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Differential Equations of Rectifying Curves and Focal Curves in $\mathbb{E}^n$

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Abstract

In this present paper, rectifying curves are re-characterized in a shorter and simpler way using harmonic curvatures and some relations between rectifying curves and focal curves are found in terms of their harmonic curvature functions in $n-$dimensional Euclidean space. Then, a rectifying Salkowski curve, which is the focal curve of a given space curve is investigated. Finally, some figures related to the theory are given in the case $n=3$.

1. Introduction

Kim and et al. consider a space curve in which the relationship between torsion and curvature is a non-constant linear function, [1]. Then, Chen characterize a special curve whose position vector always lies in its rectifying plane, [2, 3]. In other words, the position vector of a rectifying curve α with Frenet vector $\{T,N,B\}$ can be stated by

$$\alpha(s)=\lambda(s)T(s)+\mu(s)B(s) \tag{1.1}$$

for $\lambda(s)$ and $\mu(s)$ differentiable functions. The most known characterization of the rectifying curve is that the ratio of torsion to curvature is a non-constant linear function in terms of its arc-length parameter s. The authors prove that the centrode of a unit speed curve with non-zero constant curvature (or non-constant curvature) and non-constant torsion (or non-zero constant torsion) is a rectifying curve . Then, Chen obtain that a curve on a cone in $\mathbb{E}^3$ is a geodesic if and only if it is either a rectifying curve or an open portion of a ruling, [4]. Furthermore, Cambie et al. generalize rectifying curves in an arbitrary dimensional Euclidean space, [5]. In addition to these, in Minkowski space, rectifying curve is similar to in Euclidean space, [6, 7].

In 1975, authors introduced the functions of harmonic curvature, [8]. The authors generalize inclined curves thanks to the harmonic curvature in E^3 to E^n and then give a characterization for the inclined curves in E^n. This subject has been studied by many authors since then and it also has many geometric interpretations. For example, Camci et al. investigate the relations between the harmonic curvatures of a non-degenerate curve and the focal curvatures of tangent indicatrix of the curve and they give that harmonic curvature of the curve is focal curvature of the tangent indicatrix [9]. Kaya et al. give a new definition of helix strip. They study the harmonic curvatures functions of a strip by using harmonic curvature functions and give some characterizations of the strips's harmonic curvature functions and total curvature functions of a strip [10]. The authors look in a non-generated curve for a generalized helix using these curvatures in [11]. Then, Gök et al. define a new kind of helix called V_n-slant helix by using a similar approach of harmonic curvature functions in n-dimensional Euclidean space and Minkowski space, [12, 13]. Harmonic curvatures are also studied in the Lorentz-Minkowski space [14, 15, 16]. As it can be easily seen when these studies are examined, it has been very useful to use harmonic curvature functions when characterizing curves. The existing characterizations have been made very short and simple through the harmonic curvature, especially when working in high-dimensional spaces. Many geometric concepts such as helices, slant helices, strips and some other special curves have been defined by using their harmonic curvature functions.

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On the other hand, Vargas defined focal curve of α which is the centers of its osculating hyperspheres of the curve, [17]. The centers of the osculating hyperspheres of the curve are well defined only for the points of the curve where all curvatures are non-zero. Öztürk and Arslan characterized focal curves and their Darboux vectors. They have shown that if the ratios of the curvatures of a curve γ are constant, then the ratios of the curvatures of the focal curve C_γ are constant, [18]. Furthermore, Öztürk et al. studied the focal representation of $k-$slant helices in $\mathbb{E}^{m+1}$, [19].

In this study, rectifying curves with their harmonic curvature functions are re-characterized in $n-$dimensional Euclidean space. Then, some relations between rectifying curves and focal curves are investigated. Also, a necessary condition for the focal curve of any space curve to be a rectifying curve is given. Finally, the rectifying Salkowski curve whose focal curve is a rectifying curve is investigated.

2. Basic Concepts and Notations

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be an arbitrary curve in $\mathbb{E}^n$. Let $\{T, N, B_1, B_2, ..., B_{n-2}\}$ be the moving Serret-Frenet frame along the unit speed curve α. Then the Frenet formulas are given as follows

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ \vdots \\ B_{n-4}' \\ B_{n-3}' \\ B_{n-2}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ \vdots \\ B_{n-4} \\ B_{n-3} \\ B_{n-2} \end{bmatrix}$$

where all k_i curvatures denotes the i^{th} curvature function of the curve and positive, [20, 21].

Definition 2.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve. Harmonic curvatures of α is defined by*

$$H_i : I \subset \mathbb{R} \to \mathbb{R}, \quad i = 0, 1, 2, ..., n-2, \tag{2.1}$$

$$H_i(s) = \begin{cases} 0 & , \quad i = 0 \\ \dfrac{k_1(s)}{k_2(s)} & , \quad i = 1 \\ \dfrac{1}{k_{i+1}(s)} \{V_1[H_{i-1}] + H_{i-2}k_i\} & , \quad i = 2, 3, ..., n-1 \end{cases} \tag{2.2}$$

in the paper, [8].

Definition 2.2. *A curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ is a rectifying curve if the orthogonal complement of $N(s)$ contains a fixed point for all $s \in I$, [5].*

Definition 2.3. *The center of the osculating hypersphere of the curve α at a point lies in the hyperplane normal to the curve α at that point. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a curve with Frenet vectors $\{T, N, B_1, B_2, ..., B_{n-2}\}$ and k_i curvature functions. Then the focal curve of the curve α is written as follows:*

$$C_\alpha = \alpha(s) + c_1(s)N(s) + c_2(s)B_1(s) + c_3(s)B_2(s) + ... + c_{n-1}(s)B_{n-2}(s) \tag{2.3}$$

where $c_1, c_2, ..., c_{n-1}$ smooth functions called focal curvatures of the curve α. Moreover, the function c_1 never vanishes and $c_1(s) = \dfrac{1}{k_1(s)}$. Then, the focal curvature functions of the curve α have defined as

$$c_i(s) = \begin{cases} 0 & , \quad i = 0 \\ \dfrac{1}{k_1(s)} & , \quad i = 1 \\ \dfrac{1}{k_i(s)} \{c_{i-1}'(s) + c_{i-2}(s)k_{i-1}(s)\} & , \quad i = 2, 3, ..., n-1 \end{cases} \tag{2.4}$$

in the paper, [17].

Theorem 2.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be an arbitrary curve with Frenet vectors $\{T, N, B_1, ..., B_{n-2}\}$ and C_α be its focal curve with the Frenet vectors $\{\overline{T}, \overline{N}, \overline{B}_1, ..., \overline{B}_{n-2}\}$ in $\mathbb{E}^n$. Then, k_i and K_i, denotes the i^{th} curvature functions of the curve α and the curve C_α, respectively. There are following relationship between the Frenet frames and curvatures of the curves.*

$$\begin{aligned} \overline{T} &= B_{n-2}, \quad \overline{N} = B_{n-3}, \quad \overline{B}_1 = B_{n-4} \quad , \ldots, \quad \overline{B}_{n-3} = N, \quad \overline{B}_{n-2} = T, \\ K_i &= \frac{k_{n-i+1}}{c_{n-1}' + k_{n-1}c_{n-2}}, \ i \in \{1, 2, ..., n-1\} \end{aligned} \tag{2.5}$$

where c_i, $i \in \{1, 2, ..., n-1\}$ are the focal curvatures of the curve, [17].

Definition 2.5. *A curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ is the Salkowski curve if and only if it has the constant curvature but non-constant torsion with an explicit parametrization, [22, 23].*

3. Rectifying Curves in $n-$Dimensional Euclidean Space

In this first subsection, re-characterization of rectifying curve according to harmonic curvature functions is given with similar idea defined by Özdamar and Hacısalihoğlu in [8]. In the next subsection, we will look for the answer to the following question

"When does the focal curve of a given curve become rectifying curve?".

3.1. Rectifying Curves with Harmonic Curvature Functions

Let α be an arc-length parametrized rectifying curve in $\mathbb{E}^n$ as

$$\alpha(s) = \lambda(s)T(s) + \mu_1(s)B_1(s) + ... + \mu_{n-2}(s)B_{n-2}(s) \tag{3.1}$$

with $\lambda, \mu_1, ..., \mu_{n-2}$ real valued functions. If we take the derivative of α, get following equation

$$\begin{aligned} \alpha'(s) &= \lambda'(s)T(s) + \lambda(s)k_1(s)N(s) + \mu_1'(s)B_1(s) + \mu_1(s)(-k_2(s)N(s) + k_3(s)B_2(s)) + ... \\ &\quad + \mu_{n-2}'(s)B_{n-2}(s) + \mu_{n-2}(s)(-k_{n-1}(s)B_{n-3}(s)). \end{aligned}$$

Also, if we make the necessary arrangements, we have

$$\begin{aligned} T(s) &= \lambda'(s)T(s) + (\lambda(s)k_1(s) - \mu_1(s)k_2(s))N(s) + (\mu_1'(s) - \mu_2(s)k_3(s))B_1(s) + (\mu_1(s)k_3(s) + \mu_2'(s) - \mu_3(s)k_4(s))B_2(s) \\ &\quad + (\mu_2(s)k_4(s) + \mu_3'(s) - \mu_4(s)k_5(s))B_3(s) + ... + (\mu_{n-2}'(s) + \mu_{n-3}(s)k_{n-1}(s))B_{n-2}(s). \end{aligned}$$

So, we can write following equations as

$$\begin{cases} i. & \lambda'(s) = 1 \\ ii. & \lambda(s)k_1(s) - \mu_1(s)k_2(s) = 0 \\ iii. & \lambda(s)k_1(s) - \mu_1(s)k_2(s) = 0 \\ iv. & \mu_{i-1}(s)k_{i+1}(s) + \mu_i'(s) - \mu_{i+1}(s)k_{i+2}(s) = 0, \quad i \in \{2,3,...,n-3\} \\ v. & \mu_{n-2}'(s) + \mu_{n-3}(s)k_{n-1}(s) = 0. \end{cases} \tag{3.2}$$

We will try to determine λ and μ_i functions with the help of the harmonic curvature functions defined by the following definitions. In fact, we want to emphasize the similarity of the previously described $\mu_{i,k}$ functions in [5] and harmonic curvature functions.

Definition 3.1. *Let α be parameterized by an arc-length parameter curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1, k_2, ..., k_{n-1}\}$. Then, we define the harmonic curvature of rectifying curve α in terms of the curvatures using the similar idea given in the paper [8].*

$$H_i(s) = \begin{cases} 0 & , \quad i = 0 \\ (s+c)\dfrac{k_1(s)}{k_2(s)} & , \quad i = 1 \\ \dfrac{1}{k_{i+1}(s)}\left\{H_{i-1}'(s) + H_{i-2}(s)k_i(s)\right\} & , \quad i = 2,3,...,n-2 \end{cases} \tag{3.3}$$

where c is a real constant.

Definition 3.2. *Let α be an arc-lengthed regular curve in $\mathbb{E}^n$ with focal curvatures $\{c_1, c_2, ..., c_{n-1}\}$. Then the harmonic curvature functions of α in terms of the focal curvatures as follows:*

$$H_i(s) = \begin{cases} 0 & , \quad i = 0 \\ \dfrac{c_1'(s)}{c_1(s)c_2(s)} & , \quad i = 1 \\ \dfrac{2c_i(s)c_{i+1}(s)}{\delta_i(s)}\left\{\dfrac{2c_{i-1}(s)c_i(s)}{\delta_{i-1}(s)}H_{i-2}(s) + H_{i-1}'(s)\right\} & , \quad i = 2,3,...,n-2 \end{cases}$$

where $\delta_i(s) = \left(\sum_{j=1}^{i} c_j^2(s)\right)'$, [18].

Corollary 3.3. *Let α be an arc-lengthed rectifying curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1, k_2, ..., k_{n-1}\}$. Then, following equalities are obtained from equation (3.2) according to harmonic curvatures in equation (3.3).*

i. $\lambda(s) = s + c$

ii. $\mu_1(s) = \lambda(s)\dfrac{k_1(s)}{k_2(s)} = H_1(s)$

iii. $\mu_2(s) = \dfrac{\mu_1'(s)}{k_3(s)} = \dfrac{1}{k_3(s)}H_1'(s) = H_2(s)$

iv. $\mu_i(s) = \dfrac{1}{k_{i+1}(s)}\left\{\mu_{i-1}'(s) + \mu_{i-2}(s)k_i(s)\right\}$

$\mu_i(s) = \dfrac{1}{k_{i+1}(s)}\left\{H_{i-1}'(s) + H_{i-2}(s)k_i(s)\right\} = H_i(s)$

In the following Corollary, we will reconstract the Theorem 4.1 given in [5] in terms of the harmonic curvature functions.

Corollary 3.4. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1,k_2,...,k_{n-1}\}$. Then α is congruent to a rectifying curve if and only if*

$$H'_{n-2}(s)+H_{n-3}(s)k_{n-1}(s)=0 \tag{3.4}$$

where H_i are harmonic curvature functions.

Proof. Assume that α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1,k_2,...,k_{n-1}\}$. If α is a rectifying curve, we have the following equation according to item (v) in equation (3.2)

$$\mu'_{n-2}(s)+\mu_{n-3}(s)k_{n-1}(s)=0. \tag{3.5}$$

Also, from the above Corollary, we have $\mu_i(s)=H_i(s)$. If this equation is substituted in the above equation, we can easily write that

$$H'_{n-2}(s)+H_{n-3}(s)k_{n-1}(s)=0. \tag{3.6}$$

Conversely, assume that equation (3.4) is provided. Then, we can see that α is congruent to a rectifying curve. □

Corollary 3.5. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1,k_2,...,k_{n-1}\}$. The position vector of the rectifying curve α satisfies*

$$\alpha(s)=(s+c)T(s)+H_1(s)B_1(s)+...+H_{n-2}(s)B_{n-2}(s) \tag{3.7}$$

for H_i differentiable harmonic curvature functions.

Now we give a relationship between Corollary 3.4 and Theorem 4.1 in [5] with the following Corollary. The first two items are our results and the third item is the characterization of being a rectifying curve in study [5]. In other words, these theories are compatible.

Corollary 3.6. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures. Then the following equations are equivalent*
i) *α is a rectifying curve.*
ii) $H'_{n-2}(s)+H_{n-3}(s)k_{n-1}(s)=0.$
iii) $k_{n-1}(s)\sum\limits_{m=0}^{n-4}\mu_{n-3,m}(s)\frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right)+\sum\limits_{m=0}^{n-3}\left(\mu_{n-2,m}(s)\frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right)\right)'=0$

The authors gave a new approach on helices in $\mathbb{E}^n$ with harmonic curvature functions in [24]. With the help of this idea we give a relation between rectifying curve and harmonic curvature functions in the following theorem.

Theorem 3.7. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures. Then, $\sum\limits_{i=1}^{n-2}H_i^2(s)$ is non-zero constant where $H_{n-2}(s)\neq 0$ if and only if the curve α is a rectifying curve.*

Proof. Let $H_1^2(s)+H_2^2(s)+...+H_{n-2}^2(s)$ be a non-zero constant. From the equation (3.3), we have that

$$k_{i+1}(s)H_i(s)=H'_{i-1}(s)+k_i(s)H_{i-2}(s),\ 2\leq i\leq n-2 \tag{3.8}$$

If we write $i+1$ instead of i in equation (3.8), we get

$$H'_i(s)=k_{i+2}(s)H_{i+1}(s)-k_{i+1}(s)H_{i-1}(s),\ 1\leq i\leq n-3. \tag{3.9}$$

For $i=1$,

$$H'_1(s)=k_3(s)H_2(s). \tag{3.10}$$

We know that $H_1^2+H_2^2+...+H_{n-2}^2$ is constant. So we can see that

$$H_1(s)H'_1(s)+H_2(s)H'_2(s)+...+H_{n-2}(s)H'_{n-2}(s)=0$$

and

$$H_{n-2}(s)H'_{n-2}(s)=-H_1(s)H'_1(s)-H_2(s)H'_2(s)-...-H_{n-3}(s)H'_{n-3}(s). \tag{3.11}$$

If we multiply $H_i(s)$ and $H_1(s)$ both sides of the equation (3.9) and equation (3.10), respectively, we get

$$H_i(s)H'_i(s)=k_{i+2}(s)H_i(s)H_{i+1}(s)-k_{i+1}(s)H_{i-1}(s)H_i(s) \tag{3.12}$$

and

$$H_1(s)H'_1(s)=k_3(s)H_1(s)H_2(s). \tag{3.13}$$

Hence, from the equations (3.11), (3.12) and (3.13) we can easily show that

$$H_{n-2}(s)H'_{n-2}(s)=-k_{n-1}(s)H_{n-3}(s)H_{n-2}(s).$$

Since $H_{n-2}(s) \neq 0$, we have

$$H'_{n-2}(s) + k_{n-1}(s)H_{n-3}(s) = 0.$$

So, from the Corollary 3.2, the curve α is a rectifying curve.
Conversely, assume that α is a rectifying curve. From the Corollary 3.2, we know that the equality

$$H'_{n-2}(s) + H_{n-3}(s)k_{n-1}(s) = 0$$

is provided. Moreover, for $H_{n-2} \neq 0$, we can write

$$H_{n-2}(s)H'_{n-2}(s) = -k_{n-1}(s)H_{n-2}(s)H_{n-3}(s).$$

From the equations (3.13) and (3.12) , we obtain

$$H_1(s)H'_1(s) = k_3(s)H_1(s)H_2(s)$$

and

$$\begin{aligned}
\text{for } i &= n-3,\ H_{n-3}H'_{n-3} = k_{n-1}H_{n-3}H_{n-2} - k_{n-2}H_{n-4}H_{n-3},\\
\text{for } i &= n-4,\ H_{n-4}H'_{n-4} = k_{n-2}H_{n-4}H_{n-3} - k_{n-3}H_{n-5}H_{n-4},\\
\text{for } i &= n-5,\ H_{n-5}H'_{n-5} = k_{n-3}H_{n-5}H_{n-4} - k_{n-4}H_{n-6}H_{n-5},\\
&\ \vdots\\
\text{for } i &= 2,\ H_2H'_2 = k_4H_2H_3 - k_3H_1H_2.
\end{aligned}$$

Then it is easy to see that

$$H_1(s)H'_1(s) + H_2(s)H'_2(s) + \ldots + H_{n-3}(s)H'_{n-3}(s) + H_{n-2}(s)H'_{n-2}(s) = 0 \tag{3.14}$$

and

$$H_1^2(s) + H_2^2(s) + \ldots + H_{n-2}^2(s)$$

is a non-zero constant. □

Special Case for $n = 3$

In this part, we will verify the general theory for $n = 3$ because of the fact that the following characterizations are given in previous works [2] and [3]. Then, considering the definition of harmonic curvature functions of rectifying curves we show that the theory of paper is right for $n = 3$.

Let α be an arc-length parameterized rectifying curve in $\mathbb{E}^3$ as follows

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s) \tag{3.15}$$

with λ, μ real functions.
If we take the derivative of α, then we have

$$\alpha'(s) = \lambda'(s)T(s) + \lambda(s)k_1(s)N(s) + \mu'(s)B(s) + \mu(s)(-k_2(s)N(s))$$

and if the necessary arrangements are made, it is available

$$T(s) = \lambda'(s)T(s) + (\lambda(s)k_1(s) - \mu(s)k_2(s))N(s) + \mu'(s)B(s)$$

So, we can easily obtain the following equations from the above equality.

i) $\lambda'(s) = 1$
ii) $\lambda(s)k_1(s) - \mu(s)k_2(s) = 0$
iii) $\mu'(s) = 0$

We will try to determine λ and μ functions with the help of the harmonic curvature of the curve α given in the equation (3.3). Then, the functions
i) $\lambda(s) = s + c$
ii) $\mu(s) = \lambda(s)\dfrac{k_1(s)}{k_2(s)} = H_1(s)$
are easily obtained.

Corollary 3.8. *Let α be an arc-length parametrized curve in $\mathbb{E}^3$ with non-zero curvatures. Then α is congruent to a rectifying curve if and only if*

$$H'_1(s) = 0 \tag{3.16}$$

where H_1 is the 1^{th} harmonic curvature functions of the curve.

Corollary 3.9. *Let α be an arc-length parametrized curve in $\mathbb{E}^3$ with non-zero curvatures. If the curve α is a rectifying, then the position vector of the curve satisfies*

$$\alpha(s) = (s+c)T(s) + H_1(s)B(s) \tag{3.17}$$

where H_1 is 1^{th} harmonic curvature functions of the curve.

3.2. Rectifying Curves and Focal Curves

In this subsection, some relations between rectifying curve and focal curve are given in $n-$dimensional Euclidean space.

Theorem 3.10. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a curve with $\{T, N, B_1, ..., B_{n-2}\}$ and C_α be focal curve of α with $\left\{\overline{T}, \overline{N}, \overline{B}_1, ..., \overline{B}_{n-2}\right\}$. λ_i, $i \in \{1, 2, ..., n-1\}$ denotes the i^{th} function of the position vector of α and c_i, $i \in \{1, 2, ..., n-1\}$ denotes the i^{th} focal curvature of the curve α. Then, the focal curve C_α of the curve α is a rectifying curve if and only if following equation is satisfied*

$$\lambda_{n-1} = -c_{n-2}. \tag{3.18}$$

Proof. Let α be an arbitrary curve and C_α be focal curve of the curve α. Then the curve C_α can be written as follows

$$C_\alpha = \lambda_1 T + \lambda_2 N + \lambda_3 B_1 + ... + \lambda_n B_{n-2} + c_1 N + c_2 B_1 + c_3 B_2 + ... + c_{n-1} B_{n-2}.$$

If we rearrange the C_α by using $\left\{\overline{T}, \overline{N}, \overline{B}_1, \overline{B}_2, ..., \overline{B}_{n-2}\right\}$ from the equation (2.5), we get

$$\begin{aligned} C_\alpha &= \lambda_1 \overline{B}_{n-2} + \lambda_2 \overline{B}_{n-3} + ... + \lambda_n \overline{T} + c_1 \overline{B}_{n-3} + c_2 \overline{B}_{n-4} + ... + c_{n-1} \overline{T}, \\ C_\alpha &= (\lambda_n + c_{n-1})\overline{T} + (\lambda_{n-1} + c_{n-2})\overline{N} + ... + (\lambda_2 + c_1)\overline{B}_{n-3} + \lambda_1 \overline{B}_{n-2}. \end{aligned}$$

Since, C_α is a rectifying curve, following equality is available

$$\lambda_{n-1} + c_{n-2} = 0.$$

Conversely, assume that equation (3.18) is provided. Then we can easily see that C_α is a rectifying curve. □

In the following part, we will give the properties of rectifying curve with the focal curve in the $3-$dimensional Euclidean space according to the Frenet apparatus $\{T, N, B, k_1, k_2\}$.

Corollary 3.11. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary curve with $\{T, N, B, k_1, k_2\}$ and C_α be focal curve of α with $\left\{\overline{T}, \overline{N}, \overline{B}, \overline{k}_1, \overline{k}_2\right\}$ in the $3-$dimensional Euclidean space. $\lambda_1, \lambda_2, \lambda_3$ denotes the functions of the position vector of α and c_1, c_2 denotes functions of the focal curvature of the curve α. Then, the focal curve C_α of α is a rectifying curve if and only if following equality holds*

$$\lambda_2 = -c_1. \tag{3.19}$$

Corollary 3.12. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a curve with $\{T, N, B\}$ and C_α be focal curve of α with $\left\{\overline{T}, \overline{N}, \overline{B}\right\}$İf the curve α is a rectifying curve, the focal curve C_α can not be rectifying curve.*

Proof. Let α be an arbitrary curve with $\{T, N, B\}$. We can write α as

$$\alpha = \lambda_1 T + \lambda_2 N + \lambda_3 B.$$

If the curve α is a rectifying, then $\lambda_2 = 0$. But from above theorem, we know that C_α focal curve of α is a rectifying curve if and only if $\lambda_2 = -c_1 = -\dfrac{1}{k_1}$. Consequently, C_α can not be a rectifying curve.

□

Salkowski curves are defined as curves with constant curvature but non-constant torsion with an explicit parametrization. In the following two Corollaries, we give a rectifying curve which is a focal curve of a given Salkowski space curve. For this purpose, we will define the torsion of the given Salkowski curve in Euclidean $3-$space.

Corollary 3.13. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary Salkowski curve with $\{T, N, B\}$. If the focal curve C_α of α is a rectifying curve, the torsion of α is equal to $k_2(s) = \dfrac{1}{\sqrt{\frac{2s}{\lambda_1 k_1} + c}}$, $(\frac{2s}{\lambda_1 k_1} + c) > 0$.*

Proof. Since α is an arbitrary Salkowski curve, $k_1(s)$ is a constant function. Assume that the curve C_α be a rectifying curve. Then, the theory of focal curves and Theorem 3.3 give that the position vector of the curve α is

$$\alpha(s) = \lambda_1(s)T(s) + \lambda_2 N(s) + \lambda_3(s)B(s) \tag{3.20}$$

where $\lambda_2 = -c_1 = -\frac{1}{k_1}$ is a constant function. Differentiating the equation (3.20) with respect to s, we obtain

$$T(s) = (\lambda_1'(s) + 1)T(s) + (\lambda_1(s)k_1 - \lambda_3(s)k_2(s))N(s) + \left(\lambda_3'(s) - \frac{k_2(s)}{k_1}\right)B(s).$$

Then, the equality gives us the following system

$$\left.\begin{array}{c} \lambda_1'(s) = 0 \\ \lambda_1(s)k_1 - \lambda_3(s)k_2(s) = 0 \\ \lambda_3'(s) - \frac{k_2(s)}{k_1} = 0 \end{array}\right\} \tag{3.21}$$

If we consider the equation (3.21), we can easily find the following differential equation

$$\left(\frac{\lambda_1 k_1}{k_2(s)}\right)' - \frac{k_2(s)}{k_1} = 0 \tag{3.22}$$

and then the solition of the equation (3.22) is given by

$$k_2(s) = \frac{1}{\sqrt{\frac{2s}{\lambda_1 k_1} + c}}, \quad \left(\frac{2s}{\lambda_1 k_1} + c\right) > 0.$$

□

Corollary 3.14. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary Salkowski curve with the Frenet frame $\{T, N, B\}$. From the above Corollary, we can write rectifying focal curve such as*

$$C_\alpha(s) = \lambda_1 T(s) + \left(\sqrt{(2s + \lambda_1 k_1 c)\lambda_1 k_1} + c_2\right) B(s)$$

where λ_1, k_1, c, c_2 are constant functions.

Example 3.15. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary Salkowski curve and C_α be focal curve of the curve α and coefficient functions of the curve C_α be as follows;*

$$k_1 = 1, \quad k_2 = \frac{1}{\sqrt{2s}}, \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = \sqrt{2s}, \quad c_1 = 1, \quad c_2 = 0.$$

So, C_α focal curve of α is a rectifying curve such as

$$C_\alpha(s) = (f_1(s), f_2(s), f_3(s))$$

where

$$\begin{aligned} f_1(s) &= \frac{4\sqrt{s}\cos 2\sqrt{s} - \sin 2\sqrt{s} + 2\sqrt{s}}{2\sqrt{2}} \\ f_2(s) &= \frac{1}{2}\cos 2\sqrt{s} + 2\sqrt{s}\sin 2\sqrt{s} \\ f_3(s) &= \frac{-4\sqrt{s}\cos 2\sqrt{s} + \sin 2\sqrt{s} + 2\sqrt{s}}{2\sqrt{2}}. \end{aligned}$$

The figure of the rectifying focal curve C_α as follows,

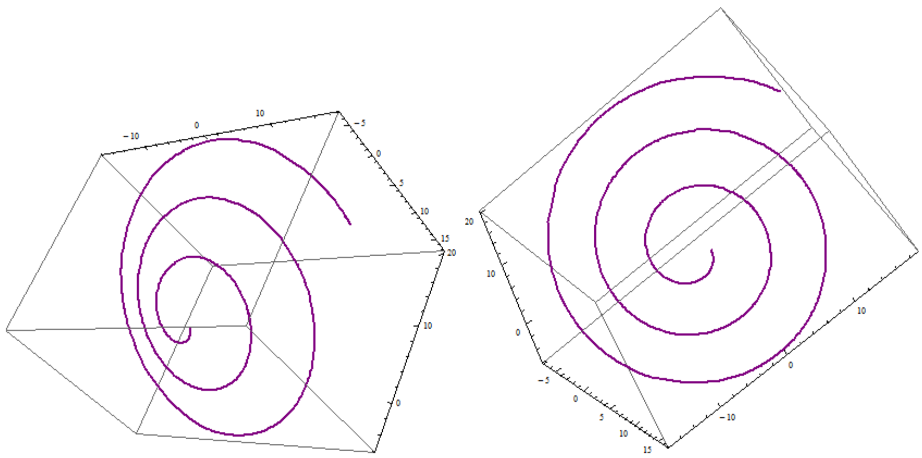

Figure 1. The focal curve C_α

4. Conclusion

Harmonic curvature functions used in several previous studies. In this study, by using harmonic curvature functions a new approach on rectifying curve is given. Characterizing rectifying curves in 3 and 4-dimensional space is easy, but calculations in $n-$dimensional space are not so easy. Harmonic curvatures have given us convenience in our operations and simplicity in characterizations. Authors in [5] characterized rectifying curve in an arbitrary dimensional Euclidean space as

$$k_{n-1}(s)\sum_{m=0}^{n-4}\mu_{n-3,m}(s)\frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right) + \sum_{m=0}^{n-3}\left(\mu_{n-2,m}(s)\frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right)\right)' = 0.$$

We have shown that the μ_i coefficients in the author's work correspond to harmonic curvatures in minor adjustments. Hence, we prove this theory for rectifying curve more simply associating with harmonic curvature functions such as

$$H'_{n-2}(s) + H_{n-3}(s)k_{n-1}(s) = 0.$$

Also, we give the relationship between rectifying curve and the focal curve in $n-$dimensional Euclidean space. And give necessary and sufficient conditions in which the focal curve of any space curve is a rectifying curve. Subsequently, we examine the these theories for special case $n = 3$. In general, our aim in this study is to examine rectifying curves and focal curves from a different perspective using harmonic curvatures.

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Author's contributions

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