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Degree of Approximation of Functions by Nörlund Summability of Double Fourier Series

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Abstract
In this research paper, the author studies some problems which are relating to harmonic summability of double Fourier series on Nörlund summability. These results constitute substantial extension and generalization of related works of F. Moricz and B.E Rhodes [1] and H.K. Nigam and K. Sharma [2].

Keywords: *Nörlund sumambility, double Fourier series, double matrix summability.*

AMS Subject Classification (2020): *40G05, 42B05,42B08.*

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1. Introduction

Let $f(\alpha,\beta)$ be Lebesgue integral in the square $R(-\pi,\pi;-\pi,\pi)$ and be of period 2π in each of the variables α and β. Then the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\gamma_{mn}\Big\{r_{mn}cos\ m\alpha\ cos\ n\beta+s_{mn}\ sin\ m\alpha\ cos\ n\beta+t_{mn}\ cos\ m\alpha\ sin\ n\beta+q_{mn}sin\ m\alpha\ sin\ n\beta\Big\} \tag{1.1}$$

is called the double Fourier series associated with the function $f(\alpha,\beta)$([2],[3]) where

$$\gamma_{mn}=\begin{cases}\frac{1}{4} & \text{for } m=0,n=0\\ \frac{1}{2} & \text{for } m=0,n>0 \text{ or } m>0,n=0\\ 1 & \text{for } m,n>0\end{cases}$$

$$r_{mn}=\frac{1}{\pi^2}\int\int_R f(\alpha,\beta)\ cos\ m\alpha\ cos\ n\beta\ d\alpha\ d\beta \tag{1.2}$$

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$$s_{mn} = \frac{1}{\pi^2}\int\int_R f(\alpha,\beta)\ sin\ m\alpha\ cos\ n\beta\ d\alpha\ d\beta \quad (1.3)$$

$$t_{mn} = \frac{1}{\pi^2}\int\int_R f(\alpha,\beta)\ cos\ m\alpha\ \ sin\ n\beta\ d\alpha\ d\beta \quad (1.4)$$

$$q_{mn} = \frac{1}{\pi^2} = \int\int_R f(\alpha,\beta)\ sin\ m\alpha\ sin\ n\beta\ d\alpha\ d\beta. \quad (1.5)$$

We have

$$\chi(\alpha,\beta) = \chi_{x,y}(\alpha,\beta) = \frac{1}{4}\Big\{f(x+\alpha,y+\beta)+f(x-\alpha,y+\beta)+f(x+\alpha,y-\beta)+f(x-\alpha,y-\beta)-4f(\alpha,\beta)\Big\}. \quad (1.6)$$

1.Definition([4],[5])

Let$\{p_m^{(1)}\}$and $\{p_n^{(2)}\}$ are two sequence of constants, real or complex.
Let

$$\begin{aligned} P_m^{(1)} &= p_0^{(1)}+p_1^{(1)}+p_2^{(1)}+...+p_m^{(1)} \\ P_n^{(2)} &= p_0^{(2)}+p_1^{(2)}+p_2^{(2)}+...+p_n^{(2)}. \end{aligned}$$

We shall also consider a double Nörlund transform of $\{a_{mn}\}$. Then the double Nörlund transform is

$$V_{mn} = \frac{1}{P_m^{(1)}P_n^{(2)}}\sum_{l=0}^{m}\sum_{g=0}^{n} p_{m-l}^{(1)}p_{g-n}^{(2)}a_{lg}. \quad (1.7)$$

2. Definition([4],[5])

The double sequence $\{\alpha_{lg}\}$ is said to be Nörlund summable to a limit V if

$$V_{mn} \to V, (m,n) \to (\infty,\infty). \quad (1.8)$$

It is also known as summable$(N, p_m^{(1)}, p_n^{(2)})$.

3. Definition([1], [2], [4], [5], [6])

If

$$\left.\begin{aligned} p_m^{(1)} &= 1\ for\ m = 0,1,2,... \\ p_n^{(2)} &= 1\ for\ m = 0,1,2,... \end{aligned}\right\} \quad (1.9)$$

then the double Nörlund transform reduces to double Cesàro transform of order one. This summability method is known as Cesàro summability (C,1,1).

4.Definition([1], [2], [5], [6])

$$p_m^{(1)} = \frac{1}{m+1}, m = 0,1,2,...\ and\ p_n^{(2)} = \frac{1}{n+1},\ n = 0,1,2,...$$

then the double Nörlund summability $(N, p_m^{(1)}, p_n^{(2)}$ becomes Harmonic summability and is denoted by $(H,1,1)$.

5. Definition([5])

If, for any $\gamma \geq 1, Vmn \to V, (m,n) \to (\infty,\infty)$in such a manner that $\gamma \geq \frac{m}{n}, \gamma \geq \frac{n}{m}$ then the sequence $\{\alpha_l g\}$ is said to be restrictedly summable N_p at(x,y) to the same limit.

There are several results on Nörlund summability of Fourier series. Nörlund summability of Fourier series has been studied by the authors[1–16]. This motivates us to study on the Nörlund summability of Fourier series in more generalized as particular cases. Therefore, an attempt to make an advance in this research work, we study on the double Fourier series and its conjugate series by Nörlund method. T. Sing [7] proved the following theorem:

Theorem 1.1. *If*

$$\int_0^v \Big|\chi(y)\Big| dy = O\Big(\frac{v}{\log\ v^{-1}}\Big),$$

where $\chi(y) = f(v+y) + f(v-y) - 2f(y)$ *a s* $v \to 0$, *then the Fourier series of* $f(u)$ *at* $v = y$ *is summable* (N, p_n) *to* $f(y)$ *where* $\{p_n\}$ *is real non-increasing sequence such that*

$$\sum_{a=2}^{n}\left(\frac{p_a}{a\log a}\right) = o(P_n).$$

In this present research paper, we established the following theorem which is the extended forms of Singh [7] and also the generalized results of [2].

2. Main Results

Theorem 2.1. *If* $(\alpha, \beta) \to (0, 0)$,

$$\int_0^{\alpha}\int_0^{\beta}\Big|\chi(s,t)\Big| dsdt = o\left(\frac{\alpha}{\log\alpha^{-1}}\frac{\beta}{\log\beta^{-1}}\right) \tag{2.1}$$

$$\int_\delta^{\pi} ds \int_0^{\beta}\Big|\chi(s,t)\Big| dt = o\left(\frac{\beta}{\log\beta^{-1}}\right), (0 < \delta < \pi) \tag{2.2}$$

and

$$\int_\delta^{\pi} dt \int_0^{\alpha}\Big|\chi(s,t)\Big| ds = o\left(\frac{\alpha}{\log\alpha^{-1}}\right), (0 < \delta < \pi) \tag{2.3}$$

then the double Fourier series of $f(\alpha, \beta)$ *at* $\alpha = x$, $\beta = y$ *is summable* $\left(N, p_m^{(1)} p_n^{(2)}\right)$ *to* $f(x, y)$ *where* $\{p_n^{(v)}\}$ *are real non-negative, non-increasing sequence of constants such that*

$$\sum_{k=2}^{n}\left(\frac{p_k^{(v)}}{k\log k}\right) = o\Big(P_n^{(v)}\Big), (v = 1, 2). \tag{2.4}$$

The following lemmas are required in the proof of our theorem.

Lemma 2.1. *If* $\{p_n\}$ *is non-negative and non-increasing, then for* $0 \le a \le b \le \infty, 0 \le t \le \pi$ *and for any n, we have*

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| \le AP_{[t^{-1}]}. \tag{2.5}$$

Lemma 2.2. *Under the condition of lemma 2.1,*

$$\left|\sum_{k=a}^{b}\frac{P_k sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = o\left(nP_n\right), 0 \le t \le \frac{1}{n}. \tag{2.6}$$

Lemma 2.3. *Under the condition of lemma 2.1,*

$$\left|\sum_{k=a}^{b} p_k\frac{sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = o\left[\frac{1}{t}P_{[t^{-1}]}\right] \qquad for\ \frac{1}{n} \le t \le \delta. \tag{2.7}$$

Lemma 2.4. *Under the condition of lemma 2.1,*

$$\left|\sum_{k=0}^{n} p_k \frac{sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = o(1) \quad for \;\; 0 \le \delta < t \le \pi. \tag{2.8}$$

They are uniformly in each of the intervals.

Proof. Let $U_{mn}(x,y;f) = U_{mn}$ denotes the rectangular $(m,n)^{th}$ partial sum of the series (1.1), then we must have

$$U_{mn}(x,y;f) - f(x,y) = \frac{1}{\pi^2}\int_0^T\int_0^\pi \chi(\alpha,\beta)D_m^1(\alpha)D_n^2(\beta)d\alpha d\beta \tag{2.9}$$

where

$$D_m^1(\alpha) = \frac{sin\left(m+\frac{1}{2}\right)\alpha}{2sin\frac{\alpha}{2}} \tag{2.10}$$

and

$$D_n^2(\beta) = \frac{sin\left(n+\frac{1}{2}\right)\beta}{2sin\frac{\beta}{2}} \tag{2.11}$$

where $D_m^1(\alpha)$ and $D_n^2(\beta)$ are respectively denote the Dirichlet kernels.
Let $\Big\{V_{mn}(x,y)\Big\}$ denote the double Nörlund transform of the sequence $\Big\{V_{mn} - f(x,y)\Big\}$ then

$$V_{mn}(x,y) = \frac{1}{P_m^{(1)}P_n^{(2)}}\sum_{l=0}^{m}\sum_{g=0}^{n} p_{m-l}^{(1)}\,p_{n-g}^{(2)}\Big\{U_{lg} - f(x,y)\Big\},$$

$$= \frac{1}{P_m^{(1)}P_n^{(2)}}\sum_{l=0}^{m}\sum_{g=0}^{n}\Big\{p_{m-l}^{(1)}p_{n-g}^{(2)}\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi(\alpha,\beta)D_l^1(\alpha)D_g^2(\beta)d\alpha d\beta\Big\}$$

$$= \int_0^\pi\int_0^\pi \chi(\alpha,\beta)\Big\{\frac{1}{2\pi P_m^{(1)}}\sum_{l=0}^{m} p_{m-l}^{(1)}\frac{sin\left(l+\frac{1}{2}\right)l}{sin\frac{\alpha}{2}}\Big\}\Big\{\frac{1}{2\pi P_n^{(2)}}\sum_{g=0}^{n} p_{g-n}^{(2)}\frac{sin\left(g+\frac{1}{2}\right)\beta}{sin\frac{\beta}{2}}\Big\}d\alpha d\beta \tag{2.12}$$

$$= \int_0^\pi\int_0^\pi \chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta \tag{2.13}$$

where

$$N_m^{(1)}(\alpha) = \frac{1}{2\pi P_m^{(1)}}\sum_{l=0}^{m} p_{m-l}^{(1)}\frac{sin\left(m-l+\frac{1}{2}\right)\alpha}{sin\frac{\alpha}{2}} \tag{2.14}$$

and

$$N_n^{(2)}(\beta) = \frac{1}{2\pi P_n^{(2)}}\sum_{g=0}^{n} p_{g-n}^{(2)}\frac{sin\left(n-g+\frac{1}{2}\right)\beta}{sin\frac{\beta}{2}}. \tag{2.15}$$

Also equation (2.13) can be written as

$$\begin{aligned} U_{mn}(x,y) - f(x,y) &= \int_0^\pi\int_0^\pi \chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta \\ &= \left(\int_0^\pi\int_0^\tau + \int_0^\delta\int_\tau^\pi + \int_\delta^\pi\int_0^\tau + \int_\delta^\pi\int_\tau^\pi\right)\chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta \\ &= I_1 + I_2 + I_3 + I_4 \;\; say \end{aligned} \tag{2.16}$$

by hypothesis and using the results of equations (2.6) and equations (2.7), we easily obtain

$$
\begin{aligned}
|I_4| &= \left|\int_{\delta}^{\pi}\int_{\tau}^{\pi} \chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta\right| \\
&= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\delta}^{\pi}\int_{\tau}^{\pi}\left|\chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)\right|d\alpha d\beta\right) \\
&= \left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{0}^{\pi}\int_{0}^{\pi}\left|\chi(\alpha,\beta)\right|d\alpha.d\beta\right)\left(as\ N_n^{(2)}(\beta),N_m^{(1)}(\alpha) \text{ are even function}\right) \\
&= o(1).
\end{aligned} \tag{2.17}
$$

Also, for I_3,

$$
\begin{aligned}
I_3 &= \int_{\delta}^{\pi} N_m^{(1)}(\alpha)d\alpha\int_{0}^{\tau}\chi(\alpha,\beta)N_n^{(2)}(\beta)d\beta \\
&= \int_{\delta}^{\pi} N_m^{(1)}(\alpha)d\alpha\left\{\int_{0}^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\delta}\right\}\chi(\alpha,\beta)N_n^{(2)}(\beta)d\beta \\
&= I_{3,1}+I_{3,2}\quad say.
\end{aligned} \tag{2.18}
$$

Thus

$$
\begin{aligned}
|I_{3,1}| &= o\left(\frac{n}{P_m^{(1)}}\int_{0}^{\pi}\int_{0}^{\frac{1}{n}}\left|\chi(\alpha,\beta)\right|d\beta\right) \\
&= o\left(\frac{n}{P_m}\right)o\left(\frac{\frac{1}{n}}{\log n}\right) \\
&= o(1).
\end{aligned} \tag{2.19}
$$

Again by equation (2.6) and equation (2.7) and hypothesis,

$$
\begin{aligned}
\left|I_{3,2}\right| &= o\left(\frac{1}{P_m^{(1)}}\int_{0}^{\pi}d\alpha\int_{\frac{1}{n}}^{\tau}\left|\chi(\alpha,\beta)\right|\frac{1}{P_n^{(2)}}\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}d\beta\right) \\
&= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\delta}^{\pi}d\alpha\left\{\frac{P_{[\beta^{-1}]}}{\beta}\chi_1(\alpha,\beta)\right\}_{\frac{1}{n}}^{\tau}-\int_{\frac{1}{n}}^{\tau}\chi_1(\alpha,\beta)d\ \left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right]\right) \\
&= (|I_{3,2,1}|)+o\left(|I_{3,2,2}|\right)\quad say
\end{aligned} \tag{2.20}
$$

where

$$\chi_1(\alpha,\beta)=\int_{0}^{\beta}|\chi(\alpha\beta)|dw$$

and $I_{3,2,1}$ and $I_{3,2,2}$ stands for two inner integrals.

$$
\begin{aligned}
|I_{3,2,1}| &= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\delta}^{\pi}d\alpha\left\{\frac{P_{[\tau^{-1}]}^{(2)}}{\tau}\phi_1(\alpha,\tau)-n\ P_n^{(2)}\phi_1(\alpha,\frac{1}{n})\right\}\right) \\
&= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\frac{p_{[\tau^{-1}]}^{(2)}}{\tau}\int_{\delta}^{\pi}d\alpha\int_{0}^{\tau}|\chi(\alpha,\beta)|d\beta\right)+o\left(\frac{n}{P_m^{(1)}}\int_{\delta}^{\pi}d\alpha\int_{0}^{\frac{1}{n}}|\chi(\alpha,\beta)|d\beta\right) \\
&= o(1)+o\left(\frac{n}{P_m^{(1)}}\frac{\frac{1}{n}}{\log\ n}\right) \\
&= o(1)
\end{aligned} \tag{2.21}
$$

and

$$\begin{aligned} |I_{3,2,2}| &= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi d\alpha\left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right]\chi(\alpha,\beta)\right) \\ &= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\frac{1}{n}}^\tau d\left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right]\int_\delta^\pi d\alpha\int_0^\beta|\chi_1(\alpha,w)|dw\right) \\ &= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\frac{1}{n}}^\tau d\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\frac{\beta}{\log(\frac{1}{\beta})}\right). \end{aligned} \tag{2.22}$$

Also,

$$\int_{\frac{1}{n}}^\tau \frac{\beta}{\log(\frac{1}{\beta})}d\left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right] = \int_{\frac{1}{\tau}}^n \frac{1}{y\log y}d\left[y\ P_{[y]}^{(2)}\right]$$

for

$$\int_j^{j+1}\frac{1}{y\log y}d\left[yP_{[y]}^{(2)}\right] < \frac{1}{j\log j}\int_j^{j+1}d\left[yP_{[y]}^{(2)}\right] = \frac{1}{j\log j}\left[yP_{[y]}^{(2)}\right]_j^{j+1}$$

$$= \frac{1}{j\log j}\left\{(j+1)P_{j+1}^{(2)} - k\ P_j^{(2)}\right\} < \frac{1}{j\log j}\left\{P_j^{(2)} + P_j^{(2)} + P_j^{(2)}\right\}$$

for

$$p_{k+1}^{(2)} \le p_k^{(2)} \ \text{ and } \ kp_k^{(2)} \le p_k^2 \le \frac{2p_j^{(2)}}{j\log j} + \frac{p_j^{(2)}}{j\log j}$$

thus,

$$\begin{aligned} \int_{\frac{1}{\tau}}^n \frac{1}{y\log y}d\left[xP_{[x]}^{(2)}\right] &< A + \sum_{j=c}^n\left(\frac{2P_k^{(2)}}{j\log j} + \frac{P_j^{(2)}}{j\log j}\right) \\ &= o\left(P_n^{(2)}\right). \end{aligned} \tag{2.23}$$

Now, by hypothesis equation (2.4) and using the equation (2.23) and equation (2.13), we get

$$|I_{3,2,1}| = o(1). \tag{2.24}$$

Combining equations (2.18), (2.19),(2.20), (2.21), (2.22), (2.23) and (2.24), we get

$$|I_3| = o(1). \tag{2.25}$$

Similarly, we can show that

$$|I_2| = o(1). \tag{2.26}$$

Now, for I_1,

$$\begin{aligned} I_1 &= \int_0^\delta\int_0^\tau \chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta \\ &= \left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} + \int_0^{\frac{1}{n}}\int_{\frac{1}{n}}^\delta + \int_{\frac{1}{m}}^\delta\int_0^{\frac{1}{n}} + \int_{\frac{1}{m}}^\delta\int_{\frac{1}{n}}^\tau\right)\chi(\alpha\,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta \\ &= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}\ \text{ say}. \end{aligned} \tag{2.27}$$

Then by (2.6) and (2.7)

$$\begin{aligned} |I_{1.1}| &= o\left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\chi(\alpha,\beta)| mn\ d\alpha d\beta\right) \\ &= o(mn) o\left(\frac{\frac{1}{m}}{\log m}\frac{\frac{1}{n}}{\log n}\right) \\ &= o(1). \end{aligned} \tag{2.28}$$

Similarly,

$$\left.\begin{array}{l} |I_{1,2}| = o(1) \\ |I_{1,3}| = o(1) \end{array}\right\} \tag{2.29}$$

and

$$\begin{aligned} \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau} \left|\chi(\alpha,\beta)\right| \frac{P_{[\alpha^{-1}]}}{\alpha}\frac{P_{[\beta-1]}}{\beta} d\alpha d\beta &= \chi(\delta,\tau)\frac{1}{\delta}P^{(1)}_{[\delta^{-1}]}\frac{1}{\tau}\ P^{(2)}_{[\tau^{-1}]} - \frac{1}{\tau}\ P^{(2)}_{[\tau^{-1}]} \\ & -\frac{1}{\tau}\ P^{(2)}_{[\tau^{-1}]}\int_{\frac{1}{m}}^{\delta} \phi(\alpha,\tau) d\frac{P^{(1)}_{[\alpha^{-1}]}}{\alpha} - \frac{1}{\delta}P^{(1)}_{[\delta^{-1}]}\int_{\frac{1}{n}}^{\tau} \chi(\alpha,\beta) d\left[\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}\right]. \end{aligned}$$

Thus,

$$\begin{aligned} |I_{1,4}| &= o\left(\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau} \left|\chi(\alpha,\beta)\right| \frac{1}{P^{(1)}_m P^{(2)}_n}\frac{P^{(1)}_{[\alpha^{-1}]}}{\alpha}\frac{P^{(2)}_{[\beta^{-1}]}}{\beta} d\alpha d\beta\right) \\ &= o(1) + o\left(\frac{1}{P^{(1)}_m P^{(2)}_n}\left(C_1 + C_2 + C_3\right)\right) \end{aligned} \tag{2.30}$$

where $o(1)$ corresponds to the integrated part in (2.29) and C_1, C_2 and C_3 are repetitively denote the remaining there integrals

$$\left.\begin{array}{l} C_2 = o(1) \\ C_3 = o(1) \end{array}\right\}. \tag{2.31}$$

Again for C_4

$$\begin{aligned} C_4 &= o\left(\int_{\frac{1}{m}}^{\delta} \frac{\alpha}{\log(\frac{1}{\alpha})} d\left(\frac{P^{(1)}_{[\alpha^{-1}]}}{\alpha}\right)\int_{\frac{1}{n}}^{\tau} \frac{\beta}{\log(\frac{1}{\beta})}\ d\left[\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}\right]\right) \\ &= o\left(P^{(1)}_m P^{(2)}_n\right) \end{aligned} \tag{2.32}$$

as in (2.23), using the estimate (2.31), we get from (2.30) that

$$|I_{1,4}| = o(1) \tag{2.33}$$

thus

$$|I_1| = O(1). \tag{2.34}$$

Combining equations (2.17),(2.23),(2.24),(2.34), we get equation (2.16). Which competes the proof of the theorem. □

Conclusion

Mathematical analysis is primarily concerned with the notion of limit of a sequences of real or complex number which forms the basis for study of infinite series. The general theory of the convergence and Summability of a double Fourier series has also been discussed by [1–16]. In 1913, in connection with the study of summation by arithmetic means of double Fourier series corresponding to function having discontinuities along a curve Moore [16] was led to the introduction of the notion of restricted summability of a double series. This differs from summability in the general sense in that the indices of the sequences whose limit is involved, become infinite in such a manner that there ratios remain bounded by two ordinary positive constants.
Corresponding to the classical tests for convergence of ordinary Fourier series, tests for pringsheim convergence of the double fourier series have been given by a number of writers. A main point of difference in which double, or multiple, Fourier series differ from ordinary series is the fact that the behavior of the former, as regards convergence, divergence, or oscillation, at a point, does not, as in the later case, depend only on the nature of the function in a neighborhood of the point, but upon its nature in cross-neighborhood of the point. The purpose of this research paper is to formulate the least conditions for Nörlund summability of double Fourier series. The main theorem for Nö"rlund summability of double Fourier series provide more stability to the system. Summability methods are used to decrease error. In this research we may find that our main theorem is a extended version by which many well known results on summabilities, can be obtained that is shown in above part. A function of two variables may be associated with a double fourier series.

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References

[1] Moricz, F., Rhodes, B. E.: *Summablity of double Fourier series by Nörlund method at a point*. Journal of Mathematical Analysis and Applications. **167**, 203-215 (1992).

[2] Nigam, H. K., Sharma, K.: *On the double summability of double conjugate Fourier series.* International Journal of Mathematics and Mathematical Science. **2012**, 104592 (2012).

[3] Sharma, P. L.: *On the harmonic summability of double Fourier series.* Proceeding of the American Mathematical Society. **9**(6), 979-986 (1958).

[4] Tripathi L. M., Singh A. P.: *A study of double Fourier series by Nörlund summability mathematics.* Proceedings A. **84** (1), 139-143 (1981).

[5] Herriot J. G.: *Nörlund summability of double Fourier series.* Transactions of the American Mathematical Society. **52**(1), 72-94 (1942).

[6] Lal, S., Tripathi, V. N.: *On the study of double Fourier series by double matrix summability method.* Tamkang Journal of Mathematics. **34**(1), 1-16 (2003).

[7] Singh, T.: *On the Nörlund summability of Fourier series and its conjugate series.* Proc. Nat. Inst. Sci. India part A. **29**, 65-73 (1963).

[8] Chow, Y. S.: *On the Cesáro summability of double Fourier series.* Tôhoku Math. J. **5**, 277-283 (1953).

[9] Nuray, F., Ulusu, U., Dündar, E.: *Cesàro summability of double sequence of sets.* Gen. Math. Notes. **25**(1), 8-18 (2014).

[10] Ulusu, U., Dündar, E., Gülle, E.: *I-Cesáro summability of double sequence of sets.* Palestine Journal of Mathematics. **9**(1), 561-568 (2020).

[11] Ersoy, M. T., Furkan, H.: *Distinguished supspaces in topological sequence spaces theory.* Aims Mathematics. **5**(4), 2858-2868 (2020).

[12] Cai, Q., Ansari, K. J., Ersoy, M. T., Özger, F.: *Statistical blending-type approximation by a class of operators that includes shape parameters λ and α.* Mathematics. **10**(7), 1149 (2022).

[13] Ersoy, M. T.: *Some Abelian, Tauberian and Core theorems related to the V, λ summability.* Universal Journal of Mathematics and Applications. **4**(2), 70-75 (2021).

[14] Kama, R.: *Spaces of vector sequences defined by the f-statistical convergence and some characterizations of normed spaces.* Revista de la Real Academia de Ciencias Exactas Físicas y Naturales. Seria A. Matemáticas. **114**, 74 (2020).

[15] Rama, R.: *On some vector valued multiplier spaces with statistical Cesàro summability.*Filomat. **33**(16), 5135-5147 (2019).

[16] Moore, C. N.: *Summability of series.* The American Mathematical Monthly. **39**(2), 62-71 (1932).

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