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Matrix Operators on the Absolute Euler Space $\left|E_{\phi}^{r}\right|(\mu)$

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Abstract. In recent paper, the space $\left|E_{\phi}^{r}\right|(\mu)$ which is the generalization of the absolute Euler Space on the space $l(\mu)$, has been introduced and studied by Gökçe and Sarıgöl [3]. In this study, we give certain characterizations of matrix transformations from the paranormed space $\left|E_{\phi}^{r}\right|(\mu)$ to one of the classical sequence spaces c_0, c, l_∞. Also, we show that such matrix operators are bounded linear operators.

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1. Introduction

The sequence spaces have an important place in summability theory, a wide field of mathematics, which have a great number of applications in approximation theory, calculus, and basically in functional analysis. Constitutively, the classical theory deals with the generalization of the concept of convergence for sequences and series. The aim is to assign a limit value for non-convergent sequences and series by making use of an operator determined by infinite matrices. The reason why matrices are used for a general linear operator is that a linear operator from a sequence space to another one can be given by an infinite matrix. The literature in the field of summability theory continues to develop not only on the generation of sequence spaces through the matrix domain of a particular limitation method and on the investigation of their topological, algebraic structures and matrix transformations but also on examinations about new series spaces derived by various absolute summability methods from a different perspective. In recent paper, the paranormed space $\left|E_{\phi}^{r}\right|(\mu)$ which is the generalization of the Absolute Euler Space on the space $l(\mu)$ has been introduced and certain matrix operators on the space have been investigated by Gökçe and Sarıgöl. In the present study, we give some characterizations of matrix transformation from the paranormed space $\left|E_{\phi}^{r}\right|(\mu)$ to one of the classical sequence spaces c_0, c, l_∞. Also, we show that such matrix operators are bounded linear operators. Finally, we express some results.

Firstly, we recall certain known concepts. Let ω be the set of all sequences of complex numbers, U, V be arbitrary sequence spaces, i.e., subsets of ω, and $\Lambda = (\lambda_{nv})$ be an infinite matrix of complex components. By $\Lambda(u) = (\Lambda_n(u))$, we stand for the Λ-transform of a sequence $u = (u_v)$, if the series

$$\Lambda_n(u) = \sum_{v=0}^{\infty} \lambda_{nv} u_v$$

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is convergent for all $n \geq 0$. If $\Lambda(u) \in V$, whenever $u \in U$, then Λ, denoted by $\Lambda : U \to V$, is called a matrix transformation from the space U into another space V, and, by (U, V), we mean the class of all infinite matrices Λ such that $\Lambda : U \to V$. The space of all convergent and bounded series are represented by c_s, b_s, respectively. Also, the concept of matrix domain of an infinite matrix Λ in a sequence space U is defined by the set

$$\Lambda_U = \{u = (u_n) \in \omega : \Lambda(u) \in U\},$$

which is also a sequence space.

If a subspace U is a Frechet space, i.e, a complete locally convex linear metric space, with continuous coordinates $p_n : U \to \mathbb{C}$ $(n \in \mathbb{N})$, where $p_n(u) = u_n$ for all $u \in U$, then it is called to be an FK space; an FK space whose metric is given by a norm is called to be a BK space. An FK-space U including the set of all finite sequences is said to have AK property if

$$\lim_{m\to\infty} u^{[m]} = \lim_{m\to\infty} \sum_{j=0}^{m} u_j e^{(j)} = u,$$

for each sequence $u \in U$ where $e^{(j)}$ is a sequence whose only non-zero term is 1 in j-th place for $j \geq 0$. For instance [8–10], it is well known that the Maddox's space

$$l(\mu) = \left\{u = (u_n) : \sum_{n=1}^{\infty} |u_n|^{\mu_n} < \infty\right\}$$

is an FK-space with AK property with respect to the paranorm

$$g(u) = \left(\sum_{n=0}^{\infty} |u_n|^{\mu_n}\right)^{1/M},$$

where $M = \max\{1, \sup_n \mu_n\}$; also the space is even a BK- space if $\mu_n \geq 1$ for all n with respect to the norm

$$\|u\| = \inf\left\{\delta > 0 : \sum_{n=0}^{\infty} |u_n/\delta|^{\mu_n} \leq 1\right\}.$$

Throughout the whole paper, we assume that $0 < \inf \mu_n \leq K < \infty$ and μ_n^* is conjugate of μ_n, *i.e.*, $1/\mu_n + 1/\mu_n^* = 1$, $\mu_n > 1$, and $1/\mu_n^* = 0$ for $\mu_n = 1$.

Assume that $\sum a_v$ is a given infinite series, s_n is its n-th partial sum, $\phi = (\phi_n)$ is a sequence of positive real numbers and $\mu = (\mu_n)$ is a bounded sequence of positive real numbers. If

$$\sum_{n=1}^{\infty} (\phi_n)^{\mu_n - 1} |U_n(s) - U_{n-1}(s)|^{\mu_n} < \infty,$$

then the series $\sum a_v$ is said to be summable $|U, \phi_n|(\mu)$ [4]. It should be point out that the summability $|U, \phi_n|(\mu)$ includes a lot of well known summability methods for special selections of the matrix U and the sequences ϕ, μ. For instance, if we take $U = E^r$ ($\mu_n = p$ for all n), then this method is reduced to the summability method $|E^r, \phi|(\mu)$ [3] ($|E^r, \phi|_p$ [5]). Here, Euler matrix E^r is described by

$$e_{nj}^r = \begin{cases} \binom{n}{j}(1-r)^{n-j} r^j, & 0 \leq j \leq n \\ 0, & j > n, \end{cases}$$

for $0 < r < 1$ and

$$e_{nj}^1 = \begin{cases} 0, & 0 \leq j < n \\ 1, & j = n, \end{cases}$$

(see also [1, 2, 7, 13, 14]).

Now, we recall some lemmas which have important places in our study.

2. Needed Lemmas

Lemma 2.1 ([6]). *Let $\mu = (\mu_v)$ be arbitrary bounded sequence of strictly positive numbers.*

(i) If $\mu_v \leq 1$, then

$$\Lambda \in (l(\mu), c) \Leftrightarrow (a) \lim_n \lambda_{nv} \text{ exists for each } v,\ (b) \sup_{n,v} |\lambda_{nv}|^{\mu_v} < \infty,$$

$$\Lambda \in (l(\mu), c_0) \Leftrightarrow (c) \lim_n \lambda_{nv} = 0 \text{ for each } v,\ (b) \text{ holds,}$$

and

$$\Lambda \in (l(\mu), l_\infty) \Leftrightarrow (b) \text{ holds.}$$

(ii) If $\mu_v > 1$ for all v, then

$$\Lambda \in (l(\mu), c) \Leftrightarrow (a') \lim_n \lambda_{nv} \text{ exists for each } v, (b') \text{ there is a number } K > 1 \text{ such that}$$

$$\sup_n \sum_{v=0}^{\infty} \left| \lambda_{nv} K^{-1} \right|^{\mu_v^*} < \infty,$$

$$\Lambda \in (l(\mu), c_0) \Leftrightarrow (c') \lim_n \lambda_{nv} = 0 \text{ for each } v,\ (b') \text{ holds,}$$

and

$$\Lambda \in (l(\mu), l_\infty) \Leftrightarrow (b') \text{ holds.}$$

Lemma 2.2 ([12]). *Let U be an FK-space with AK property, R be triangle, S be its inverse and V be any subset of ω. Then, we have $\Lambda \in (U_R, V)$ if and only if $\widehat{\Lambda} \in (U, V)$ and $W^{(n)} \in (U, c)$ for all n, where*

$$\hat{\lambda}_{nv} = \sum_{j=v}^{\infty} \lambda_{nj} s_{jv};\ n, v = 0, 1, ...$$

and

$$w_{mv}^{(n)} = \begin{cases} \sum_{j=v}^{m} \lambda_{nj} s_{jv}, & 0 \leq v \leq m \\ 0, & v > m. \end{cases}$$

Lemma 2.3 ([11]). *Let R be a triangle. Then, for $U, V \subset \omega$, $\Lambda \in (U, V_R)$ if and only $B = R\Lambda \in (U, V)$.*

3. Main Results

In this section, we give certain characterizations of matrix transformations between the paranormed series space $\left| E_\phi^r \right| (\mu)$ which is defined as the set of all series summable by the absolute summability method of Euler matrix and classical sequence spaces c, c_0, l_∞. Also, we show that the operators are linear bounded operators.

As a beginning, we remind that, using the definition of the summability $|\Lambda, \phi_n| (\mu)$, the space $\left| E_\phi^r \right| (\mu)$ can be written as

$$\left| E_\phi^r \right| (\mu) = \left\{ a \in \omega : \sum_{n=0}^{\infty} \phi_n^{\mu_n - 1} \left| \triangle \Lambda_n^r (s) \right|^{\mu_n} < \infty \right\},$$

where

$$\triangle \Lambda_n^r (s) = \Lambda_n^r (s) - \Lambda_{n-1}^r (s)$$

and

$$\Lambda_n^r (s) = \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k s_k,\ \ n \geq 0,\ \ \Lambda_{-1}^r (s) = 0.$$

After a few calculation, we can introduce the space $\left| E_\phi^r \right| (\mu)$ as follows:

$$\left| E_\phi^r \right| (\mu) = \left\{ a = (a_k) : \sum_{n=1}^{\infty} \left| \phi_n^{1/\mu_n^*} \sum_{k=1}^{n} \binom{n-1}{k-1} (1-r)^{n-k} r^k a_k \right|^{\mu_n} < \infty \right\}.$$

On the other hand, by considering $T_n^r(\phi,\mu)(a) = \phi_n^{1/\mu_n^*} \triangle \Lambda_n^r(s)$, we get $T_0^r(\phi,\mu)(a) = a_0\phi_0^{1/\mu_0^*}$ and

$$\begin{aligned} T_n^r(\phi,\mu)(a) &= \phi_n^{1/\mu_n^*} \sum_{k=1}^{n} \binom{n-1}{k-1} r^k (1-r)^{n-k} a_k \\ &= \sum_{k=1}^{n} t_{nk}^r(\phi,\mu) a_k, \end{aligned}$$

where

$$t_{nk}^r(\phi,\mu) = \begin{cases} \phi_0^{1/\mu_0^*}, & k=n=0 \\ \phi_n^{1/\mu_n^*}\binom{n-1}{k-1} r^k(1-r)^{n-k}, & 1 \le k \le n \\ 0, & k > n. \end{cases}$$

According to the notation of domain, it is written that

$$\left|E_\phi^r\right|(\mu) = [l(\mu)]_{T^r(\phi,\mu)}$$

and so, the connection between Maddox's space and the space $\left|E_\phi^r\right|(\mu)$ is established. Also, it is obvious that if $r = 1$ and $\phi_n = 1$ for all $n \ge 0$, the space $\left|E_\phi^r\right|(\mu)$ is reduced to the space $l(\mu)$.

It is noted that since every triangle matrix has a unique inverse which is a triangle (see [15]), the matrix $T^r(\phi,\mu)$ has a unique inverse $S^r(\phi,\mu) = (s_{nk}^r(\phi,\mu))$ given by

$$s_{nk}^r(\phi,\mu) = \begin{cases} \phi_0^{-1/\mu_0^*}, & k=n=0 \\ \phi_k^{-1/\mu_k^*}\binom{n-1}{k-1}(r-1)^{n-k} r^{-n}, & 1 \le k \le n \\ 0, & k > n. \end{cases}$$

Lemma 3.1 ([3]). *Let $0 < r < 1$ and $\mu = (\mu_n)$ be a bounded sequence of non-negative numbers. Then, the set $\left|E_\phi^r\right|(\mu)$ is a linear space with the coordinate-wise addition and scalar multiplication. Beside, this space is an FK-space with the paranorm given by*

$$g(x) = \left(\sum_{n=0}^{\infty} \left| T_n^r(\phi,\mu)(x) \right|^{\mu_n} \right)^{1/M},$$

where $M = \max\{1, \sup \mu_n\}$.

Moreover, the space $\left|E_\phi^r\right|(\mu)$ is isometrically isomorphic to the space $l(\mu)$, i.e., $\left|E_\phi^r\right|(\mu) \cong l(\mu)$.

Theorem 3.2. *Let $\Lambda = (\lambda_{nv})$ be an infinite matrix of complex numbers, (ϕ_n) be sequence of positive numbers, $\mu = (\mu_n)$ be arbitrary bounded sequence of positive numbers with $\mu_n \le 1$ for all n. Beside, let the matrix $\hat{\Lambda}$ be defined by*

$$\hat{\lambda}_{nv} = \sum_{j=v}^{\infty} \lambda_{nj} b_j^{(v)},$$

where $b_j^{(v)} = \phi_v^{-1/\mu_v^*}\binom{j-1}{v-1}(r-1)^{j-v} r^{-j}, v \ge 1; b_0^{(0)} = \phi_0^{-1/\mu_0^*}$. Then, for $n = 0, 1, ...$
(a) $\Lambda \in \left(\left|E_\phi^r\right|(\mu), c\right)$ if and only if

$$\sum_{j=v}^{\infty} b_j^{(v)} \lambda_{nj} \text{ converges for each } v, \tag{3.1}$$

$$\sup_{m,v} \left| \sum_{j=v}^{m} b_j^{(v)} \lambda_{nj} \right|^{\mu_v} < \infty, \tag{3.2}$$

$$\lim_{n\to\infty} \sum_{j=v}^{\infty} \lambda_{nj} b_j^{(v)} \text{ exists for each } v, \tag{3.3}$$

and

$$\sup_{n,v}\left|\sum_{j=v}^{\infty} b_j^{(v)}\lambda_{nj}\right|^{\mu_v} < \infty, \tag{3.4}$$

(*b*) $\Lambda \in \left(\left|E_\phi^r\right|(\mu), c_0\right)$ *if and only if (3.1), (3.2), (3.4) hold and*

$$\lim_{n\to\infty}\sum_{j=v}^{\infty}\lambda_{nj}b_j^{(v)} = 0 \ \textit{ for each } \ v,$$

(*c*) $\Lambda \in \left(\left|E_\phi^r\right|(\mu), l_\infty\right)$ *if and only if (3.1), (3.2), (3.4) hold.*

Proof. Since the remaining part can be proved similar way, we just prove (a) to avoid repetition.
(*a*) Let $\mu_v \leq 1$ for all v. Note that $\left|E_\phi^r\right|(\mu) = [l(\mu)]_{T^r(\phi,\mu)}$. By Lemma 2.2, it is said that $\Lambda \in \left(\left|E_\phi^r\right|(\mu), c\right)$ if and only if $\hat{\Lambda} \in (l(\mu), c)$ and $W^{(n)} \in (l(\mu), c)$, where the matrix $W^{(n)}$ is described by

$$w_{mv}^{(n)} = \begin{cases} \sum\limits_{j=v}^{m} b_j^{(v)}\lambda_{nj}, & 0 \leq v \leq m \\ 0, & v > m. \end{cases}$$

Now, if we apply Lemma 2.1 (*i*) to the matrices $\hat{\Lambda}$ and $W^{(n)}$, it follows that $W^{(n)} \in (l(\mu), c)$ if and only if the conditions (3.1) and (3.2) hold, also $\hat{\Lambda} \in (l(\mu), c)$ if and only if the conditions (3.3) and (3.4) hold which concludes the proof of (*a*).

□

Theorem 3.3. *Assume that $\Lambda = (\lambda_{nv})$ is an infinite matrix of complex numbers and (ϕ_n) is any sequence of positive numbers. If $\mu = (\mu_n)$ is any bounded sequence of positive numbers such that $\mu_n > 1$ for all n. Then,*

(*a*) *The necessary and sufficient conditions for $\Lambda \in \left(\left|E_\phi^r\right|(\mu), c\right)$ are*

$$\sum_{k=v}^{\infty} b_k^{(v)}\lambda_{nk} \ \textit{ converges for each } \ v, \tag{3.5}$$

$$\sup_{m}\sum_{v=0}^{\infty}\left|\sum_{k=v}^{m} b_k^{(v)}\lambda_{nk}K^{-1}\right|^{\mu_v^*} < \infty, \exists K > 1, \tag{3.6}$$

$$\lim_{n\to\infty}\sum_{j=v}^{\infty}\lambda_{nj}b_j^{(v)} \ \textit{ exists for each } \ v,$$

and

$$\sup_{n}\sum_{v=0}^{\infty}\left|\sum_{k=v}^{\infty} b_k^{(v)}\lambda_{nk}K^{-1}\right|^{\mu_v^*} < \infty, \exists K > 1, \tag{3.7}$$

for $n = 0, 1, ...,$
(*b*) $\Lambda \in \left(\left|E_\phi^r\right|(\mu), c_0\right)$ *if and only if (3.5), (3.6), (3.7) hold and*

$$\lim_{n\to\infty}\sum_{k=v}^{\infty} b_k^{(v)}\lambda_{nk} = 0 \ \textit{ for each } \ v, \tag{3.8}$$

(*c*) $\Lambda \in \left(\left|E_\phi^r\right|(\mu), l_\infty\right)$ *if and only if (3.5), (3.6), (3.7) hold.*

Proof. (b) Let $\mu_n > 1$ for each $n \in \mathbb{N}$. It is obvious that $\left|E_{\phi}^{r}\right|(\mu) = [l(\mu)]_{T^r(\phi,\mu)}$. So, by Lemma 2.2, it is seen that $\Lambda \in \left(\left|E_{\phi}^{r}\right|(\mu), c_0\right)$ is equivalent that $\hat{\Lambda} \in (l(\mu), c_0)$ and $W^{(n)} \in (l(\mu), c)$, where $\hat{\Lambda}$ and $W^{(n)}$ are given in Theorem 3.2. Hence, applying Lemma 2.1 (ii) to the matrix $W^{(n)}$, it is obtained that $W^{(n)} \in (l(\mu), c)$ if and only if the conditions (3.5) and (3.6) are satisfied. Also, if we apply Lemma 2.1 (ii) to the matrix $\hat{\Lambda}$, then we get $\hat{\Lambda} \in (l(\mu), c_0)$ if and only if the conditions (3.8) and (3.7) hold. This completes the proof of (b).
The other parts can be proved similarly. □

Theorem 3.4. *Let $\Lambda = (\lambda_{nv})$ be an infinite matrix of complex numbers, $\phi = (\phi_n)$ be any sequence of positive real numbers, $\mu = (\mu_v)$ be any bounded sequence of positive numbers and $\Lambda \in \{c, c_0, l_\infty\}$. If $\Lambda \in (\left|E_{\phi}^{r}\right|(\mu), \Lambda)$, then Λ defines a bounded linear operator.*

Proof. Because of matrix transformations between FK-spaces are continuous, it can be immediately seen that the matrix operators between the spaces c, c_0, l_∞ which are BK-spaces, i.e, normed FK-spaces, and $\left|E_{\phi}^{r}\right|(\mu)$ are bounded linear operators. □

Take the matrix $L = (l_{nj})$ defined by

$$l_{nj} = \begin{cases} 1, & 0 \le j \le n \\ 0, & j > n. \end{cases}$$

Then, since $b_s = \{l_\infty\}_L$ and $c_s = \{c\}_L$, the matrix classes $\left(\left|E_{\phi}^{r}\right|(\mu), c_s\right)$ and $\left(\left|E_{\phi}^{r}\right|(\mu), b_s\right)$ can be characterized as follows with Lemma 2.3:

Corollary 3.5. *Let $0 < r < 1$, $\mu = (\mu_n)$ be a bounded sequence of positive numbers and $\lambda(n, j) = \sum_{k}^{n} \lambda_{kj}$.*

(i) *If $\mu_n \le 1$ for all n, $\Lambda \in \left(\left|E_{\phi}^{r}\right|(\mu), c_s\right)$ if and only if*

$$(a) \sum_{j=v}^{\infty} b_j^{(v)} \lambda(n, j) \text{ converges for each } v,$$

$$(b) \sup_{m,v} \left| \sum_{j=v}^{m} b_j^{(v)} \lambda(n, j) \right|^{\mu_v} < \infty,$$

$$(c) \lim_{n\to\infty} \sum_{j=v}^{\infty} \lambda(n, j) b_j^{(v)} \text{ exists for each } v,$$

$$(d) \sup_{n,v} \left| \sum_{j=v}^{\infty} b_j^{(v)} \lambda(n, j) \right|^{\mu_v} < \infty,$$

and $A \in \left(\left|E_{\phi}^{r}\right|(\mu), b_s\right) \Leftrightarrow (a), (b)$ and (d) hold.

(ii) *If $\mu_n > 1$ for all n, $\Lambda \in \left(\left|E_{\phi}^{r}\right|(\mu), c_s\right)$ if and only if*

$$(a') \sum_{j=v}^{\infty} b_j^{(v)} \lambda(n, j) \text{ converges for each } v,$$

$$(b') \sup_{m} \sum_{v=0}^{\infty} \left| \sum_{j=v}^{m} b_j^{(v)} \lambda(n, j) M^{-1} \right|^{\mu_v^*} < \infty,$$

$$(c') \lim_{n\to\infty} \sum_{j=v}^{\infty} \lambda(n, j) b_j^{(v)} \text{ exists for each } v,$$

$$(d') \sup_{n} \sum_{v=0}^{\infty} \left| \sum_{j=v}^{\infty} b_j^{(v)} \lambda(n, j) M^{-1} \right|^{\mu_v^*} < \infty,$$

and $\Lambda \in \left(\left|E_{\phi}^{r}\right|(\mu), b_s\right) \Leftrightarrow (a'), (b')$ and (d') hold.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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