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Coupled system of ψ–Caputo fractional differential equations without and with delay in generalized Banach spaces

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Abstract

The main objective of this research manuscript is to establish various existence and uniqueness results as well as the Ulam–Hyers stability of solutions to a Coupled system of ψ–Caputo fractional differential equations without and with delay in generalized Banach spaces. Existence and uniqueness results are obtained by applying Krasnoselskii's type fixed point theorem, Schauder's fixed point theorem in generalized Banach spaces, and Perov's fixed point theorem combined with the Bielecki norm. While Urs's approach is used to analyze the Ulam–Hyers stability of solutions for the proposed problem. Finally, Some examples are given to illustrate the obtained results.

Keywords: ψ–Caputo fractional derivative Coupled system Existence Uniqueness Fixed point Bielecki norm Ulam stability Generalized Banach spaces.
2010 MSC: 34A08, 26A33, 34A34.

1. Introduction

Nonlinear fractional differential equations (NFDEs) play an important role in describing many phenomena in applied sciences and engineering applications the reader can consult [20, 29, 34, 43]. Some recent results on the topic can be found in the following monographs [1, 2, 3, 24, 34, 47]. During the last few years, different variant of fractional differential operators have been introduced [7, 9, 14, 22, 27, 41]. Existence and uniqueness

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of solutions as well as the Ulam stability for fractional differential equations without and with delay in scalar and abstract Banach spaces have been wildly considered; see for instance [4, 5, 8, 10, 11, 12, 13, 16, 17, 21, 23, 25, 28, 30, 39, 40, 42, 44]. In this regard, different methods have been employed to verify the qualitative properties of their solutions. Moreover, the study of coupled systems of fractional order is also important in various problems of applied sciences, see [15, 18, 19, 26, 33]. Additionally, in papers [6, 31, 32, 35, 36, 38, 46], the authors studied the existence and uniqueness of solutions for a system of ordinary or fractional differential equations using some well-known fixed point theorems in generalized Banach spaces. As far as we know, there are no contributions associated with the solutions of a system of fractional differential equations without and with delay in generalized Banach spaces in the frame of ψ–Caputo derivative. Therefore, this paper comes to fill this gap in the literature. Motivated by aforementioned reasons, in this research article first, we deal with the existence and uniqueness results as well as the Ulam–Hyers stability of solutions for the following system of differential equations involving the ψ-Caputo derivative of fractional order:

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\nu;\psi} x)(\tau) = \mathbb{A}_1 x(\tau) + \mathbb{G}_1(\tau, x(\tau), y(\tau)), \\ ({}^c\mathbb{D}_{a^+}^{\mu;\psi} y)(\tau) = \mathbb{A}_2 y(\tau) + \mathbb{G}_2(\tau, x(\tau), y(\tau)), \end{cases} \quad \tau \in \mathrm{J} := [a, b], \tag{1}$$

subject to the initial conditions

$$\begin{cases} x(a) = \phi_1, \\ y(a) = \phi_2, \end{cases} \tag{2}$$

where ${}^c\mathbb{D}_{a^+}^{\nu;\psi}$, ${}^c\mathbb{D}_{a^+}^{\mu;\psi}$ are the ψ-Caputo fractional derivative of order $\nu, \mu \in (0, 1]$, respectively which was recently proposed by Almeida[7]. $\mathbb{G}_1, \mathbb{G}_2 : \mathrm{J} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are a given continuous functions, a and b are positive constants such that $a < b, \phi_1, \phi_2 \in \mathbb{R}^n$ and $\mathbb{A}_1, \mathbb{A}_2 \in \mathbb{R}^{n\times n}$.

Next, we turn our attention to study the existence and uniqueness of solutions to the following delayed coupled system of the form:

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\nu;\psi} x)(\tau) = \mathbb{F}_1(\tau, x_\tau, y_\tau), \\ ({}^c\mathbb{D}_{a^+}^{\mu;\psi} y)(\tau) = \mathbb{F}_2(\tau, x_\tau, y_\tau), \end{cases} \quad \tau \in \mathrm{J}, \tag{3}$$

along with the initial conditions

$$\begin{cases} x(\tau) = \alpha(\tau), \\ y(\tau) = \beta(\tau), \end{cases} \quad \tau \in [a - \delta, a], \tag{4}$$

where $\delta > 0$ is a constant delay and $\mathbb{F}_1, \mathbb{F}_2 : \mathrm{J} \times C([-\delta, 0], \mathbb{R}^n) \times C([-\delta, 0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$, are given continuous functions and $\alpha, \beta : [a - \delta, a] \longrightarrow \mathbb{R}^n$ are two continuous functions. For any function z defined on $[a - \delta, a]$ and any $\tau \in \mathrm{J}$, we denote by z_τ the element of $C([-\delta, 0], \mathbb{R}^n)$ defined by

$$z_\tau(\rho) = z(\tau + \rho), \quad \rho \in [-\delta, 0].$$

Hence $z_\tau(\cdot)$ represents the history of the state from times $\tau - \delta$ up to the present time τ.

Our paper is organized as follows: In Section 2, we provide some basic definitions needed for our work. In Section 3, we prove the existence and uniqueness of solutions for problem (1)–(2) by using Perov's and Krasnoselskii's fixed point theorems in generalized Banach spaces together with the Bielecki norm. Further, the Ulam–Hyers's stability of the above-mentioned problem is also investigated. In Section 4, we discuss the existence and uniqueness of solutions for problem (3) subjected to initial conditions (4) via fixed point techniques of Schauder's and Perov's in generalized Banach spaces. Finally, we close up this paper by providing some examples to illustrate the applicability of the obtained results.

2. Background material

In this section, we provide some fundamental material about fractional calculus, matrix analysis, and fixed-point theorems that will be used throughout this paper.

First, we introduce the essential functional spaces that we will adopt in this paper. We denote by $C([a,b],\mathbb{R}^n)$ the Banach space of all continuous functions z from $[a,b]$ into $\mathbb{R}^n$ with the supremum norm

$$\|z\|_{[a,b]} = \sup_{\tau\in[a,b]} \|z(\tau)\|.$$

Let $\mathfrak{C} = C([a-\delta,b],\mathbb{R}^n)$, denote the Banach space of functions from $[a-\delta,b]$ into $\mathbb{R}^n$ equipped with the supremum norm $\|z\|_{\mathfrak{C}}$. In addition, Let us denote by $\mathfrak{C}_\delta := C([-\delta,0],\mathbb{R}^n)$ the Banach space of functions w from $[-\delta,0]$ into $\mathbb{R}^n$, endowed with the norm

$$\|w\|_{\mathfrak{C}_\delta} = \sup_{\rho\in[-\delta,0]} \|w(\rho)\|.$$

Now, we present some facts from the theory of fractional calculus.

Definition 2.1 ([7, 24]). *For $\nu > 0$, the left-sided ψ–Riemann–Liouville fractional integral of order ν for an integrable function $z\colon [a,b] \longrightarrow \mathbb{R}$ with respect to another function $\psi\colon [a,b] \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(\tau) \neq 0$, for all $\tau \in \mathrm{J}$ is defined as follows*

$$\mathbb{I}_{a^+}^{\nu;\psi} z(\tau) = \frac{1}{\Gamma(\nu)} \int_a^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\nu-1} z(s)\mathrm{d}s, \tag{5}$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\nu) = \int_0^{+\infty} \tau^{\nu-1} e^{-\tau}\mathrm{d}\tau, \quad \nu > 0.$$

Definition 2.2 ([7]). *Let $n \in \mathbb{N}$ and let $\psi, z \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(\tau) \neq 0$, for all $\tau \in \mathrm{J}$. The left-sided ψ–Riemann–Liouville fractional derivative of a function z of order ν is defined by*

$$\begin{aligned}
\mathbb{D}_{a^+}^{\nu;\psi} z(\tau) &= \left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^n \mathbb{I}_{a^+}^{n-\nu;\psi} z(\tau) \\
&= \frac{1}{\Gamma(n-\nu)} \left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^n \int_a^\tau \psi'(s)(\psi(\tau)-\psi(s))^{n-\nu-1} z(s)\mathrm{d}s,
\end{aligned}$$

where $n = [\nu]+1$.

Definition 2.3 ([7]). *Let $n \in \mathbb{N}$ and let $\psi, z \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(\tau) \neq 0$, for all $\tau \in \mathrm{J}$. The left-sided ψ-Caputo fractional derivative of z of order ν is defined by*

$${}^c\mathbb{D}_{a^+}^{\nu;\psi} z(\tau) = \mathbb{I}_{a^+}^{n-\nu;\psi} \left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^n z(\tau),$$

where $n = [\nu]+1$ for $\nu \notin \mathbb{N}$, $n = \nu$ for $\nu \in \mathbb{N}$.
For the sake of brevity, let us take

$$z_\psi^{[n]}(\tau) = \left(\frac{1}{\psi'(\tau)}\frac{d}{dt}\right)^n z(\tau).$$

From the definition, it is clear that

$${}^c\mathbb{D}_{a^+}^{\nu;\psi} z(\tau) = \begin{cases} \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{n-\nu-1}}{\Gamma(n-\nu)} z_\psi^{[n]}(s)\mathrm{d}s &, \quad \text{if } \nu \notin \mathbb{N}, \\ z_\psi^{[n]}(\tau) &, \quad \text{if } \nu \in \mathbb{N}. \end{cases}$$

Some basic properties are listed in the following Lemma.

Lemma 2.4 ([7]). *Let $\nu, \beta > 0$, and $z \in C([a,b],\mathbb{R})$. Then for each $\tau \in \mathrm{J}$ we have*

(1) ${}^c\mathbb{D}_{a^+}^{\nu;\psi}\mathbb{I}_{a^+}^{\nu;\psi}z(\tau)=z(\tau)$,

(2) $\mathbb{I}_{a^+}^{\nu;\psi}{}^c\mathbb{D}_{a^+}^{\nu;\psi}z(\tau)=z(\tau)-z(a), \quad 0<\nu\leq 1$,

(3) $\mathbb{I}_{a^+}^{\nu;\psi}(\psi(\tau)-\psi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\nu)}(\psi(\tau)-\psi(a))^{\beta+\nu-1}$,

(4) ${}^c\mathbb{D}_{a^+}^{\nu;\psi}(\psi(\tau)-\psi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\nu)}(\psi(\tau)-\psi(a))^{\beta-\nu-1}$,

(5) ${}^c\mathbb{D}_{a^+}^{\nu;\psi}(\psi(\tau)-\psi(a))^{k}=0$, *for all* $k\in\{0,\dots,n-1\}, n\in\mathbb{N}$.

The following lemma has an important role in proving our main results.

Lemma 2.5 ([11]). *Let $\omega,\theta>0$. Then for all $\tau\in[a,b]$ we have*

$$\mathbb{I}_{a^+}^{\omega;\psi}e^{\theta(\psi(\tau)-\psi(a))}\leq\frac{e^{\theta(\psi(\tau)-\psi(a))}}{\theta^{\omega}}.$$

Remark 2.6 ([40, 42]). *On the space $C(\mathrm{J},\mathbb{R}^n)$ we define a Bielecki type norm $\|\cdot\|_{\mathfrak{B}}$ as below*

$$\|z\|_{\mathfrak{B}}:=\sup_{\tau\in\mathrm{J}}\frac{\|z(\tau)\|}{e^{\theta(\psi(\tau)-\psi(a))}},\quad \theta>0. \tag{6}$$

Consequently, we have the following proprieties

1. $\big(C(\mathrm{J},\mathbb{R}^n),\|\cdot\|_{\mathfrak{B}}\big)$ *is a Banach space.*
2. *The norms $\|\cdot\|_{\mathfrak{B}}$ and $\|\cdot\|_{\infty}$ are equivalent on $C(\mathrm{J},\mathbb{R}^n)$, where $\|\cdot\|_{\infty}$ denotes the Chebyshev norm on $C(\mathrm{J},\mathbb{R}^n)$, i.e;*
$$\iota_1\|\cdot\|_{\mathfrak{B}}\leq\|\cdot\|_{\infty}\leq\iota_2\|\cdot\|_{\mathfrak{B}},$$
where
$$\iota_1=1,\quad \iota_2=e^{\theta(\psi(b)-\psi(a))}.$$

For more properties on Bielecki type norm see [16, 40, 42]

Let $x,y\in\mathbb{R}^m$ with $x=(x_1,x_2,\dots,x_m), y=(y_1,y_2,\dots,y_m)$.
By $x\leq y$ we mean $x_i\leq y_i, i=1,\dots,m$. Also,

$$|x|=(|x_1|,|x_2|,\dots,|x_m|),$$
$$\max(x,y)=(\max(x,y),\max(\bar{x},\bar{y}),\dots,\max(x_m,y_m)),$$

and

$$\mathbb{R}^m_+=\big\{x\in\mathbb{R}^m\colon x_i\in\mathbb{R}_+, i=1,\dots,m\big\}.$$

If $c\in\mathbb{R}$, then $x\leq c$ means $x_i\leq c, i=1,\dots,m$.

Definition 2.7 ([31]). *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d\colon X\times X\to\mathbb{R}^m$ with the following properties:*

(*i*) $\mathrm{d}(x,y)\geq 0$ *for all* $x,y\in X$, *and if* $\mathrm{d}(x,y)=0$, *then* $x=y$;

(*ii*) $\mathrm{d}(x,y)=\mathrm{d}(y,x)$ *for all* $x,y\in X$;

(*iii*) $\mathrm{d}(x,z)\leq\mathrm{d}(x,y)+\mathrm{d}(y,z)$ *for all* $x,y,z\in X$.

We call the pair (X, d) *a generalized metric space with*

$$\mathrm{d}(x,y) := \begin{pmatrix} \mathrm{d}_1(x,y) \\ \mathrm{d}_2(x,y) \\ \vdots \\ \mathrm{d}_m(x,y) \end{pmatrix}.$$

Notice that d *is a generalized metric space on* X *if and only if* $\mathrm{d}_i, i = 1, \dots, m$, *are metrics on* X.

Definition 2.8 ([45]). *A square matrix* $\mathbb{A}$ *of real numbers is said to be convergent to zero if and only if its spectral radius* $\rho(\mathbb{A})$ *is strictly less than* 1. *In other words, this means that all the eigenvalues of* $\mathbb{A}$ *are in the open unit disc, i.e.,* $|\lambda| < 1$ *for every* $\lambda \in \mathbb{C}$ *with* $\det(\mathbb{A} - \lambda\mathbb{I}) = 0$, *where* $\mathbb{I}$ *denotes the unit matrix of* $\mathbb{A}_{m\times m}(\mathbb{R})$.

Theorem 2.9 ([45]). *For any nonnegative square matrix* $\mathbb{A}$, *the following properties are equivalent*

(*i*) $\mathbb{A}$ *is convergent to zero;*

(*ii*) $\rho(\mathbb{A}) < 1$;

(*iii*) *the matrix* $\mathbb{I} - \mathbb{A}$ *is nonsingular and*

$$(\mathbb{I} - \mathbb{A})^{-1} = \mathbb{I} + \mathbb{A} + \cdots + \mathbb{A}^n + \cdots ;$$

(*iv*) $\mathbb{I} - \mathbb{A}$ *is nonsingular and* $(\mathbb{I} - \mathbb{A})^{-1}$ *is a nonnegative matrix.*

Example 2.10 ([35]). *The matrix* $\mathbb{A} \in \mathbb{A}_{2\times 2}(\mathbb{R})$ *defined by*

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

converges to zero in the following cases:

(1) $b = c = 0, a, d > 0,$ *and* $\max\{a, d\} < 1$.

(2) $c = 0, a, d > 0, a + d < 1,$ *and* $-1 < b < 0$.

(3) $a + b = c + d = 0, a > 1, c > 0,$ *and* $|a - c| < 1$.

Definition 2.11 ([36, 37]). *Let* $(\mathbb{E}, \mathrm{d})$ *be a generalized metric space. An operator* $\mathbb{T}\colon \mathbb{E} \to \mathbb{E}$ *is said to be contractive if there exists a matrix* $\mathbb{A}$ *convergent to zero such that*

$$\mathrm{d}(\mathbb{T}(x), \mathbb{T}(y)) \le \mathbb{A}\mathrm{d}(x,y), \quad \textit{for all } x, y \in \mathbb{E}.$$

We close this section by introducing the following fixed-point theorems that will be employed in the sequel.

Theorem 2.12 ([32, 36]). *Let* $(\mathbb{E}, \mathrm{d})$ *be a complete generalized metric space and* $\mathbb{T}\colon \mathbb{E} \to \mathbb{E}$ *be a contractive operator with Lipschitz matrix* $\mathbb{A}$. *Then* $\mathbb{T}$ *has a unique fixed point* x_0, *and for each* $x \in \mathbb{E}$, *we have*

$$\mathrm{d}(\mathbb{T}^k(x), x_0) \le \mathbb{A}^k(\mathbb{I} - \mathbb{A})^{-1}\mathrm{d}(x, \mathbb{T}(x)) \quad \textit{for all } k \in \mathbb{N}.$$

Theorem 2.13 ([31]). *Let* Ω *be a closed, convex, non-empty subset of a generalized Banach spaces* $\mathbb{X}$. *Suppose that* $\mathbb{U}$ *and* $\mathbb{V}$ *map* Ω *into* $\mathbb{X}$ *and that*

(*i*) $\mathbb{U}x + \mathbb{V}y \in \Omega$ *for all* $x, y \in \Omega$;

(ii) $\mathbb{U}$ *is compact and continuous;*

(iii) $\mathbb{V}$ *is an* $\mathbb{A}$*-contraction mapping.*

Then the operator equation $\mathbb{U}z + \mathbb{V}z = z$ *has at least one solution on* Ω.

Theorem 2.14 ([31]). *Let* $\mathbb{X}$ *be a generalized Banach space,* $D \subset \mathbb{X}$ *be a nonempty closed convex subset of* $\mathbb{X}$*, and* $\mathbb{K}\colon D \to D$ *be a continuous operator with relatively compact range. Then* $\mathbb{K}$ *has at least a fixed point in* D.

3. Existence, uniqueness and stability results for problem (1)–(2).

In this section, we prove the existence and uniqueness of solutions for the given problem (1)–(2). We also study the Ulam–Hyers stability of the mentioned system.

Before starting and proving our main result, let us define what we mean by a solution of the problem (1)–(2).

Definition 3.1. *By a solution of problem* (1)–(2) *we mean a coupled function* $(x, y) \in C(\mathrm{J}, \mathbb{R}^n) \times C(\mathrm{J}, \mathbb{R}^n)$ *that satisfies the system*

$$\begin{cases} ({}^c\mathbb{D}^{\nu;\psi}_{a^+}x)(\tau) = \mathbb{A}_1 x(\tau) + \mathbb{G}_1(\tau, x(\tau), y(\tau)), \\ ({}^c\mathbb{D}^{\mu;\psi}_{a^+}y)(\tau) = \mathbb{A}_2 y(\tau) + \mathbb{G}_2(\tau, x(\tau), y(\tau)), \end{cases} \quad \tau \in \mathrm{J},$$

and the initial conditions

$$\begin{cases} x(a) = \phi_1, \\ y(a) = \phi_2. \end{cases}$$

For the existence of solutions for the problem (1)–(2), we need the following lemma:

Lemma 3.2. *Let* $\omega \in (0, 1]$ *be fixed,* $\mathbb{A} \in \mathbb{R}^{n \times n}$ *and* $h \in C(\mathrm{J} \times \mathbb{R}^n, \mathbb{R}^n)$*. Then the Cauchy problem*

$$\begin{cases} ({}^c\mathbb{D}^{\omega;\psi}_{a^+}z)(\tau) = \mathbb{A}z(\tau) + f(\tau, z(\tau)), \quad \tau \in \mathrm{J}, \\ z(a) = \phi \in \mathbb{R}^n, \end{cases} \tag{7}$$

is equivalent to the following integral equation,

$$z(\tau) = \phi + \int_a^\tau \frac{\psi'(s)(\psi(\tau) - \psi(s))^{\omega - 1}}{\Gamma(\omega)} (\mathbb{A}z(s) + f(s, z(s)))\mathrm{ds}. \tag{8}$$

Proof. Let $z(\tau)$ be a solution of the problem (7). Define $h(\tau) = \mathbb{A}z(\tau) + f(\tau, z(\tau))$. Then

$$({}^c\mathbb{D}^{\omega;\psi}_{a^+}z)(\tau) = h(\tau),\ 0 < \omega \le 1,$$

that is

$$({}^c\mathbb{D}^{\omega;\psi}_{a^+}z)(\tau) = \mathbb{I}^{1-\omega;\psi}_{a^+}\left(\frac{1}{\psi'(\tau)}\frac{\mathrm{d}}{\mathrm{dt}}z\right)(\tau) = h(\tau),\ 0 < \omega \le 1.$$

Taking the ψ–Riemann–Liouville fractional integral of order ω to the above equation, we get

$$\mathbb{I}^{1;\psi}_{a^+}\left(\frac{1}{\psi'(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}z\right)(\tau) = \mathbb{I}^{\omega;\psi}_{a^+}h(\tau),\ 0 < \omega \le 1.$$

Since

$$\mathbb{I}^{1;\psi}_{a^+}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}\tau}z\right)(\tau) = \mathbb{I}^{1}_{a^+}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}z\right)(\tau) = z(\tau) - z(a),$$

we get

$$z(\tau) = \phi + \mathbb{I}_{a^+}^{\omega;\psi} h(\tau).$$

Using the definition of $h(\tau)$, we obtain (8). Conversely, suppose that $z(\tau)$ is the solution of the Eq. (8). Then it can be written as

$$z(\tau) = \phi + \mathbb{I}_{a^+}^{\omega;\psi} h(\tau), \tag{9}$$

where $h(\tau) = \mathbb{A}z(\tau) + f(\tau, z(\tau))$. Since $h(\tau)$ is continuous and ϕ is a constant vector, operating the the ψ–Caputo fractional differential operator ${}^{c}\mathbb{D}_{a^+}^{\omega;\psi}$ on both sides of Eq. (9) we obtain

$$({}^{c}\mathbb{D}_{a^+}^{\omega;\psi} z)(\tau) = {}^{c}\mathbb{D}_{a^+}^{\omega;\psi} \phi + ({}^{c}\mathbb{D}_{a^+}^{\omega;\psi} z)\mathbb{I}_{a^+}^{\omega;\psi} h(\tau).$$

Using Lemma 2.4, it yields

$$({}^{c}\mathbb{D}_{a^+}^{\omega;\psi} z)(\tau) = \mathbb{A}z(\tau) + f(\tau, z(\tau)).$$

From (9), we get $z(a) = \phi$. This proves that $z(\tau)$ is the solution of Cauchy problem (7) which completes the proof. □

As a consequence of Lemma 3.2 we have the following result which is useful in our main results.

Lemma 3.3. *Let $\nu, \mu \in (0, 1]$ be fixed, $\mathbb{A}_1, \mathbb{A}_2 \in \mathbb{R}^{n\times n}$ and $\mathbb{G}_1, \mathbb{G}_2 \in C(\mathrm{J} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$. Then the coupled systems* (1)–(2) *is equivalent to the following integral equations*

$$\begin{cases} x(\tau) = & \phi_1 + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \big(\mathbb{A}_1 x(s) + \mathbb{G}_1(s, x(s), y(s))\big)\mathrm{ds}, \\ y(\tau) = & \phi_2 + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)} \big(\mathbb{A}_2 y(s) + \mathbb{G}_2(s, x(s), y(s))\big)\mathrm{ds}, \end{cases}, \quad \tau \in \mathrm{J}. \tag{10}$$

Our first result on the uniqueness is based on the Perov's fixed point theorem combined with the Bielecki norm.

Theorem 3.4. *Let the following assumptions hold:*

(H1) *$\mathbb{G}_1, \mathbb{G}_2 : \mathrm{J} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous functions.*

(H2) *There exist continuous functions $p_i, q_i \colon \mathrm{J} \to \mathbb{R}_+, i = 1, 2,$ such that*

$$\|\mathbb{G}_i(\tau, x_1, y_1) - \mathbb{G}_i(\tau, x_2, y_2)| \le p_i(\tau)\|x_1 - x_2\| + q_i(\tau)\|y_1 - y_2\|,$$

for all $\tau \in \mathrm{J}$ and each $x_1, y_1, x_2, y_2 \in \mathbb{R}^n$.

Then the coupled system (3)–(4) *has a unique solution.*

For computational convenience, we introduce the following notations:

$$p_i^* := \sup_{\tau\in \mathrm{J}} p_i(\tau),\ q_i^* := \sup_{\tau\in \mathrm{J}} q_i(\tau),\ \phi_i^* := \|\phi_i\|,\ \mathbb{A}_i^* = \|\mathbb{A}_i\|,$$

$$\mathbb{G}_i^* := \sup_{\tau\in \mathrm{J}} \|\mathbb{G}_i(\tau, 0, 0)\|,\ \ell_\psi^\nu := \frac{(\psi(b) - \psi(a))^\nu}{\Gamma(\nu + 1)}, \quad \ell_\psi^\mu := \frac{(\psi(b) - \psi(a))^\mu}{\Gamma(\mu + 1)}.$$

Proof. Consider the Banach space $C(\mathrm{J}, \mathbb{R}^n)$ equipped with a Bielecki norm type $\|\cdot\|_{\mathfrak{B}}$ defined in (6). Consequently, the product space $\mathbb{X} := C(\mathrm{J}, \mathbb{R}^n) \times C(\mathrm{J}, \mathbb{R}^n)$ is a generalized Banach space, endowed with the Bielecki vector-valued norm

$$\|(x, y)\|_{\mathbb{X},\mathfrak{B}} = \begin{pmatrix} \|x\|_{\mathfrak{B}} \\ \|y\|_{\mathfrak{B}} \end{pmatrix}.$$

We define an operator $\mathbb{T} = \big(\mathbb{T}_1, \mathbb{T}_2\big) \colon \mathbb{X} \to \mathbb{X}$ by:

$$\mathbb{T}(x,y) = \big(\mathbb{T}_1(x,y), \mathbb{T}_2(x,y)\big). \tag{11}$$

Where

$$(\mathbb{T}_1(x,y))(\tau) = \phi_1 + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\big(\mathbb{A}_1 x(s) + \mathbb{G}_1(s,x(s),y(s))\big)\mathrm{ds}, \tag{12}$$

and

$$(\mathbb{T}_2(x,y))(\tau) = \phi_2 + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)}\big(\mathbb{A}_2 y(s) + \mathbb{G}_2(s,x(s),y(s))\big)\mathrm{ds}. \tag{13}$$

Now, we apply Perov's fixed point theorem to prove that $\mathbb{T}$ has a unique fixed point. Indeed, it enough to show that $\mathbb{T}$ is $\mathbb{A}_\theta$-contraction mapping on $\mathbb{X}$ via the Bielecki's vector-valued norm. For this end, given $(x_1,y_1),(x_2,y_2) \in \mathbb{X}$ and $\tau \in \mathrm{J}$, using (H2), and Lemma 2.5, we can get

$$\begin{aligned}
&\big\|\big(\mathbb{T}_1(x_1,y_1)\big)(\tau) - \big(\mathbb{T}_1(x_2,y_2)\big)(\tau)\big\| \\
&\le \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\left(p_1(s)\|x_1(s)-x_2(s)\| + q_1(s)\|y_1(s)-y_2(s)\|\right)\mathrm{ds} \\
&\quad + \|\mathbb{A}_1\| \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\|x_1(s)-x_2(s)\|\mathrm{ds} \\
&\le \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\frac{p_1(s)\|x_1(s)-x_2(s)\| + q_1(s)\|y_1(s)-y_2(s)\|}{e^{\theta(\psi(s)-\psi(a))}}e^{\theta(\psi(s)-\psi(a))}\mathrm{ds} \\
&\quad + \|\mathbb{A}_1\| \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\frac{\|x_1(s)-x_2(s)\|}{e^{\theta(\psi(s)-\psi(a))}}e^{\theta(\psi(s)-\psi(a))}\mathrm{ds} \\
&\le \big((p_1^* + \mathbb{A}_1^*)\|x_1-x_2\|_{\mathfrak{B}} + q_1^*\|y_1-y_2\|_{\mathfrak{B}}\big)\int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}e^{\theta(\psi(s)-\psi(a))}\mathrm{ds} \\
&\le \frac{e^{\theta(\psi(\tau)-\psi(a))}}{\theta^\nu}\big((p_1^* + \mathbb{A}_1^*)\|x_1-x_2\|_{\mathfrak{B}} + q_1^*\|y_1-y_2\|_{\mathfrak{B}}\big).
\end{aligned}$$

Hence

$$\big\|\mathbb{T}_1(x_1,y_1) - \mathbb{T}_1(x_2,y_2)\big\|_{\mathfrak{B}} \le \frac{p_1^* + \mathbb{A}_1^*}{\theta^\nu}\|x_1-x_2\|_{\mathfrak{B}} + \frac{q_1^*}{\theta^\nu}\|y_1-y_2\|_{\mathfrak{B}}.$$

By the same technique, we can also get

$$\big\|\mathbb{T}_2(x_1,y_1) - \mathbb{T}_2(x_2,y_2)\big\|_{\mathfrak{B}} \le \frac{p_2^*}{\theta^\mu}\|x_1-x_2\|_{\mathfrak{B}} + \frac{q_2^* + \mathbb{A}_2^*}{\theta^\mu}\|y_1-y_2\|_{\mathfrak{B}}.$$

This implies that

$$\big\|\mathbb{T}(x_1,y_1) - \mathbb{T}(x_2,y_2)\big\|_{\mathbb{X},\mathfrak{B}} \le \mathbb{A}_\theta\|(x_1,y_1)-(x_2,y_2)\|_{\mathbb{X},\mathfrak{B}},$$

where

$$\mathbb{A}_\theta = \begin{pmatrix} \frac{p_1^*+\mathbb{A}_1^*}{\theta^\nu} & \frac{q_1^*}{\theta^\nu} \\ \frac{p_2^*}{\theta^\mu} & \frac{q_2^*+\mathbb{A}_2^*}{\theta^\mu} \end{pmatrix}. \tag{14}$$

Taking θ large enough it follows that the matrix $\mathbb{A}$ is convergent to zero and thus, an application of Perov's theorem shows that $\mathbb{T}$ has a unique fixed point. So the coupled system (1)–(2) has a unique solution in $\mathbb{X}$. □

Now we give our existence result for problem (1)–(2). The arguments are based on the Krasnoselskii's type fixed point theorem in generalized Banach spaces.

Theorem 3.5. *Let the assumptions* (H1) *and* (H2) *are satisfied. Then the coupled system* (1)–(2) *has at least one solution.*

Proof. In order to use the Krasnoselskii's fixed point theorem to prove our main result, we define a subset $\mathbb{B}_\xi$ of $\mathbb{X}$ by

$$\mathbb{B}_\xi = \{(x, y) \in \mathbb{X} : \|(x, y)\|_{\mathbb{X},\mathfrak{B}} \leq \xi\},$$

with $\xi := (\xi_1, \xi_2) \in \mathbb{R}^2_+$ such that

$$\begin{cases} \xi_1 \geq \gamma_1\mathbb{M}_1 + \gamma_2\mathbb{M}_2, \\ \xi_2 \geq \gamma_3\mathbb{M}_1 + \gamma_4\mathbb{M}_2, \end{cases}$$

where $\mathbb{M}_1, \mathbb{M}_2$ and $\gamma_i, i = \overline{1, 4}$ are positive real numbers that will be specified later. Moreover, notice that $\mathbb{B}_\xi$ is closed, convex and bounded subset of the generalized Banach space $\mathbb{X}$, and construct the operators $\mathbb{U} = (\mathbb{U}_1, \mathbb{U}_2)$ and $\mathbb{V} = (\mathbb{V}_1, \mathbb{V}_2)$ on $\mathbb{B}_\xi$ as

$$\begin{cases} \mathbb{U}_1(x, y)(\tau) = & \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathbb{G}_1(s, x(s), y(s))\mathrm{d}s, \\ \mathbb{U}_2(x, y)(\tau) = & \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)}\mathbb{G}_2(s, x(s), y(s))\mathrm{d}s, \end{cases}$$

and

$$\begin{cases} \mathbb{V}_1(x, y)(\tau) = & \phi_1 + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathbb{A}_1 x(s)\mathrm{d}s, \\ \mathbb{V}_2(x, y)(\tau) = & \phi_2 + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)}\mathbb{A}_2 y(s)\mathrm{d}s. \end{cases}$$

Obviously, both $\mathbb{U}$ and $\mathbb{V}$ are well defined due to (H1) and (H2). Furthermore, by Lemma 3.3 the operators form of system (10) may be written as

$$(x, y) = (\mathbb{U}_1(x, y), \mathbb{U}_2(x, y)) + (\mathbb{V}_1(x, y), \mathbb{V}_2(x, y)) := \mathbb{T}(x, y). \tag{15}$$

Thus, the fixed point of operator $\mathbb{T}$ coincides with the solution of the coupled system (1)–(2). We shall prove that $\mathbb{U}$ and $\mathbb{V}$, satisfy all conditions of Theorem 2.13. For better readability, we break the proof into a sequence of steps.

Step 1 : $\mathbb{U}(x, y) + \mathbb{V}(\bar{x}, \bar{y}) \in \mathbb{B}_\xi$, for any $(x, y), (\bar{x}, \bar{y}) \in \mathbb{B}_\xi$. Indeed, for $(x, y), (\bar{x}, \bar{y}) \in \mathbb{X}$ and for each $\tau \in \mathrm{J}$, from the definition of the operator $\mathbb{U}_1$ and assumption (H2), we can get

$$\begin{aligned} &\|\mathbb{U}_1(x, y)(\tau)\| \\ &\leq \int_a^\tau \frac{\psi'(s)(\psi(\tau) - \psi(s))^{\nu-1}}{\Gamma(\nu)}\left(\left\|\mathbb{G}_1(s, x(s), y(s)) - \mathbb{G}_1(s, 0, 0)\right\| + \left\|\mathbb{G}_1(s, 0, 0)\right\|\right)\mathrm{d}s \\ &\leq \frac{e^{\theta(\psi(\tau)-\psi(a))}}{\theta^\nu}(p_1^*\|x\|_{\mathfrak{B}} + q_1^*\|y\|_{\mathfrak{B}}) + \mathbb{G}_1^*\int_a^\tau \frac{\psi'(s)(\psi(\tau) - \psi(s))^{\nu-1}}{\Gamma(\nu)}\mathrm{d}s \\ &= \frac{e^{\theta(\psi(\tau)-\psi(a))}}{\theta^\nu}(p_1^*\|x\|_{\mathfrak{B}} + q_1^*\|y\|_{\mathfrak{B}}) + \mathbb{G}_1^*\ell_\psi^\nu. \end{aligned}$$

Hence

$$\|\mathbb{U}_1(x, y)\|_{\mathfrak{B}} \leq \frac{p_1^*}{\theta^\nu}\|x\|_{\mathfrak{B}} + \frac{q_1^*}{\theta^\nu}\|y\|_{\mathfrak{B}} + \mathbb{G}_1^*\ell_\psi^\nu.$$

By similar procedure, we get

$$\|\mathbb{U}_2(x, y)\|_{\mathfrak{B}} \leq \frac{p_2^*}{\theta^\mu}\|x\|_{\mathfrak{B}} + \frac{q_2^*}{\theta^\mu}\|y\|_{\mathfrak{B}} + \mathbb{G}_2^*\ell_\psi^\mu.$$

Thus the above inequalities can be written in the vectorial form as follows

$$\left\|\mathbb{U}(x,y)\right\|_{\mathbb{X},\mathfrak{B}} := \begin{pmatrix} \left\|\mathbb{U}_1(x,y)\right\|_{\mathfrak{B}} \\ \left\|\mathbb{U}_2(x,y)\right\|_{\mathfrak{B}} \end{pmatrix} \leq \mathbb{B}_\theta \begin{pmatrix} \|x\|_{\mathfrak{B}} \\ \|y\|_{\mathfrak{B}} \end{pmatrix} + \begin{pmatrix} \mathbb{G}_1^* \ell_\psi^\nu \\ \mathbb{G}_2^* \ell_\psi^\mu \end{pmatrix}, \tag{16}$$

where

$$\mathbb{B}_\theta = \begin{pmatrix} \frac{p_1^*}{\theta^\nu} & \frac{q_1^*}{\theta^\nu} \\ \frac{p_2^*}{\theta^\mu} & \frac{q_2^*}{\theta^\mu} \end{pmatrix}.$$

In a similar way, we get

$$\left\|\mathbb{V}(\bar{x},\bar{y})\right\|_{\mathbb{X},\mathfrak{B}} := \begin{pmatrix} \left\|\mathbb{V}_1(\bar{x},\bar{y})\right\|_{\mathfrak{B}} \\ \left\|\mathbb{V}_2(\bar{x},\bar{y})\right\|_{\mathfrak{B}} \end{pmatrix} \leq \mathbb{D}_\theta \begin{pmatrix} \|\bar{x}\|_{\mathfrak{B}} \\ \|\bar{y}\|_{\mathfrak{B}} \end{pmatrix} + \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}, \tag{17}$$

where

$$\mathbb{D}_\theta = \begin{pmatrix} \frac{\mathbb{A}_1^*}{\theta^\nu} & 0 \\ 0 & \frac{\mathbb{A}_2^*}{\theta^\mu} \end{pmatrix}.$$

Combining (16) and (17), it follows that

$$\left\|\mathbb{U}(x,y)\right\|_{\mathbb{X},\mathfrak{B}} + \left\|\mathbb{V}(\bar{x},\bar{y})\right\|_{\mathbb{X},\mathfrak{B}} \leq \mathbb{B}_\theta \begin{pmatrix} \|x\|_{\mathfrak{B}} \\ \|y\|_{\mathfrak{B}} \end{pmatrix} + \mathbb{D}_\theta \begin{pmatrix} \|\bar{x}\|_{\mathfrak{B}} \\ \|\bar{y}\|_{\mathfrak{B}} \end{pmatrix} + \begin{pmatrix} \ell_\psi^\nu \mathbb{G}_1^* + \phi_1^* \\ \ell_\psi^\mu \mathbb{G}_2^* + \phi_2^* \end{pmatrix}. \tag{18}$$

Now we look for $\xi = (\xi_1, \xi_2) \in \mathbb{R}_+^2$ such that $\mathbb{U}(x,y) + \mathbb{V}(\bar{x},\bar{y}) \in \mathbb{B}_\xi$ for any $(x,y), (\bar{x},\bar{y}) \in \mathbb{B}_\xi$. To this end, according to (18), it is sufficient to show

$$\mathbb{A}_\theta \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \mathbb{M}_1 \\ \mathbb{M}_2 \end{pmatrix} \leq \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

where

$$\begin{pmatrix} \mathbb{M}_1 \\ \mathbb{M}_2 \end{pmatrix} = \begin{pmatrix} \ell_\psi^\nu \mathbb{G}_1^* + \phi_1^* \\ \ell_\psi^\mu \mathbb{G}_2^* + \phi_2^* \end{pmatrix}.$$

Equivalently

$$\begin{pmatrix} \mathbb{M}_1 \\ \mathbb{M}_2 \end{pmatrix} \leq (\mathbb{I} - \mathbb{A}_\theta) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{19}$$

For a sufficiently large θ, matrix $\mathbb{A}_\theta$ is convergent to zero. It yields, from Theorem 2.9 that the matrix $(\mathbb{I} - \mathbb{A}_\theta)$ is nonsingular and $(\mathbb{I} - \mathbb{A}_\theta)^{-1}$ has nonnegative elements. Therefore, (19) is equivalent to

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \geq (\mathbb{I} - \mathbb{A}_\theta)^{-1} \begin{pmatrix} \mathbb{M}_1 \\ \mathbb{M}_2 \end{pmatrix}.$$

Moreover, if we denote

$$(\mathbb{I} - \mathbb{A}_\theta)^{-1} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix},$$

then we obtain

$$\begin{cases} \xi_1 \geq \gamma_1 \mathbb{M}_1 + \gamma_2 \mathbb{M}_2, \\ \xi_2 \geq \gamma_3 \mathbb{M}_1 + \gamma_4 \mathbb{M}_2. \end{cases}$$

Which means that $\mathbb{G}(x,y) + \mathbb{H}(\bar{x},\bar{y}) \in \mathbb{B}_\xi$.

Step 2 : $\mathbb{V}$ is $\mathbb{D}_\theta$-contraction mapping on $\mathbb{B}_\xi$. In fact for each $\tau \in \mathrm{J}$ and for any $(x_1,y_1),(x_2,y_2) \in \mathbb{B}_\xi$. By the same way of the proof of Theorem 3.4, we can easily show that

$$\left\|\mathbb{V}(x_1,y_1) - \mathbb{V}(x_2,y_2)\right\|_{\mathbb{X},\mathfrak{B}} \leq \mathbb{D}_\theta \|(x_1,y_1) - (x_2,y_2)\|_{\mathbb{X},\mathfrak{B}}.$$

Taking θ large enough it follows that the matrix $\mathbb{D}_\theta$ is convergent to zero and thus, $\mathbb{V}$ is an $\mathbb{D}_\theta$-contraction mapping on $\mathbb{B}_\xi$ with respect to the Bielecki norm.

Step 3 : $\mathbb{U}$ is compact and continuous. Firstly, the continuity of $\mathbb{U}$ follows from the continuity of $\mathbb{G}_1$ and $\mathbb{G}_2$. Next we prove that $\mathbb{U}$ is uniformly bounded on $\mathbb{B}_\xi$. From (16), and for each $(x,y) \in \mathbb{B}_\xi$ we can get

$$\left\|\mathbb{U}(x,y)\right\|_{\mathbb{X},\mathfrak{B}} := \begin{pmatrix} \left\|\mathbb{U}_1(x,y)\right\|_{\mathfrak{B}} \\ \left\|\mathbb{U}_2(x,y)\right\|_{\mathfrak{B}} \end{pmatrix} \leq \mathbb{B}_\theta \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \mathbb{G}_1^* \ell_\psi^\nu \\ \mathbb{G}_2^* \ell_\psi^\mu \end{pmatrix} < \infty.$$

This proves that $\mathbb{U}$ is uniformly bounded.

Finally, it remains to show that $\mathbb{U}(\mathbb{B}_\xi)$ is equicontinuous. Let $(x,y) \in \mathbb{B}_\xi$ and any $\tau_1, \tau_2 \in \mathrm{J}$, with $\tau_1 \leq \tau_2$. Taking assumption (H2), into consideration, together with Remark 2.6, we can find

$$\begin{aligned}
&\|\mathbb{U}_1(x,y)(\tau_2) - \mathbb{U}_1(x,y)(\tau_1)\| \\
\leq &\int_a^{\tau_1} \frac{\psi'(s)\left[(\psi(\tau_2)-\psi(s))^{\nu-1} - (\psi(\tau_1)-\psi(s))^{\nu-1}\right]}{\Gamma(\nu)} \|\mathbb{G}_1(s,x(s),y(s))\| \mathrm{ds} \\
&+ \int_{\tau_1}^{\tau_2} \frac{\psi'(s)(\psi(\tau_2)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \|\mathbb{G}_1(s,x(s),y(s))\| \,\mathrm{ds} \\
\leq &(p_1^*\|x\|_\infty + q_1^*\|y\|_\infty + \mathbb{G}_1^*) \int_a^{\tau_1} \frac{\psi'(s)\left[(\psi(\tau_2)-\psi(s))^{\nu-1} - (\psi(\tau_1)-\psi(s))^{\nu-1}\right]}{\Gamma(\nu)} \mathrm{ds} \\
&+ (p_1^*\|x\|_\infty + q_1^*\|y\|_\infty + \mathbb{G}_1^*) \int_{\tau_1}^{\tau_2} \frac{\psi'(s)(\psi(\tau_2)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \,\mathrm{ds} \\
\leq &\frac{p_1^*\iota_2\|x\|_{\mathfrak{B}} + q_1^*\iota_2\|y\|_{\mathfrak{B}} + \mathbb{G}_1^*}{\Gamma(\nu+1)} \left[(\psi(\tau_1)-\psi(a))^\nu + 2(\psi(\tau_2)-\psi(\tau_1))^\nu - (\psi(\tau_2)-\psi(a))^\nu\right] \\
\leq &2\frac{p_1^*\iota_2\xi_1 + q_1^*\iota_2\xi_2 + \mathbb{G}_1^*}{\Gamma(\nu+1)} (\psi(\tau_2)-\psi(\tau_1))^\nu.
\end{aligned}$$

Similarly,

$$\|\mathbb{U}_2(x,y)(\tau_2) - \mathbb{U}_2(x,y)(\tau_1)\| \leq 2\frac{p_2^*\iota_2\xi_1 + q_2^*\iota_2\xi_2 + \mathbb{G}_2^*}{\Gamma(\mu+1)} (\psi(\tau_2)-\psi(\tau_1))^\mu.$$

Therefore,

$$\begin{aligned}
\|\mathbb{U}(x,y)(\tau_2) - \mathbb{U}(x,y)(\tau_1)\| &:= \begin{pmatrix} \|\mathbb{U}_1(x,y)(\tau_2) - \mathbb{U}_1(x,y)(\tau_1)\| \\ \|\mathbb{U}_2(x,y)(\tau_2) - \mathbb{U}_2(x,y)(\tau_1)\| \end{pmatrix} \\
&\leq 2 \begin{pmatrix} \frac{p_1^*\iota_2\xi_1 + q_1^*\iota_2\xi_2 + \mathbb{G}_1^*}{\Gamma(\nu+1)} (\psi(\tau_2)-\psi(\tau_1))^\nu \\ \frac{p_2^*\iota_2\xi_1 + q_2^*\iota_2\xi_2 + \mathbb{G}_2^*}{\Gamma(\mu+1)} (\psi(\tau_2)-\psi(\tau_1))^\mu \end{pmatrix}.
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequalities tends to zero independently of $(x,y) \in \mathbb{B}_\xi$. Hence, we conclude that $\mathbb{T}(\mathbb{B}_\xi)$ is equicontinuous. By Arzelà–Ascoli's theorem, we deduce that $\mathbb{U}$ is a compact operator. Thus all the assumptions of Theorem 2.13 are satisfied. As a consequence of Krasnoselskii's fixed point theorem, we conclude that the operator $\mathbb{T} = \mathbb{U} + \mathbb{V}$ defined by (15) has at least one fixed point $(x,y) \in \mathbb{B}_\xi$, which is just the solution of system (1)–(2). This completes the proof of the Theorem. 3.5. □

Now, We close this section by studying the Ulam–Hyers stability for problem (1)–(2) by means of integral representation of its solution given by $x(\tau) = \mathbb{T}_1(x,y)(\tau)$, $y(\tau) = \mathbb{T}_2(x,y)(\tau)$, where $\mathbb{T}_1$ and $\mathbb{T}_2$ are defined by (12) and (13).

Define the following nonlinear operators $\mathbb{S}_1, \mathbb{S}_2 : \mathbb{X} \to C(\mathrm{J}, \mathbb{R})$:

$$\begin{cases} ({}^c\mathbb{D}_{a^+}^{\nu;\psi}\tilde{x})(\tau) - \mathbb{A}_1\tilde{x}(\tau) - \mathbb{G}_1(\tau,\tilde{x}(\tau),\tilde{y}(\tau)) = \mathbb{S}_1(\tilde{x},\tilde{y})(\tau), \\ ({}^c\mathbb{D}_{a^+}^{\mu;\psi}\tilde{y})(\tau) - \mathbb{A}_2\tilde{y}(\tau) - \mathbb{G}_2(\tau,\tilde{x}(\tau),\tilde{y}(\tau)) = \mathbb{S}_2(\tilde{x},\tilde{y})(\tau), \end{cases} \quad \tau \in \mathrm{J}.$$

For some $\varepsilon_1, \varepsilon_2 > 0$, we consider the following inequality:

$$\begin{cases} \big\|\mathbb{S}_1(\tilde{x},\tilde{y})(\tau)\big\| \le \varepsilon_1, \\ \big\|\mathbb{S}_2(\tilde{x},\tilde{y})(\tau)\big\| \le \varepsilon_2, \end{cases} \quad \tau \in \mathrm{J}. \tag{20}$$

Definition 3.6 ([38?]). *The coupled system* (1)–(2) *is Ulam–Hyers stable if we can find a positive constants* $\omega_i, i = \overline{1,4}$ *such that for every* $\varepsilon_1, \varepsilon_1 > 0$ *and for each solution* $(\tilde{x},\tilde{y}) \in \mathbb{X}$ *of inequality* (20), *there exists a solution* $(x,y) \in \mathbb{X}$ *of* (1)–(2) *with*

$$\begin{cases} \big\|\tilde{x}(\tau) - x(\tau)\big\| \le \omega_1\varepsilon_1 + \omega_2\varepsilon_2, \\ \big\|\tilde{y}(\tau) - y(\tau)\big\| \le \omega_3\varepsilon_1 + \omega_4\varepsilon_2, \end{cases} \quad \tau \in \mathrm{J}.$$

Theorem 3.7. *Let the assumptions of Theorem 3.4 hold. Then problem* (1)–(2) *is Ulam–Hyers stable with respect to the Bielecki's norm.*

Proof. Let $(x,y) \in \mathbb{X}$ be the solution of problem (1)–(2) satisfying (12) and (13). Let $(\tilde{x},\tilde{y})$ be any solution satisfying (20):

$$\begin{cases} ({}^{c}\mathbb{D}^{\nu;\psi}_{a^+}\tilde{x})(\tau) = \mathbb{A}_1\tilde{x}(\tau) + \mathbb{G}_1(\tau,\tilde{x}(\tau),\tilde{y}(\tau)) + \mathbb{S}_1(\tilde{x},\tilde{y})(\tau), \\ ({}^{c}\mathbb{D}^{\mu;\psi}_{a^+}\tilde{y})(\tau) = \mathbb{A}_2\tilde{y}(\tau) + \mathbb{G}_2(\tau,\tilde{x}(\tau),\tilde{y}(\tau)) + \mathbb{S}_2(\tilde{x},\tilde{y})(\tau). \end{cases} \quad \tau \in \mathrm{J}.$$

So

$$\tilde{x}(\tau) = \mathbb{T}_1(\tilde{x},\tilde{y})(\tau) + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathbb{S}_1(\tilde{x},\tilde{y})(s)\mathrm{d}s, \tag{21}$$

and

$$\tilde{y}(\tau) = \mathbb{T}_2(\tilde{x},\tilde{y})(\tau) + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)}\mathbb{S}_2(\tilde{x},\tilde{y})(s)\mathrm{d}s. \tag{22}$$

It follows from (21) and (22) that

$$\big\|\tilde{x}(\tau) - \mathbb{T}_1(\tilde{x},\tilde{y})(\tau)\big\| \le \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\big\|\mathbb{S}_1(\tilde{x},\tilde{y})(s)\big\|\mathrm{d}s \le \ell^{\nu}_{\psi}\varepsilon_1, \tag{23}$$

and

$$\big\|\tilde{y}(\tau) - \mathbb{T}_2(\tilde{x},\tilde{y})(\tau)\big\| \le \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)}\big\|\mathbb{S}_2(\tilde{x},\tilde{y})(s)\big\|\mathrm{d}s \le \ell^{\mu}_{\psi}\varepsilon_2. \tag{24}$$

Thus, by (H2), Lemma 2.5 and inequalities (23), (24), we get

$$\begin{aligned} \big\|\tilde{x}(\tau) - x(\tau)\big\| &= \big\|\tilde{x}(\tau) - \mathbb{T}_1(\tilde{x},\tilde{y})(\tau) + \mathbb{T}_1(\tilde{x},\tilde{y})(\tau) - x(\tau)\big\| \\ &\le \big\|\tilde{x}(\tau) - \mathbb{T}_1(\tilde{x},\tilde{y})(\tau)\big\| + \big\|\mathbb{T}_1(\tilde{x},\tilde{y})(\tau) - \mathbb{T}_1(x,y)(\tau)\big\| \\ &\le \ell^{\nu}_{\psi}\varepsilon_1 + \left(\frac{p_1^* + \mathbb{A}_1^*}{\theta^\nu}\|\tilde{x} - x\|_{\mathfrak{B}} + \frac{q_1^*}{\theta^\nu}\|\tilde{y} - y\|_{\mathfrak{B}}\right)e^{\theta(\psi(\tau)-\psi(a))}. \end{aligned}$$

Hence we get

$$\|\tilde{x} - x\|_{\mathfrak{B}} \le \ell^{\nu}_{\psi}\varepsilon_1 + \frac{p_1^* + \mathbb{A}_1^*}{\theta^\nu}\|\tilde{x} - x\|_{\mathfrak{B}} + \frac{q_1^*}{\theta^\nu}\|\tilde{y} - y\|_{\mathfrak{B}}. \tag{25}$$

Similarly, we have

$$\|\tilde{y} - y\|_{\mathfrak{B}} \le \ell^{\mu}_{\psi}\varepsilon_1 + \frac{p_2^*}{\theta^\mu}\|\tilde{x} - x\|_{\mathfrak{B}} + \frac{q_2^* + \mathbb{A}_2^*}{\theta^\nu}\|\tilde{y} - y\|_{\mathfrak{B}}. \tag{26}$$

Inequalities (25) and(25) can be rewritten in matrix form as

$$(\mathbb{I}-\mathbb{A}_\theta)\begin{pmatrix}\|\tilde{x}-x\|_{\mathfrak{B}}\\ \|\tilde{y}-y\|_{\mathfrak{B}}\end{pmatrix}\le\begin{pmatrix}\ell_\psi^\nu\varepsilon_1\\ \ell_\psi^\mu\varepsilon_2\end{pmatrix}, \tag{27}$$

where $\mathbb{A}_\theta$ is the matrix given by (14). Since the matrix $\mathbb{A}_\theta$ is convergent to zero for sufficiently large θ, it yields, from Theorem 2.7 that the matrix $(\mathbb{I}-\mathbb{A}_\theta)$ is nonsingular and $(\mathbb{I}-\mathbb{A}_\theta)^{-1}$ has nonnegative elements. Therefore, (27) is equivalent to

$$\begin{pmatrix}\|\tilde{x}-x\|_{\mathfrak{B}}\\ \|\tilde{y}-y\|_{\mathfrak{B}}\end{pmatrix}\le(\mathbb{I}-\mathbb{A}_\theta)^{-1}\begin{pmatrix}\ell_\psi^\nu\varepsilon_1\\ \ell_\psi^\mu\varepsilon_2\end{pmatrix},$$

which yields that

$$\begin{cases}\|\tilde{x}-x\|_{\mathfrak{B}}\le\gamma_1\ell_\psi^\nu\varepsilon_1+\gamma_2\ell_\psi^\mu\varepsilon_2,\\ \|\tilde{y}-y\|_{\mathfrak{B}}\le\gamma_3\ell_\psi^\nu\varepsilon_1+\gamma_4\ell_\psi^\mu\varepsilon_2,\end{cases}$$

where $\gamma_i, i=\overline{1,4}$ are the elements of the matrix $(\mathbb{I}-\mathbb{A}_\theta)^{-1}$.

Hence, the coupled system (1)–(2) is Ulam–Hyers stable with respect to Bielecki's norm $\|\cdot\|_{\mathfrak{B}}$. □

4. Existence and uniqueness solutions for problem (3)–(4).

In this section, we focus on the existence and uniqueness of solutions for the given problem (3)–(4). Before proceeding to the main results, we start by the following definition.

Definition 4.1. *By a solution of problem* (3)–(4) *we mean a coupled function* $(x,y)\in C([a-\delta,b],\mathbb{R}^n)\times C([a-\delta,b],\mathbb{R}^n)$ *that satisfies the system*

$$\begin{cases}({}^c\mathbb{D}_{a^+}^{\nu;\psi}x)(\tau)=\mathbb{F}_1(\tau,x_\tau,y_\tau),\\ ({}^c\mathbb{D}_{a^+}^{\mu;\psi}y)(\tau)=\mathbb{F}_2(\tau,x_\tau,y_\tau),\end{cases}\quad\tau\in\mathrm{J},$$

and the initial conditions

$$\begin{cases}x(\tau)=\alpha(\tau),\\ y(\tau)=\beta(\tau),\end{cases}\quad\tau\in[a-\delta,a].$$

To prove the existence of solutions to (3)–(4), we need the following lemma that was proven in the recent work of Almeida [8].

Lemma 4.2 ([8])**.** *Let* $g:[a,b]\times C([-\delta,0],\mathbb{R}^n)\longrightarrow\mathbb{R}^n$ *be a continuous function. Then* $z\in C([a-\delta,b],\mathbb{R}^n)$ *is the solution of*

$$\begin{cases}{}^c\mathbb{D}_{a^+}^{\nu;\psi}z(\tau)=g(\tau,z_\tau), & ,\quad\tau\in[a,b],\\ z(\tau)=\alpha(\tau) & ,\quad\tau\in[a-\delta,a],\end{cases}$$

if and only if it is the solution of the integral equation

$$z(\tau)=\begin{cases}\alpha(a)+\int_a^\tau\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}g(s,z_s)\mathrm{d}s & ,\quad\tau\in[a,b],\\ \alpha(\tau) & ,\quad\tau\in[a-\delta,a].\end{cases} \tag{28}$$

As a consequence of Lemma 4.2 we have the following result which will be used in the sequel in the proofs of the main results.

Lemma 4.3. *Let $\nu, \mu \in (0,1]$ be fixed and $\mathbb{F}_1, \mathbb{F}_2 : \mathrm{J} \times C([-\delta,0],\mathbb{R}^n) \times C([-\delta,0],\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ are a given continuous functions. Then the coupled systems* (3)–(4) *is equivalent to the following integral equations*

$$x(\tau) = \begin{cases} \alpha(a) + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \mathbb{F}_1(s, x_s, y_s)\mathrm{d}s & , \quad \tau \in [a,b], \\ \alpha(\tau) & , \quad \tau \in [a-\delta, a], \end{cases} \tag{29}$$

and

$$y(\tau) = \begin{cases} \beta(a) + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)} \mathbb{F}_2(s, x_s, y_s)\mathrm{d}s & , \quad \tau \in [a,b], \\ \beta(\tau), & , \quad \tau \in [a-\delta, a]. \end{cases} \tag{30}$$

We assume that the conditions given below stands hold throughout the remainder of the paper:

(C1) $\mathbb{F}_1, \mathbb{F}_2 : \mathrm{J} \times C([-\delta,0],\mathbb{R}^n) \times C([-\delta,0],\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ are continuous functions.

(C2) There exist positive constants $\mathbb{L}_i, \mathbb{M}_i, i = 1, 2$, such that

$$\|\mathbb{F}_i(\tau, x, y) - \mathbb{F}_i(\tau, \bar{x}, \bar{y})\| \leq \mathbb{L}_i \|x - \bar{x}\|_{\mathfrak{C}_\delta} + \mathbb{M}_i \|y - \bar{y}\|_{\mathfrak{C}_\delta}$$

for all $\tau \in \mathrm{J}$ and each $(x, y), (\bar{x}, \bar{y}) \in \mathfrak{C}_\delta \times \mathfrak{C}_\delta$.

For the sake of brevity, we set

$$\mathbb{F}_i^* := \sup_{\tau \in \mathrm{J}} \|\mathbb{F}_i(\tau, 0, 0)\|.$$

Define a square matrix $\mathbb{A}_\psi$ as

$$\mathbb{A}_\psi = \begin{pmatrix} \ell_\psi^\nu \mathbb{L}_1 & \ell_\psi^\nu \mathbb{M}_1 \\ \ell_\psi^\mu \mathbb{L}_2 & \ell_\psi^\mu \mathbb{M}_2 \end{pmatrix}. \tag{31}$$

Firstly, we prove the uniqueness result by means of the Perov's fixed point theorem.

Theorem 4.4. *If the assumptions* (C1) *and* (C2) *are true along with the matrix $\mathbb{A}_\psi$ defined in* (31) *converges to zero. Then the coupled system* (3)–(4) *possesses a unique solution in the space $C([a-\delta,b],\mathbb{R}^n) \times C([a-\delta,b],\mathbb{R}^n)$*

Proof. Consider the Banach space $\mathfrak{C} = C([a-\delta,b],\mathbb{R}^n)$ equipped with the norm

$$\|z\|_{\mathfrak{C}} := \sup_{\tau \in [a-\delta,b]} \|z(\tau)\|.$$

Consequently, the product space $\mathcal{X} := \mathfrak{C} \times \mathfrak{C}$ is a generalized Banach space, endowed with the vector-valued norm

$$\|(x,y)\|_{\mathcal{X}} = \begin{pmatrix} \|x\|_{\mathfrak{C}} \\ \|y\|_{\mathfrak{C}} \end{pmatrix}.$$

We define an operator $\mathbb{K} = \big(\mathbb{K}_1, \mathbb{K}_2\big) : \mathcal{X} \to \mathcal{X}$ by:

$$\mathbb{K}(x,y) = \big(\mathbb{K}_1(x,y), \mathbb{K}_2(x,y)\big). \tag{32}$$

where

$$\mathbb{K}_1(x,y)(\tau) = \begin{cases} \alpha(a) + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \mathbb{F}_1(s, x_s, y_s)\mathrm{d}s & , \quad \tau \in [a,b], \\ \alpha(\tau) & , \quad \tau \in [a-\delta, a]. \end{cases} \tag{33}$$

and

$$\mathbb{K}_2(x,y)(\tau) = \begin{cases} \beta(a) + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\mu-1}}{\Gamma(\mu)} \mathbb{F}_2(s, x_s, y_s)\mathrm{d}s & , \quad \tau \in [a,b], \\ \beta(\tau), & , \quad \tau \in [a-\delta, a]. \end{cases} \tag{34}$$

Now, we apply Perov's fixed point theorem to prove that $\mathbb{K}$ has a unique fixed point. To do this, it enough to show that $\mathbb{K}$ is $\mathbb{A}_\psi$-contraction mapping on $\mathcal{X}$. In fact, for all $\tau \in [a-\delta, b], (x,y), (\bar{x},\bar{y}) \in \mathcal{X}$. When $\tau \in [a-\delta, a]$, we have

$$\|\mathbb{K}_1(x,y)(\tau) - \mathbb{K}_1(\bar{x},\bar{y})(\tau)\| = 0.$$

On the other hand, keeping in mind the definition of the operator $\mathbb{K}_1$ on $[a,b]$ together with assumption (C2), we can get

$$\begin{aligned}
&\big\|\mathbb{K}_1(x,y)(\tau) - \mathbb{K}_1(\bar{x},\bar{y})(\tau)\big\| \\
\leq& \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\|\mathbb{F}_1(s,x_s,y_s) - \mathbb{F}_1(s,\bar{x}_s,\bar{y}_s)\|\mathrm{ds} \\
\leq& \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\big(\mathbb{L}_1\|x_s-\bar{x}_s\|_{\mathfrak{C}_\delta} + \mathbb{M}_1\|y_s-\bar{y}_s\|_{\mathfrak{C}_\delta}\big)\mathrm{ds} \\
\leq& \big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}} + \mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big)\int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)}\mathrm{ds} \\
\leq& \frac{(\psi(b)-\psi(a))^\nu}{\Gamma(\nu+1)}\big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}} + \mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big) \\
=& \ell_\psi^\nu\big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}} + \mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big).
\end{aligned}$$

Hence

$$\big\|\mathbb{K}_1(x,y) - \mathbb{K}_1(\bar{x},\bar{y})\big\|_{[a,b]} \leq \ell_\psi^\nu\big(\mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}} + \mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}}\big).$$

By similar procedure, we get

$$\begin{cases} \|\mathbb{K}_2(x,y) - \mathbb{K}_2(\bar{x},\bar{y})\|_{[a-\delta,a]} = 0 \\ \|\mathbb{K}_2(x,y) - \mathbb{K}_2(\bar{x},\bar{y})\|_{[a,b]} \leq \ell_\psi^\mu\big(\mathbb{L}_2\|x-\bar{x}\|_{\mathfrak{C}} + \mathbb{M}_2\|y-\bar{y}\|_{\mathfrak{C}}\big). \end{cases}$$

Consequently,

$$\big\|\mathbb{K}(x,y) - \mathbb{K}(\bar{x},\bar{y})\big\|_{\mathcal{X}} := \begin{pmatrix} \|\mathbb{K}_1(x,y)\|_{\mathfrak{C}} \\ \|\mathbb{K}_2(x,y)\|_{\mathfrak{C}} \end{pmatrix} \leq \mathbb{A}_\psi\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{X}},$$

where $\mathbb{A}_\psi$ is the matrix given by (31). Since the matrix $\mathbb{A}_\psi$ converges to zero, then Theorem 2.12 implies that coupled system (3)–(4) has a unique solution in $\mathcal{X}$. □

Next, the following result is based on Schauder's type fixed point theorem in generalized Banach spaces.

Theorem 4.5. *Let the assumptions* (C1) *and* (C2) *are satisfied. Then the coupled system* (3)–(4) *has at least one solution, provided that the spectral radius of the matrix* $\mathbb{A}_\psi$ *is less than one.*

Proof. In order to apply Schauder's fixed point theorem type in a generalized Banach space, we need to construct a nonempty closed bounded convex set $\mathbb{B}_r \subset \mathcal{X}$ such that

$$\mathbb{K}(\mathbb{B}_r) \subseteq \mathbb{B}_r, \tag{35}$$

where the operator $\mathbb{K}: \mathcal{X} \to \mathcal{X}$ defined in (11). Let us consider the set

$$\mathbb{B}_r = \{(x,y) \in \mathcal{X} : \|(x,y)\|_{\mathcal{X}} \leq r\},$$

where $r := (r_1, r_2) \in \mathbb{R}_+^2$ will be specified later. Now we try to find $r_1, r_2 \geq 0$ such that (35) holds. Indeed, for all $\tau \in [a-\delta, b], (x,y), \in \mathcal{X}$. When $\tau \in [a-\delta, a]$, we have

$$\|\mathbb{K}_1(x,y)(\tau)\| \leq \|\alpha\|_{[a-\delta,a]},$$

which yields

$$\|\mathbb{K}_1(x,y)\|_{[a-\delta,a]} \leq \|\alpha\|_{[a-\delta,a]}, \tag{36}$$

and if $\tau \in [a,b]$, we have

$$\begin{aligned}\|\mathbb{K}_1(x,y)(\tau)\| \leq & \|\alpha(a)\| + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \big(\|\mathbb{F}_1(s,x_s,y_s) - \mathbb{F}_1(s,0,0)\| \\ & + \|\mathbb{F}_1(s,0,0)\|\big)\mathrm{ds} \\ \leq & \|\alpha(a)\| + \ell_\psi^\nu\big(\mathbb{L}_1\|x\|_{\mathfrak{C}} + \mathbb{M}_1\|y\|_{\mathfrak{C}} + \mathbb{F}_1^*\big).\end{aligned}$$

Hence, we get

$$\|\mathbb{K}_1(x,y)\|_{[a,b]} \leq \|\alpha(a)\| + \ell_\psi^\nu\big(\mathbb{L}_1\|x\|_{\mathfrak{C}} + \mathbb{M}_1\|y\|_{\mathfrak{C}} + \mathbb{F}_1^*\big). \tag{37}$$

So from (36) and (37), we get

$$\begin{aligned}\|\mathbb{K}_1(x,y)\|_{\mathfrak{C}} &\leq \|\mathbb{K}_1(x,y)\|_{[a-\delta,a]} + \|\mathbb{K}_1(x,y)\|_{[a,b]} \\ &\leq \|\alpha\|_{[a-\delta,a]} + \|\alpha(a)\| + \ell_\psi^\nu\big(\mathbb{L}_1\|x\|_{\mathfrak{C}} + \mathbb{M}_1\|y\|_{\mathfrak{C}} + \mathbb{F}_1^*\big).\end{aligned}$$

In a similar way, we get

$$\|\mathbb{K}_2(x,y)\|_{\mathfrak{C}} \leq \|\beta\|_{[a-\delta,a]} + \|\beta(a)\| + \ell_\psi^\nu\big(\mathbb{L}_2\|x\|_{\mathfrak{C}} + \mathbb{M}_2\|y\|_{\mathfrak{C}} + \mathbb{F}_2^*\big).$$

Thus the above inequalities can be written in the vectorial form as follows

$$\|\mathbb{K}(x,y)\|_{\mathcal{X}} := \begin{pmatrix} \|\mathbb{K}_1(x,y)\|_{\mathfrak{C}} \\ \|\mathbb{K}_2(x,y)\|_{\mathfrak{C}} \end{pmatrix} \leq \mathbb{A}_\psi \begin{pmatrix} \|x\|_{\mathfrak{C}} \\ \|y\|_{\mathfrak{C}} \end{pmatrix} + \begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix}, \tag{38}$$

where $\mathbb{A}_\psi$ is the matrix given by (31), and

$$\begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix} = \begin{pmatrix} \ell_\psi^\nu \mathbb{F}_1^* + \|\alpha(a)\| + \|\alpha\|_{[a-\delta,a]} \\ \ell_\psi^\mu \mathbb{F}_2^* + \|\beta(a)\| + \|\beta\|_{[a-\delta,a]} \end{pmatrix}.$$

Now we look for $r = (r_1, r_2) \in \mathbb{R}_+^2$ such that $\|\mathbb{K}(x,y)\|_{\mathfrak{C}} \leq r$, for any $(x,y) \in \mathbb{B}_r$. To this end, according to (38), it is sufficient to show

$$\mathbb{A}_\psi \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix} \leq \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

Equivalently

$$\begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix} \leq (\mathbb{I} - \mathbb{A}_\psi) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{39}$$

Since the matrix $\mathbb{A}_\psi$ is convergent to zero. It yields, from Theorem 2.7 that the matrix $(\mathbb{I}-\mathbb{A}_\psi)$ is nonsingular and $(\mathbb{I}-\mathbb{A}_\psi)^{-1}$ has nonnegative elements. Therefore, (39) is equivalent to

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \geq (\mathbb{I} - \mathbb{A}_\psi)^{-1} \begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix}. \tag{40}$$

Furthermore, if we denote

$$(\mathbb{I} - \mathbb{A}_\psi)^{-1} = \begin{pmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{pmatrix}.$$

Then (40) becomes

$$\begin{cases} r_1 \geq \kappa_1 \mathbb{P}_1 + \kappa_2 \mathbb{P}_2, \\ r_2 \geq \kappa_3 \mathbb{P}_1 + \kappa_4 \mathbb{P}_2. \end{cases}$$

Which means that $\mathbb{K}(\mathbb{B}_r) \subseteq \mathbb{B}_r$. Moreover, by a similar process used in [8], it is easy to show that the operator $\mathbb{K}$ is continuous and, $\mathbb{K}(\mathbb{B}_r)$ is relatively compact. Combining this facts, with Arzelà–Ascoli's theorem, we conclude that $\mathbb{K}$ is a compact operator. Invoking Theorem 2.14 we get a fixed point of $\mathbb{K}$ in $\mathbb{B}_r$, which is just a solution of system (3)–(4). This completes the proof of the Theorem 4.5. $\square$

5. Applications

In this section, we provide some examples to illustrate our results constructed in the previous two sections

Example 5.1. *Consider the following fractional relaxation differential systems*

$$\begin{cases} ({}^{c}\mathbb{D}_{0^+}^{0.5}x)(\tau) = 0.5x(\tau) + \mathbb{G}_1(\tau, x(\tau), y(\tau)), \\ ({}^{c}\mathbb{D}_{0^+}^{0.5}y)(\tau) = 0.5y(\tau) + \mathbb{G}_2(\tau, x(\tau), y(\tau)), \end{cases} \quad \tau \in \mathrm{J} := [0, 1], \tag{41}$$

with initial conditions

$$\begin{cases} x(0) = 1, \\ y(0) = 1, \end{cases} \tag{42}$$

where

$$\nu = \mu = 0.5, \mathbb{A}_1 = \mathbb{A}_2 = c = 0.5, a = 0, b = 1, \psi(\tau) = \tau, n = 1.$$

and

$$\mathbb{G}_1(\tau, x(\tau), y(\tau)) = (\tau + 1)\ln(1 + |x(\tau)|) + e^{\tau} \arctan y(\tau),$$
$$\mathbb{G}_2(\tau, x(\tau), y(\tau)) = \frac{\tau^2}{1 + |x(\tau)| + |y(\tau)|}.$$

Clearly, the functions $\mathbb{G}_1, \mathbb{G}_2 : \mathrm{J} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous. Moreover, for any $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $\tau \in \mathrm{J}$ we have

$$|\mathbb{G}_1(\tau, x_1, y_1) - \mathbb{G}_1(\tau, x_2, y_2)| \le p_1(\tau)|x_1 - x_2| + q_1(\tau)|y_1 - y_2|$$
$$|\mathbb{G}_2(\tau, x_1, y_1) - \mathbb{G}_2(\tau, x_2, y_2)| \le p_2(\tau)|x_1 - x_2| + q_2(\tau)|y_1 - y_2|,$$

where

$$p_1(\tau) = \tau + 1, \quad q_1(\tau) = e^{\tau}, \quad p_2(\tau) = q_2(\tau) = \tau^2.$$

Obviously,

$$p_1^* := 2,\ q_1^* := e^2,\ p_2^* = q_2^* := 1,$$

Furthermore, the matrix $\mathbb{A}_\theta$ given by (14) has the following form

$$\mathbb{A}_\theta = \frac{1}{\sqrt{\theta}} \begin{pmatrix} 2.5 & e^2 \\ 1 & 1.5 \end{pmatrix}.$$

Taking θ large enough it follows that the matrix $\mathbb{A}_\theta$ is convergent to zero and thus, an application of Theorem 4.4 shows that the coupled system (41)–(42) has a unique solution and is Ulam–Hyers stable.

Example 5.2. *Let us consider problem (3)–(4) with specific data:*

$$\begin{gathered} \nu = 0.8, \mu = 0.9, a = 0, b = 1, n = 2, \\ \mathbb{A}_1 = \begin{pmatrix} 2.5 & e^2 \\ 1 & 1.5 \end{pmatrix}, \ \mathbb{A}_2 = \begin{pmatrix} 2.5 & e^2 \\ 1 & 1.5 \end{pmatrix}. \end{gathered} \tag{43}$$

In order to illustrate Theorem 3.5, we take $\psi(\tau) = \sigma(\tau)$ where $\sigma(\tau)$ is the Sigmoid function [27] which can be expressed as in the following form

$$\sigma(\tau) = \frac{1}{1 + e^{-\tau}}, \tag{44}$$

and a convenience of the Sigmoid function is its derivative

$$\sigma'(\tau) = \sigma(\tau)(1-\sigma(\tau)).$$

Taking also $\mathbb{G}_1, \mathbb{G}_2 : \mathrm{J}\times\mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *such that,* $x=(x_1,x_2), y=(y_1,y_2)$ *with*

$$\mathbb{G}_1(\tau,x(\tau),y(\tau)) = \begin{pmatrix} (x_1(\tau)+x_2(\tau))e^{\tau} \\ \tau\ln(1+|y_1(\tau)|+|y_2(\tau)|) \end{pmatrix}. \tag{45}$$

$$\mathbb{G}_2(\tau,x(\tau),y(\tau)) = \begin{pmatrix} (1+\tau)e^{-(y_1(\tau)+y_2(\tau))} \\ e^{2\tau}\sin(x_1(\tau)+x_2(\tau)) \end{pmatrix}.$$

Clearly, the functions $\mathbb{G}_1, \mathbb{G}_2$ *are continuous. Moreover, for any* $x,y,\bar{x},\bar{y}\in\mathbb{R}^2$ *and* $\tau\in\mathrm{J}$ *we have*

$$\|\mathbb{G}_1(\tau,x,y)-\mathbb{G}_1(\tau,\bar{x},\bar{y})\|_1 \le p_1(\tau)\|x-\bar{x}\|_1 + q_1(\tau)\|y-\bar{y}\|_1$$
$$\|\mathbb{G}_2(\tau,x,y)-\mathbb{G}_2(\tau,\bar{x},\bar{y})\|_1 \le p_2(\tau)\|x-\bar{x}\|_1 + q_2(\tau)\|y-\bar{y}\|_1,$$

where $\|\cdot\|_1$ *is a norm in* $\mathbb{R}^2$ *defined as follows*

$$\|x\|_1 = |x_1|+|x_2|, \quad x=(x_1,x_2).$$

Hence the hypothesis (H2) *is satisfied with*

$$p_1(\tau)=e^{\tau}, \quad q_1(\tau)=\tau, \quad p_2(\tau)=\tau+1, \quad q_2(\tau)=e^{2\tau}.$$

It follows from Theorem 3.5 that the system (1)–(2) *with the data* (43), (44) *and* (45) *has at least one solution.*

Example 5.3. *Consider the following fractional delayed coupled system of the form:*

$$\begin{cases} ({}^{CH}\mathbb{D}^{0.5}_{1^+}x)(\tau) = \mathbb{F}_1(\tau,x_\tau,y_\tau), \\ ({}^{CH}\mathbb{D}^{0.5}_{1^+}y)(\tau) = \mathbb{F}_2(\tau,x_\tau,y_\tau), \end{cases} \quad \tau\in\mathrm{J}:=[1,e], \tag{46}$$

with initial conditions

$$\begin{cases} x(\tau)=\alpha(\tau)=(\alpha_1(\tau),\alpha_2(\tau)), \\ y(\tau)=\beta(\tau)=(\beta_1(\tau),\beta_2(\tau)), \end{cases} \quad \tau\in[1-\delta,1], \tag{47}$$

where

$$\nu=\mu=0.5, \psi(\tau)=\ln\tau, a=1, b=e, \ell^{\nu}_{\psi}=\ell^{\mu}_{\psi}=\frac{2}{\sqrt{\pi}}.$$

and

$$\mathbb{F}_1(\tau,x_\tau,y_\tau) = \begin{pmatrix} \frac{|x_{1,\tau}|+|x_{2,\tau}|}{e^{\tau+1}} \\ \frac{\sin(|y_{1,\tau}|+|y_{2,\tau}|)}{\tau+9} \end{pmatrix}.$$

$$\mathbb{F}_2(\tau,x_\tau,y_\tau) = \frac{1}{(\tau+1)^2}\begin{pmatrix} \ln(1+|y_{1,\tau}|+|y_{2,\tau}|) \\ |x_{1,\tau}|+|x_{2,\tau}| \end{pmatrix}.$$

Clearly, the functions $\mathbb{F}_1, \mathbb{F}_2$ *are continuous. Moreover, for any* $x,y,\bar{x},\bar{y}\in\mathfrak{C}_\delta$ *and* $\tau\in\mathrm{J}$ *we have*

$$\|\mathbb{F}_1(\tau,x,y)-\mathbb{F}_1(\tau,\bar{x},\bar{y})\|_1 \le \mathbb{L}_1\|x-\bar{x}\|_{\mathfrak{C}_\delta} + \mathbb{M}_1\|y-\bar{y}\|_{\mathfrak{C}_\delta}$$
$$\|\mathbb{F}_2(\tau,x,y)-\mathbb{F}_2(\tau,\bar{x},\bar{y})\|_1 \le \mathbb{L}_2\|x-\bar{x}\|_{\mathfrak{C}_\delta} + \mathbb{M}_2\|y-\bar{y}\|_{\mathfrak{C}_\delta},$$

Hence the hypothesis (C2) *holds with*

$$\mathbb{L}_1=e^{-2}, \quad \mathbb{M}_1=0.1, \quad \mathbb{L}_2=\mathbb{M}_2=0.25.$$

Furthermore, the matrix $\mathbb{A}_\psi$ *given by* (31) *has the following form*

$$\mathbb{A}_\psi = \frac{2}{\sqrt{\pi}}\begin{pmatrix} e^{-2} & 0.1 \\ 0.25 & 0.25 \end{pmatrix}.$$

Using the Matlab program we can get the eigenvalues of $\mathbb{A}_\psi$ *as follows* $\sigma_1=0.0276, \sigma_2=0.4072$, *which show that* $\mathbb{A}_\psi$ *is converging to zero. Therefore, by Theorem 4.5 the coupled system* (46)–(47) *has a unique solution.*

6. Conclusion

The present work addressed some basic results on the existence, uniqueness, and Ulam–Hyers stability of solutions for a new problem of FDEs containing ψ–Caputo fractional derivative without and with delay in generalized Banach spaces. The results are obtained through the techniques of fixed point theory and nonlinear analysis. At the last, we yielded some examples that fulfill our findings. It will be also interesting to look for some qualitative properties of solutions for a coupled system of nonlinear fractional differential equations involving ψ–Riesz-Caputo fractional derivatives. This case will be taken into account in our future work.

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