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# Bertrand Curves and B-Lift Curves in Lorentzian 3-Space

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Abstract. In this article, based on Thorpe's definition, we define a new curve called the B-lift curve in Lorentzian 3-space and examine the Frenet vectors of the B-lift curve. Furthermore, we examine the relationship between the Frenet vectors of the Bertrand curve and the Frenet vectors of the natural lift curve. Finally, we give an example on these results.

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**Keywords:** B-lift curve, Bertrand curve, Lorentzian space.

## 1. Introduction

The Lorentzian space is a field that is extensively studied by those interested in theoretical physics and mathematics. This field increased its popularity in the 20th century when it was used in Einstein's special and general relativity theories and therefore it had an opportunity to develop. Today, it has become a structure used in every field of science whether theoretical or applied. Lorentzian space is often compared to Euclidean space. The spacetime interval corresponding to the geometric length in Lorentzian space is spacelike, timelike or lightlike (null).

The concept of curves is one of the fundemental topics of differential geometry. One of these curves is the natural lift curve. Natural lift curve was introduced in Thorpe's "Elementary Topics in Differential Geometry" book [11]. The natural lift curve is defined as the curve drawn by the endpoints of the unit tangent vector at each point of a given curve. Many mathematicians have studied on natural lift curves [2–6]. The Frenet vectors of the natural lift curve were introduced by Çalışkan and Ergün with regard to the Frenet vectors of the main curve [5].

A space curve can be defined depending on the parameter and the Frenet operators of the curve can be characterized. Some special curve definitions are given by establishing a relationship between Frenet vectors at the mutual points of two curves in space. Bertrand curves are one of these curves.

The emergence of the Bertrand curve is due to the problem posed by Venant in 1845. Venant posed the problem of whether there is another curve whose normals are linearly dependent on a surface produced on the principal normal of a curve. Bertrand solved this problem in 1850. A basic study on the Bertrand curves was examined by Ekmekci and İlarslan in 2018. They characterized the Bertrand curves in Lorentzian space [1].

In this study, we defined the B-Lift curve and obtained the Frenet vectors of this curve in the Lorentzian 3-space. Besides, we introduce the equations of the Frenet vectors between the Bertrand curve and the B-Lift curve. Eventually, we give an examples and draw our curves with Mathematica program.

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## 2. Preliminaries

The Lorentzian 3-space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3$ supplied by Lorentzian inner product defined as

$$< x, y >_{IL} = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $x = (x_1, x_2, x_3)$ , $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ [7].

Let $x = (x_1, x_2, x_3)$ be a vector in $\mathbb{R}^3_1$. Then, x is called spacelike if $\langle$ x,x $\rangle_{IL} > 0$ or x=0, timelike if $\langle$ x,x $\rangle_{IL} < 0$, and lightlike (null) if $\langle$ x,x $\rangle_{IL} = 0$ and x≠0 [7].

A curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ is spacelike, timelike or lightlike (null), if $\gamma^{'}(s)$ is spacelike, timelike or lightlike (null) at any $s \in I$, respectively. Using the Lorentzian inner product, the norm of the vector $x = (x_1, x_2, x_3)$ is defined as [7]

$$\|x\| = \sqrt{|\langle x, x\rangle_{IL}|}.$$

If $\|x\| = 1$, then the vector $x$ is called unit vector. For the vectors $x$ and $y$ in $\mathbb{R}^3_1$, the Lorentzian vector product of the vectors $x$ and $y$ is defined as [8]

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Suppose that $\gamma$ is a unit speed curve. The set $\{T(s), N(s), B(s)\}$ is called Frenet frame given by tangent, principal normal and binormal vectors, respectively. Now, we examine Frenet formulas depending on the Lorentzian character of the curve [12]:

**i**) Let $\gamma$ be unit speed spacelike curve with spacelike binormal. Then, $T$ and $B$ are spacelike vectors, $N$ is a timelike vector. In this condition, we have

$$N \times B = -T, \quad T \times N = -B, \quad B \times T = -N.$$

Frenet formulas are following as

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= \kappa T + \tau B, \\ B^{'} &= \tau N. \end{aligned}$$

**ii**) Let $\gamma$ be unit speed spacelike curve with timelike binormal. Then, $T$ and $N$ are spacelike vectors, $B$ is a timelike vector. In this case, we have

$$N \times B = -T, \quad T \times N = B, \quad B \times T = -N.$$

Frenet formulas are as follows

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= -\kappa T + \tau B, \\ B^{'} &= \tau N. \end{aligned}$$

**iii**) Let $\gamma$ be unit speed timelike curve. Then, $N$ and $B$ are spacelike vectors and $T$ is a timelike vector. In that case, we have

$$N \times B = T, \quad T \times N = -B \quad B \times T = -N.$$

Frenet formulas are as follows

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= \kappa T + \tau B, \\ B^{'} &= -\tau N. \end{aligned}$$

**Lemma 2.1** ( [10]). *Assume that $x$ and $y$ are linearly independent spacelike vectors which span a spacelike vector subspace in $\mathbb{R}^3_1$. In that case, we get the following inequality:*

$$|\langle x, y\rangle_{IL}| \leq \|x\| \cdot \|y\|.$$

*Hence, we can write*

$$\langle x, y\rangle_{IL} = \|x\|\cdot\|y\| cos\varphi,$$

*where $\varphi$ is a Lorentzian spacelike angle amongst x and y.*

**Lemma 2.2** ( [10]). *Let x and y be linearly independent spacelike vectors which span a timelike vector subspace in $\mathbb{R}^3_1$. Then we have*

$$|\langle x, y\rangle_{IL}| > \|x\|\cdot\|y\|.$$

*Hence, we can write*

$$|\langle x, y\rangle_{IL}| = \|x\|\cdot\|y\| cosh\varphi,$$

*where $\varphi$ is a Lorentzian timelike angle amongst x and y.*

**Lemma 2.3** ( [10]). *Suppose that x is a spacelike vector and y is a timelike vector in $\mathbb{R}^3_1$. In this situation, we can write*

$$|\langle x, y\rangle_{IL}| = \|x\|\cdot\|y\| sinh\varphi,$$

*where $\varphi$ is a Lorentzian timelike angle amongst x and y.*

**Lemma 2.4** ( [10]). *Imagine that x and y are timelike vectors in $\mathbb{R}^3_1$. In that case, we can write*

$$\langle x, y\rangle_{IL} = \|x\|\cdot\|y\| cosh\varphi,$$

*where $\varphi$ is a Lorentzian timelike angle amongst x and y.*

**Definition 2.5** ( [1]). Assume that $\gamma = (\gamma(s); T(s), N(s), B(s))$ and $\gamma^* = (\gamma^*(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ are regular curves in $\mathbb{R}^3_1$. $\gamma(s)$ and $\gamma^*(s^*)$ are called the Bertrand curve if $N(s)$ and $N^*(s^*)$ are linearly independent. In that case, $(\gamma, \gamma^*)$ is called Bertrand mate.

**Proposition 2.6** ( [9]). *Let $(\gamma, \gamma^*)$ be a timelike-spacelike Bertrand mate. We know the following equations amongst the Frenet frame $\{T^*, N^*, B^*\}$ of the curve $\gamma^*$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} sinh\theta & 0 & cosh\theta \\ 0 & 1 & 0 \\ -cosh\theta & 0 & -sinh\theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**Proposition 2.7** ( [9]). *Let $(\gamma, \gamma^*)$ be a timelike Bertrand couple. We know the following equation:*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} cosh\theta & 0 & sinh\theta \\ 0 & 1 & 0 \\ -sinh\theta & 0 & cosh\theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**Proposition 2.8** ( [9]). *Let $\gamma$ and $\gamma^*$ be spacelike curves with spacelike binormal. We know the following equation among the Frenet frame $\{T^*, N^*, B^*\}$ of the curve $\gamma^*$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & -cos\theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**Proposition 2.9** ( [9]). *Let $\gamma$ be a spacelike curve with timelike binormal. Then, the Bertrand curve of the curve $\gamma$ is the spacelike curve and $B^*$ is a timelike vector. We know the following equation among the Frenet frame $\{T^*, N^*, B^*\}$ of the curve $\gamma^*$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} cosh\theta & 0 & sinh\theta \\ 0 & 1 & 0 \\ sinh\theta & 0 & -cosh\theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**Proposition 2.10** ( [9]). *Assume that $\gamma$ is a spacelike curve and B is a timelike vector. Then, the Bertrand curve of the curve $\gamma$ is the timelike curve. We know the following equation among the Frenet frame $\{T^*, N^*, B^*\}$ of the curve $\gamma^*$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} sinh\theta & 0 & cosh\theta \\ 0 & 1 & 0 \\ cosh\theta & 0 & -sinh\theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

## 3. Bertrand Curves and B-Lift Curves in Lorentzian 3-Space

**Definition 3.1.** Let $\gamma : I \to P$ be a unit speed curve, then $\gamma_B : I \to TP$ is called the B-Lift curve and ensures the following equation:

$$\gamma_B(s) = (\gamma(s), B(s)) = B(s)|_{\gamma(s)},$$

where $B$ is the binormal vector of the curve $\gamma$.

**Proposition 3.2.** *Let $\gamma$ be a timelike curve. Then, $\gamma_B$ is a spacelike curve with spacelike or timelike binormal.*

**i**) *Assume that $\gamma_B$ is a spacelike curve and $B_B$ is timelike vector. We know the following equations among the Frenet frame $\{T_B, N_B, B_B\}$ of the curve $\gamma_B$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

**a**) *If W is spacelike vector, we get*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -cosh\varphi & 0 & -sinh\varphi \\ sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**b**) *If W is timelike vector, we get*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -sinh\varphi & 0 & -cosh\varphi \\ cosh\varphi & 0 & sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**ii**) *Assume that $\gamma_B$ is a spacelike curve and $B_B$ is spacelike vector. We know the following equations among the Frenet frame $\{T_B, N_B, B_B\}$ of the curve $\gamma_B$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

**a**) *If W is spacelike vector, we know*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ cosh\varphi & 0 & sinh\varphi \\ sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**b**) *If W is timelike vector, we know*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ sinh\varphi & 0 & cosh\varphi \\ cosh\varphi & 0 & sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**Proposition 3.3.** *Suppose that $\gamma$ is a spacelike curve and $B$ is a spacelike vector. Then, $\gamma_B$ is a timelike curve. We know the following equation among the Frenet frame $\{T_B, N_B, B_B\}$ of the curve $\gamma_B$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ cos\varphi & 0 & sin\varphi \\ sin\varphi & 0 & -cos\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**Proposition 3.4.** *Assume that $\gamma$ is a spacelike curve and $B$ is timelike vector. Then, $\gamma_B$ is a spacelike curve and $B_B$ is timelike or spacelike vector.*

**i**) *Let $\gamma_B$ be a spacelike curve with timelike binormal. We know the following equations among the Frenet frame $\{T_B, N_B, B_B\}$ of the curve $\gamma_B$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

**a**) *If W is spacelike vector, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -sinh\varphi & 0 & cosh\varphi \\ cosh\varphi & 0 & -sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**b**) *If W is timelike vector, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -cosh\varphi & 0 & sinh\varphi \\ sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**ii**) *Let $\gamma_B$ be a spacelike curve with spacelike binormal. We know the following equations among the Frenet frame $\{T_B, N_B, B_B\}$ of the curve $\gamma_B$ and the Frenet frame $\{T, N, B\}$ of the curve $\gamma$ :*

**a**) *If W is spacelike vector, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ sinh\varphi & 0 & -cosh\varphi \\ cosh\varphi & 0 & -sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**b**) *If $W$ is timelike vector, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ cosh\varphi & 0 & sinh\varphi \\ sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

**Proposition 3.5.** *Imagine that $\gamma$ is a spacelike curve and $B$ is a spacelike vector. In that case, $\gamma_B$ is a timelike curve, $\gamma^*$ is a spacelike curve and $B^*$ is a spacelike vector. We know the following equations:*

$$\begin{aligned} T^* &= cos(\theta - \varphi)N_B - sin(\theta - \varphi)B_B, \\ N^* &= T_B, \\ B^* &= sin(\theta + \varphi)N_B - cos(\theta + \varphi)B_B. \end{aligned}$$

**Proposition 3.6.** *Let $\gamma$ be a spacelike curve with timelike binormal. Then, $\gamma_B$ and $\gamma^*$ are spacelike curve with timelike binormal.*

**a**) *If $W$ Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= sinh(\theta + \varphi)N_B + cosh(\theta + \varphi)B_B, \\ N^* &= T_B, \\ B^* &= cosh(\theta + \varphi)N_B + sinh(\theta + \varphi)B_B. \end{aligned}$$

**b**) *If $W$ Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= -cosh(\theta + \varphi)N_B - sinh(\theta + \varphi)B_B, \\ N^* &= T_B, \\ B^* &= -sinh(\theta + \varphi)N_B - cosh(\theta + \varphi)B_B. \end{aligned}$$

**Proposition 3.7.** *Assume that $\gamma$ is a spacelike curve and $B$ is a timelike vector. In that case, $\gamma_B$ is a spacelike curve with timelike binormal and $\gamma^*$ is a spacelike curve with spacelike binormal.*

**a**) *If $W$ Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= sinh(\theta + \varphi)N_B - cosh(\theta + \varphi)B_B, \\ N^* &= T_B, \\ B^* &= cosh(\theta + \varphi)N_B - sinh(\theta + \varphi)B_B. \end{aligned}$$

**b**) *If $W$ Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= -cosh(\theta + \varphi)N_B + sinh(\theta + \varphi)B_B, \\ N^* &= T_B, \\ B^* &= -sinh(\theta + \varphi)N_B + cosh(\theta + \varphi)B_B. \end{aligned}$$

**Proposition 3.8.** *Suppose that $\gamma$ is a spacelike curve and $B$ is a timelike vector. Then, $\gamma_B$ is a spacelike curve, $B_B$ is timelike vector and $\gamma^*$ is a timelike curve.*

**a**) *If $W$ Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= cosh(\theta + \varphi)N_B + sinh(\theta + \varphi)B_B, \\ N^* &= T_B, \\ B^* &= sinh(\theta + \varphi)N_B + cosh(\theta + \varphi)B_B. \end{aligned}$$

**b**) *If $W$ Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= -sinh(\theta + \varphi)N_B - cosh(\theta + \varphi)B_B, \\ N^* &= T_B, \\ B^* &= -cosh(\theta + \varphi)N_B - sinh(\theta + \varphi)B_B. \end{aligned}$$

**Proposition 3.9.** *Imagine that $\gamma$ is a spacelike curve and $B$ is a timelike vector. Then, $\gamma_B$ is a spacelike curve, $B_B$ is a spacelike vector and $\gamma^*$ is a timelike curve.*

**a**) *If W Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= -cosh(\theta+\varphi)N_B + sinh(\theta+\varphi)B_B, \\ N^* &= T_B, \\ B^* &= -sinh(\theta+\varphi)N_B + cosh(\theta+\varphi)B_B. \end{aligned}$$

**b**) *If W Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= sinh(\theta+\varphi)N_B - cosh(\theta+\varphi)B_B, \\ N^* &= T_B, \\ B^* &= cosh(\theta+\varphi)N_B - sinh(\theta+\varphi)B_B. \end{aligned}$$

**Proposition 3.10.** *Assume that $\gamma$ is a timelike curve. Then, $\gamma_B$ is a spacelike curve, $B_B$ is a timelike vector and $\gamma^*$ is a timelike curve.*

**a**) *If W Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= -cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= -sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B. \end{aligned}$$

**b**) *If W Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= -sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= -cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B. \end{aligned}$$

**Proposition 3.11.** *Assume that $\gamma$ is a timelike curve. Then, $\gamma_B$ is a spacelike curve, $B_B$ is a spacelike vector and $\gamma^*$ is a timelike curve.*

**a**) *If W Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B. \end{aligned}$$

**b**) *If W Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B. \end{aligned}$$

**Proposition 3.12.** *Imagine that $\gamma$ is a timelike curve. Then, $\gamma_B$ is a spacelike curve, $B_B$ is timelike vector and $\gamma^*$ is a spacelike curve.*

**a**) *If W Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= -sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= -cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B. \end{aligned}$$

**b**) *If W Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= -cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= -sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B. \end{aligned}$$

**Proposition 3.13.** *Suppose that $\gamma$ is a timelike curve. Then, $\gamma_B$ is a spacelike curve, $B_B$ is a spacelike vector and $\gamma^*$ is a spacelike curve.*

**a**) *If W Darboux vector is spacelike, we get*

$$\begin{aligned} T^* &= sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B. \end{aligned}$$

**b**) *If W Darboux vector is timelike, we get*

$$\begin{aligned} T^* &= cosh(\theta-\varphi)N_B + sinh(\theta-\varphi)B_B, \\ N^* &= -T_B, \\ B^* &= sinh(\theta-\varphi)N_B + cosh(\theta-\varphi)B_B. \end{aligned}$$

**Corollary 3.14.** *Let $\gamma_B$ be a B-Lift curve of $\gamma$ and $\gamma^*$ be a Bertrand mate of $\gamma$, then the set $\{T_B, N^*\}$ is linearly independent.*

**Example 3.15.** Let $\gamma$ be a unit speed spacelike circular helix curve that is given by $\gamma(s) = \left(\frac{4}{3}s, \frac{5}{3}coss, \frac{5}{3}sins\right)$.

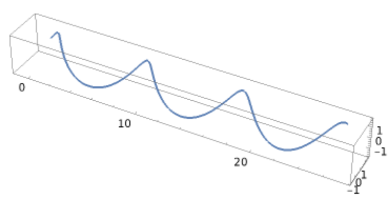


Figure 1. Circular helix $\gamma(s)$

After some calculations the Frenet vectors of the curve $\gamma$ are as follows:

$$\begin{aligned} T(s) &= \left(\frac{4}{3}, -\frac{5}{3}sins, \frac{5}{3}coss\right), \\ N(s) &= (0, coss, -sins), \\ B(s) &= \left(\frac{5}{3}, \frac{4}{3}sins, \frac{4}{3}coss\right). \end{aligned}$$

Since $\gamma_B(s) = B(s)$, we have $\gamma_B(s) = \left(\frac{5}{3}, \frac{4}{3}sins, \frac{4}{3}coss\right)$.

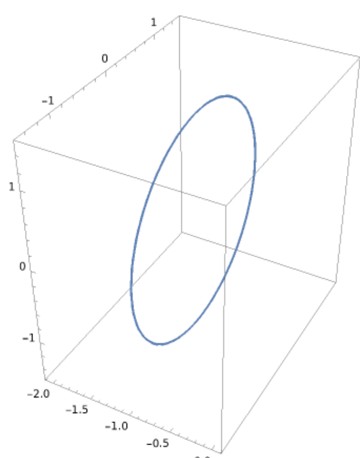


Figure 2. The curve $\gamma_B(s)$

Bertrand couple of the curve $\gamma(s)$ is given as

$$\begin{aligned}\gamma^*(s) &= \gamma(s) + \lambda \cdot N(s), \quad \lambda \in \mathbb{R} \\ &= \left(\frac{4}{3}s, \frac{5}{3}coss, \frac{5}{3}sins\right) + \lambda \cdot (0, coss, sins)\end{aligned}$$

For $\lambda = \frac{-2}{3}$, we have $\gamma^*(s) = \left(\frac{4}{3}s, coss, sins\right)$.

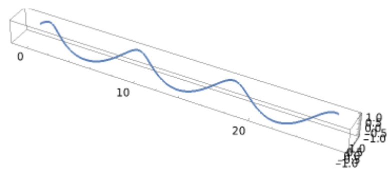


Figure 3. The curve $\gamma^*(s)$

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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