i=\dfrac{2i-1}{2\tilde{m}}, i=1,2,...,\tilde{m}$ into equation (4), we obtain the following Ultraspherical wavelet matrix $\Phi_{\tilde{m}\times\tilde{m}}$:

$$\Phi_{\tilde{m}\times\tilde{m}} = \left[\Psi^{\beta}\left(\frac{1}{2\tilde{m}}\right), \Psi^{\beta}\left(\frac{3}{2\tilde{m}}\right), ..., \Psi^{\beta}\left(\frac{2\tilde{m}-1}{2\tilde{m}}\right)\right]. \tag{6}$$

**Theorem 2.2 (Convergence Theorem)** *A function* $u(\eta,t) \in L^2(\mathbb{R}\times\mathbb{R})$ *defined on* $[0,1]\times[0,1]$ *can be expanded as an infinite series of Ultraspherical wavelets, which converges uniformly to* $u(\eta,t)$, *provided* $u(\eta,t)$ *has bounded mixed fourth partial derivative* $\left|\frac{\partial^4 u(\eta,t)}{\partial\eta^2\partial t^2}\right| \le M$.

**Proof: See([12]).**

## 4. BLOCK PULSE FUNCTIONS(BPFS)

Block pulse functions (BPFs) constitute a complete set of orthogonal functions [22], which are defined on the interval $[0,b)$ by

$$b_i(\eta) = \begin{cases} 1, & \frac{i-1}{\hat{m}} \le \eta < \frac{i}{\hat{m}}b \\ 0, & otherwise \end{cases}, i = 1, 2, ..., \hat{m}.$$

An arbitrary function $u(\eta)$ on the interval $[0,b)$ can be represented by BPFs as:

$$u(\eta) \simeq \zeta^T B_{\hat{m}}(\eta)$$

where

$$\zeta^T = [u_1, u_2, ..., u_{\hat{m}}]$$
$$B_{\hat{m}} = [b_1(\eta), b_2(\eta), ..., b_{\tilde{m}}(\eta)]$$

in which $u_i$ variables are the coefficients of the block pulse function which are calculated by using the following relation:

$$u_i = \frac{\tilde{m}}{b}\int_0^b u(\eta)b_i(\eta)d\eta = \frac{m}{b}\int_{((i-1)/m)b}^{(i/m)b} u(\eta)b_i(\eta)d\eta.$$

**Lemma 1.** *Suppose that* $f(\eta)$ *and* $g(\eta)$ *are two absolutely integrable functions, and these functions may be represented in terms of block pulse functions as:*

$$f(\eta) = F^T B(\eta)$$
$$g(\eta) = G^T B(\eta).$$

*Then,*

$$f(\eta)g(\eta) = F^T B(\eta)B^T(\eta)G = HB(\eta)$$

*where* $H = F^T \otimes G^T$ [23].

**Lemma 2.** *Suppose that $f(\eta,t)$ and $g(\eta,t)$ are two absolutely integrable functions, and these functions may be represented in terms of block pulse functions as:*

$$f(\eta,t)g(\eta,t) = B^T(\eta)HB(\eta)$$

*where $H = F \otimes G$ [23].*

### 4.1. Nonlinear Term Approximation by Ultraspherical Wavelets

Ultraspherical wavelets may be represented [23] with an $\tilde{m}$ -set of block pulse functions as:

$$\Psi^{\beta}(t) = \Phi_{\tilde{m}\times\tilde{m}} B_{\tilde{m}}(t). \tag{7}$$

The operational matrix of the product of Ultraspherical wavelets can be calculated by using the properties of BPFs. The absolutely integrable $f_1(\eta,t)$ and $f_2(\eta,t)$ functions can be represented by Ultraspherical wavelets as:

$$f_1(\eta,t) = \left(\Psi^{\beta}\right)^T(\eta) F_1 \Psi^{\beta}(t) \tag{8}$$

and

$$f_2(\eta,t) = \left(\Psi^{\beta}\right)^T(\eta) F_2 \Psi^{\beta}(t). \tag{9}$$

From equation (7), equations (8)- (9) are rewritten as:

$$\begin{aligned} f_1(\eta,t) &= \left(\Psi^{\beta}\right)^T(\eta) F_1 \Psi^{\beta}(t) = B^T(\eta)\Phi^T_{\tilde{m}x\tilde{m}} F_1 \Phi_{\tilde{m}x\tilde{m}} B(t) = B^T(\eta) F_a B(t) \\ f_2(\eta,t) &= \left(\Psi^{\beta}\right)^T(\eta) F_2 \Psi^{\beta}(t) = B^T(\eta)\Phi^T_{\tilde{m}x\tilde{m}} F_2 \Phi_{\tilde{m}x\tilde{m}} B(t) = B^T(\eta) F_b B(t) \end{aligned} \tag{10}$$

where $F_a = \Phi^T_{\tilde{m}x\tilde{m}} F_1 \Phi_{\tilde{m}x\tilde{m}}$ and $F_b = \Phi^T_{\tilde{m}x\tilde{m}} F_2 \Phi_{\tilde{m}x\tilde{m}}$. Let $F_3 = F_a \otimes F_b$, then

$$\begin{aligned} f_1(\eta,t) f_2(\eta,t) &= B^T(\eta) F_3 B(t) \\ &= B^T(\eta)\Phi^T_{\tilde{m}x\tilde{m}} inv(\Phi^T_{\tilde{m}x\tilde{m}}) F_3 inv\left(\Phi_{\tilde{m}x\tilde{m}}\right)\Phi_{\tilde{m}x\tilde{m}} B(t) \\ &= \left(\Psi^{\beta}\right)^T(\eta) F_4 \Psi^{\beta}(t) \end{aligned}$$

where $F_4 = inv(\Phi^T_{\tilde{m}x\tilde{m}}) F_3 inv\left(\Phi_{\tilde{m}x\tilde{m}}\right)$.

## 5. OPERATIONAL MATRIX OF THE GENERAL-ORDER INTEGRATION

The integration of the vector $\Psi^{\beta}(\eta)$, which is given in (4), can be approximated as:

$$\int_0^{\eta} \Psi^{\beta}(\xi)\, d\xi \approx P\Psi^{\beta}(\eta)$$

where $P$ is called the operational matrix of integration for Ultraspherical wavelets. As given in [15], the matrix $P$ is defined as:

$$P \simeq \Phi_{\hat{m}\times\hat{m}} \tilde{P} \Phi^{-1}_{\hat{m}\times\hat{m}}$$

where the $\hat{m}\times\hat{m}$ matrix $\tilde{P}$ is called the operational matrix of integration for BPFs and is taken in references in [24-25] as:

$$\tilde{P}=\frac{1}{2\tilde{m}}\begin{bmatrix}1 & 2 & 2 & \dots & 2\\ 0 & 1 & 2 & \dots & 2\\ 0 & 0 & 1 & \dots & 2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}.$$

## 6. ULTRASPHERICAL WAVELET GALERKIN METHOD (UWGM)

The Ultraspherical wavelet expansion, together with the operational matrix of integration, is used to solve the following Klein-Gordon equation:

$$\frac{\partial^2 u(\eta,t)}{\partial t^2}+\gamma_1\frac{\partial^2 u(\eta,t)}{\partial \eta^2}+\gamma_2 u(\eta,t)+\gamma_3 u^2(\eta,t)=f(\eta,t),\qquad \eta\in[0,1],\ t\in[0,1]$$

with initial and boundary conditions

$$u(\eta,0)=0,\quad u_t(\eta,0)=0\qquad \eta\in\left[0,1\right]$$

and

$$u(0,t)=h_1(t),\quad u(1,t)=h_2(t),\quad t\in[0,1].$$

For solving this system, by integrating this equation two times with respect to $t$ and consider initial conditions, the integral form of the Klein-Gordon is obtained as follows:

$$\frac{\partial u(\eta,t)}{\partial t}=\left.\frac{\partial u(\eta,t)}{\partial t}\right|_{t=0}-\gamma_1\int_0^t\frac{\partial^2 u(\eta,\tau)}{\partial\eta^2}d\tau-\gamma_2\int_0^t u(\eta,\tau)d\tau-\gamma_3\int_0^t u^2(\eta,\tau)d\tau+\int_0^t f(\eta,\tau)d\tau$$

$$u(\eta,t)=-\gamma_1\int_0^t\int_0^t\frac{\partial^2 u(\eta,\tau)}{\partial\eta^2}d\tau-\gamma_2\int_0^t\int_0^t u(\eta,\tau)d\tau-\gamma_3\int_0^t\int_0^t u^2(\eta,\tau)d\tau+\int_0^t\int_0^t f(\eta,\tau)d\tau \tag{11}$$

Now, we approximate $\frac{\partial^2 u(x,t)}{\partial\eta^2}$ by the Ultraspherical wavelets as follows:

$$\frac{\partial^2 u(\eta,t)}{\partial\eta^2}\simeq\sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}}u_{ij}^{\beta}\psi_i^{\beta}(\eta)\psi_j^{\beta}(t)=\left(\Psi^{\beta}(\eta)\right)^T U\Psi^{\beta}(t). \tag{12}$$

Here, $U=\left[u_{ij}\right]_{\hat{m}\times\hat{m}}$ is an unknown matrix which should be found. When we integrate Equation (12) two times with respect to $\eta$, we obtain:

$$\frac{\partial u(\eta,t)}{\partial\eta}=\left.\frac{\partial u(\eta,t)}{\partial\eta}\right|_{\eta=0}+\left(\Psi^{\beta}(\eta)\right)^T P^T U\Psi^{\beta}(t) \tag{13}$$

and

$$u(\eta,t) = u(0,t) + \eta\left(\frac{\partial u(\eta,t)}{\partial \eta}\Big|_{\eta=0}\right) + \left(\Psi^{\beta}(\eta)\right)^{T}\left(P^{2}\right)^{T} U\Psi^{\beta}(t). \tag{14}$$

When we put $\eta = 1$ in Equation (14) and the boundary conditions is used, we have:

$$\frac{\partial u(\eta,t)}{\partial \eta}\Big|_{\eta=0} = h_2(t) - h_1(t) - \Psi^{\beta}(1)^{T}\left(P^{2}\right)^{T} U\Psi^{\beta}(t). \tag{15}$$

$h_1(t)$ and $h_2(t)$ can be expressed by a terminated Ultraspherical wavelet series at the value $\hat{m}$ as follows:

$$\begin{aligned} h_1(t) &= H_1^T\Psi^{\beta}(t) \\ h_2(t) &= H_2^T\Psi^{\beta}(t) \end{aligned} \tag{16}$$

in which $H_1$ and $H_2$ are the Ultraspherical wavelet coefficients vectors. If we substitute Equation (16) into Equation (15), we obtain:

$$\frac{\partial u(\eta,t)}{\partial \eta}\Big|_{\eta=0} \simeq \left(H_2^T - H_1^T - \Psi^{\beta}(1)^{T}\left(P^{2}\right)^{T} U\right)\Psi^{\beta}(t) = \widehat{U}^{T}\Psi^{\beta}(t). \tag{17}$$

By substituting Equation (17) into Equations (13) and (14), we get:

$$\frac{\partial u(\eta,t)}{\partial \eta} \simeq \Psi^{\beta}(\eta)^{T}\left(E\widehat{U} + P^{T}U\right)\Psi^{\beta}(t) = \Psi^{\beta}(\eta)^{T} A_1\Psi^{\beta}(t) \tag{18}$$

$$u(\eta,t) \simeq \left(\Psi^{\beta}(\eta)\right)^{T}\left(EH_1^T + X\widehat{U} + \left(P^{2}\right)^{T} U\right)\Psi^{\beta}(t) = \left(\Psi^{\beta}(\eta)\right)^{T} A_2\Psi^{\beta}(t) \tag{19}$$

in which $\eta = \Psi(\eta)^T \mathrm{X}$ and $1 = \Psi(\eta)^T E$. Furthermore, we can be expressed by a terminated Ultraspherical wavelet series at the value $\hat{m}$ as follows:

$$f(\eta,t) \simeq \left(\Psi^{\beta}(\eta)\right)^{T} F\Psi^{\beta}(t) \tag{20}$$

where $F$ is the Ultraspherical wavelet coefficient matrix.

Now by substituting Equations (12), (19) and (20) into Equation (11) , then using operational matrices of integration, we obtain the residual function $R(x,t)$ for this equation as follows:

$$R(\eta,t) = \left(\Psi^{\beta}(\eta)\right)^{T}\left[A_2 + \gamma_1 UP^2 + \gamma_2 A_2 P^2 + \gamma_3 A_3 P^2 - FP^2\right]\Psi^{\beta}(t)$$

in which

$$\left[\left(\Psi^{\beta}(\eta)\right)^{T} A_2\Psi^{\beta}(t)\right]\left[\left(\Psi^{\beta}(\eta)\right)^{T} A_2\Psi^{\beta}(t)\right] = \left(\Psi^{\beta}(\eta)\right)^{T} A_3\Psi^{\beta}(t)$$

As in Galerkin method [26], for $u_{ij},, i = 1,2,\ldots,\hat{m}$ we obtain $\hat{m}^2$ non-linear algebraic equations as follows:

$$\int_0^1\int_0^1 R(\eta,t)\psi_i^{\beta}(\eta)\psi_j^{\beta}(t)\omega_n(\eta)\omega_n(t)\,d\eta dt = 0, \quad i, j = 1,2,\ldots,\hat{m}$$

Eventually, by solving this system for the unknown matrix $U$ , approximate solution for the Klein-Gordon equation is obtained.

## 7. ILLUSTRATIVE EXAMPLES

In this section, two test problems were examined to show the accuracy and efficiency of the presented method. Such type of problems that have exact solutions wase selected. In order to measure the difference between the analytic and numerical solutions, we used the following error function defined as

$$E(\eta_i,t_i)=\left|u_{exactsol}(\eta_i,t_i)-u_{numsol}(\eta_i,t_i)\right|.$$

The obtained errors are shown in tables.

**Example 1.** Suppose the following Linear Klein-Gordon equation as

$$\frac{\partial^2 u(\eta,t)}{\partial t^2}+\gamma_1\frac{\partial^2 u(\eta,t)}{\partial \eta^2}+\gamma_2 u(\eta,t)+\gamma_3 u^2(\eta,t)=f(\eta,t),$$

subject to the initial and boundary conditions

$$u(\eta,0)=0, u_t(\eta,0)=0$$

and

$$u(0,t)=0, \quad u(1,t)=t^3.$$

Here $f(\eta,t)=6\eta^3 t+(\eta^3-6\eta)t^3$ and $\gamma_1=-1, \gamma_2=1$ ve $\gamma_3=0$ .

The exact solution of this problem is $u(\eta,t)=\eta^3 t^3$ [27].

**Table 1.** Absolute error at different values of $(\eta,t)$

| $\left\|u_{exactsol}(\eta_i,t_i)-u_{numsol}(\eta_i,t_i)\right\|$ | | |
|---|---|---|
| $(\eta_i,t_i)$ | $\beta=1/2, M=10, k=1$ | $\beta=3/2, M=10, k=1$ |
| $(0.1,0.1)$ | $3.91257122448327\times10^{-7}$ | $4.73644590538997\times10^{-4}$ |
| $(0.2,0.2)$ | $8.30695799578360\times10^{-6}$ | $5.74906482856706\times10^{-5}$ |
| $(0.3,0.3)$ | $4.03940390231403\times10^{-5}$ | $1.44138344344434\times10^{-7}$ |
| $(0.4,0.4)$ | $1.23945768756534\times10^{-4}$ | $5.95056528304065\times10^{-5}$ |
| $(0.5,0.5)$ | $2.93440091529318\times10^{-4}$ | $5.98350532921694\times10^{-4}$ |
| $(0.6,0.6)$ | $5.56942460131049\times10^{-4}$ | $7.34183981250411\times10^{-3}$ |
| $(0.7,0.7)$ | $7.80649978712467\times10^{-4}$ | $1.07737555560637\times10^{-2}$ |
| $(0.8,0.8)$ | $8.81314297932367\times10^{-4}$ | $1.95158399595203\times10^{-2}$ |
| $(0.9,0.9)$ | $5.63384785369325\times10^{-4}$ | $5.42959117653197\times10^{-3}$ |

Table 1 shows the maximum errors obtained by the Ultraspherical wavelet Galerkin method for $M=10, k=1, \beta=\frac{1}{2}$ ve $\beta=\frac{3}{2}$.

**Example 2.** Consider the following Klein-Gordon equation with quadratic nonlinearity:

$$\frac{\partial^2 u(\eta,t)}{\partial t^2}+\gamma_1\frac{\partial^2 u(\eta,t)}{\partial \eta^2}+\gamma_2 u(\eta,t)+\gamma_3 u^2(\eta,t)=f(\eta,t),\eta\in[0,1],0<t\le T$$

The above problem is associated with the initial and boundary conditions

$$u(\eta,0)=0,u_t(\eta,0)=0,0\le\eta\le 1$$

and

$$u(0,t)=0,\quad u(1,t)=t^3.$$

Here $f(\eta,t)=6\eta t(\eta^2-t^2)+\eta^6 t^6$ and $\gamma_1=-1,\gamma_2=0$ ve $\gamma_3=1$ .

The exact solution of this problem is $u(\eta,t)=\eta^3 t^3$ [28].

**Table 2.** Absolute error at different values of $(\eta,t)$

| $\lvert u_{exactsol}(\eta_i,t_i)-u_{numsol}(\eta_i,t_i)\rvert$ | | |
|---|---|---|
| $(\eta_i,t_i)$ | $\beta=1/2, M=5, k=1$ | $\beta=3/2, M=4, k=1$ |
| $(0.1,0.1)$ | $5.15210321495812\times10^{-5}$ | $2.41458877189035\times10^{-5}$ |
| $(0.2,0.2)$ | $4.56256704309143\times10^{-5}$ | $3.86845800750016\times10^{-5}$ |
| $(0.3,0.3)$ | $7.01936225233186\times10^{-5}$ | $1.84576025220513\times10^{-4}$ |
| $(0.4,0.4)$ | $3.50160067745614\times10^{-4}$ | $9.83339405666225\times10^{-4}$ |
| $(0.5,0.5)$ | $9.01264539172070\times10^{-4}$ | $2.22640732044785\times10^{-3}$ |
| $(0.6,0.6)$ | $1.64215612239218\times10^{-3}$ | $3.43938435342541\times10^{-3}$ |
| $(0.7,0.7)$ | $1.89581244001211\times10^{-3}$ | $4.00740592116726\times10^{-3}$ |
| $(0.8,0.8)$ | $9.81767510053244\times10^{-4}$ | $3.31446792402951\times10^{-3}$ |
| $(0.9,0.9)$ | $4.58661367632618\times10^{-4}$ | $1.80231079429083\times10^{-3}$ |

Table 2 shows the maximum errors obtained by the Ultraspherical wavelet Galerkin method for $M=10, k=1, \beta=\frac{1}{2}$ ve $\beta=\frac{3}{2}$.

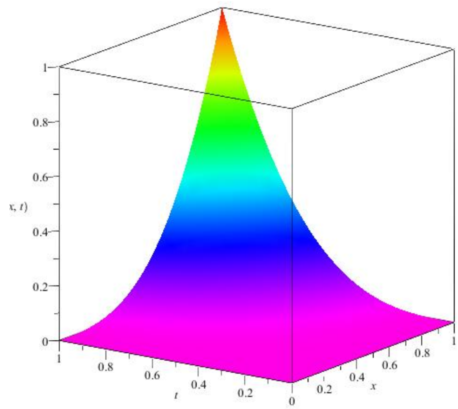


**Figure 1.** Exact solution

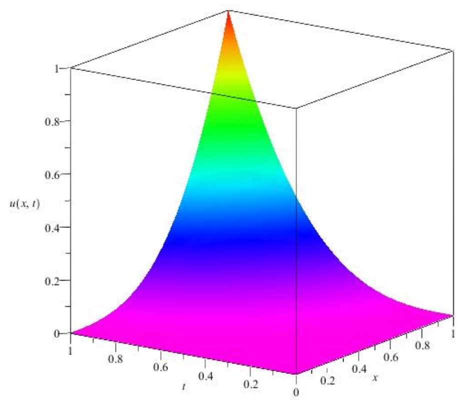


**Figure 2.** Wavelet solution using UWGM for $\beta = \frac{1}{2}$.

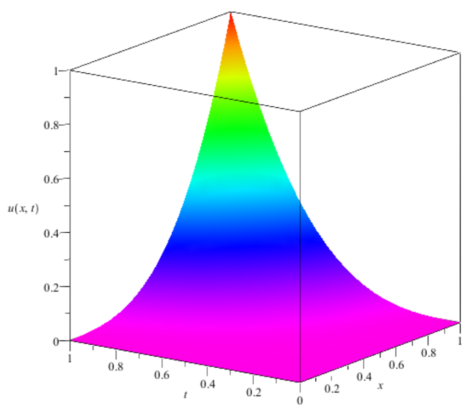


**Figure 3.** Wavelet solution using UWG $\beta = \frac{3}{2}$.

## 8. CONCLUSION

In the present study, a scheme to get numerical solutions of linear and nonlinear KG equations using the Ultraspherical wavelet Galerkin method ,which is combined Ultraspherical wavelets with their operational matrices of integration, is presented. The method is very convenient for solving boundary value problems, because the boundary conditions are taken into account automatically. Also the implementation of the method is very simple and as the numerical results indicate the method is very useful technique to find numerical solutions of such type of problems . As a result, the presented method can be employed to obtain numerical solutions of various partial differential equations in the literature.

## REFERENCES

[1] Dehghan M. and Shokri A., (2009) Numerical solution of the nonlinear Klein–Gordon equation using radial basis functions, *Journal of Computational and Applied Mathematics*, 230(2), 400-410.

[2] Wazwaz A. M., (2006) The modified decomposition method for analytic treatment of differential equations, *Applied Mathematics and Computation*, 173(1), 165-176.

[3] Bülbül B., & Sezer M., (2013) A new approach to numerical solution of nonlinear Klein-Gordon equation, *Mathematical Problems in Engineering*, 2013.

[4] Sadigh B. S., (2011), Numerical Solution Of Klein-Gordon Equation By Using The Adomian's Decomposition And Variational Iterative Methods, *Int. J. Industrial Mathematics*,79-89.

[5] Hepson O. E., Korkmaz A., & Dag I., (2018) On the numerical solution of the Klein-Gordon equation by exponential cubic B-spline collocation method, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 68(1), 412-421.

[6] Bildik N., & Deniz S., (2020) New approximate solutions to the nonlinear Klein-Gordon equations using perturbation iteration techniques, *Discrete & Continuous Dynamical Systems-S*, 13(3), 503.

[7] Yusufoğlu E., (2008) The variational iteration method for studying the Klein–Gordon equation, *Applied Mathematics Letters*, 21(7), 669-674.

[8] Han H., & Zhang Z., (2009) An analysis of the finite-difference method for one-dimensional Klein–Gordon equation on unbounded domain, *Applied numerical mathematics*, 59(7), 1568-1583.

[9] Kaya D., & El-Sayed S. M., (2004) A numerical solution of the Klein–Gordon equation and convergence of the decomposition method, *Applied mathematics and computation*, 156(2), 341-353.

[10] Ebaid A., (2009) Exact solutions for the generalized Klein–Gordon equation via a transformation and Exp-function method and comparison with Adomian's method, *Journal of Computational and Applied Mathematics*, 223(1), 278-290.

[11] Wang T. M., & Zhu J. M., (2009) New explicit solutions of the Klein–Gordon equation using the variational iteration method combined with the Exp-function method, *Computers & Mathematics with Applications*, 58(11-12), 2444-2448.

[12] Secer A., & Ozdemir N., (2019) An effective computational approach based on Gegenbauer wavelets for solving the time-fractional Kdv-Burgers-Kuramoto equation, *Advances in Difference Equations*, 2019(1), 386.

[13] Ozdemir N., Secer A., & Bayram M., (2019) The Gegenbauer Wavelets-Based Computational Methods for the Coupled System of Burgers' Equations with Time-Fractional Derivative, *Mathematics*, 7(6), 486.

[14] Seçer A., Altun S., & Bayram M., (2019) Legendre Wavelet Operational Matrix Method for Solving Fractional Differential Equations in Some Special Conditions, *Thermal Science*, 23(Suppl. 1), 203-2014.

[15] Secer A., & Bakir Y., (2019) Chebyshev wavelet collocation method for Ginzburg-Landau equation, Thermal Science, 23(Suppl. 1), 57-65.

[16] Heydari M.H., Maalek Ghaini, F.M., Hooshmandasl M.R., (2014) Legendre wavelets method for numerical solution of time-fractional heat equation, Wavelet Linear Algebra, 1, 19–31.

[17] Heydari M.H., Hooshmandasl M.R., Ghaini F.M., (2014) A new approach of the Chebyshev wavelets method for partial differential equations with boundary conditions of the telegraph type, *Appl. Math. Model*., 38, 1597–1606.

[18] Secer A., & Ozdemirn N., (2019) Modified Laguerre wavelet based Galerkin method for fractional and fractional-order delay differential equations, *Thermal Science*, 23(Suppl. 1), 13-21.

[19] Dehestani H., Ordokhani Y., & Razzaghi M., (2019) On the applicability of Genocchi wavelet method for different kinds of fractional-order differential equations with delay, *Numerical Linear Algebra with Applications*, 26(5), e2259.

[20] Zheng X, Wei Z., (2015) Discontinuous Legendre Wavelet Galerkin Method for One-Dimensional Advection-Diffusion Equation, *Springer P. Math. Stat*., 6(09), 1581.

[21] Giordano C, Laforgia A., (2013) On the Bernstein-type inequalities for ultraspherical polynomials, *J. comput. Appl. Math*, 153(1-2), 243-248.

[22] Chi-Hsu W., (1983) On the generalization of block pulse operational matrices for fractional and operational calculus, *J. Frankl. Inst*., 315(2), 91-102.

[23] Yin F., Song J., Cao X., Lu F., (2013) Couple of the variational iteration method and Legendre wavelets for nonlinear partial differential equations, *J. Appl. Math*..

[24] Maleknejad K., Khodabin M., Rostami M., (2012) Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions, *Math. Comput. Model,* 55(3-4), 791-800.

[25] Maleknejad K., Khodabin M., Rostami, M., (2012) A numerical method for solving m-dimensional stochastic Itô–Volterra integral equations by stochastic operational matrix, *Comput. Math. Appl*., 63(1), 133-143.

[26] Canuto C., Hussaini M.Y., Quarteroni A., Thomas Jr., (2012) A. Spectral methods in fluid dynamics, Springer Science & Business Media.

[27] Odibat Z., & Momani S., (2009) The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics, *Computers & Mathematics with Applications*, 58(11-12), 2199-2208.

[28] Abbasbandy S., (2007) Numerical solution of non-linear Klein–Gordon equations by variational iteration method, *International Journal for Numerical Methods in Engineering*, 70(7), 876-881.