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The Five Dimensional Transformed Weyl–Yang–Kaluza–Klein Theory of Gravity

Beş Boyutlu Dönüştürülmüş Weyl–Yang–Kaluza–Klein Kütleçekim Teorisi

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Abstract

We study the case in which the five dimensional theory is the transformed Weyl–Yang–Kaluza–Klein gravity. The dimensionally reduced equations of motion are derived by considering an alternative form of the main equation of the theory in the coordinate basis. The conformal transformation rules are applied to the invariants. We also discuss the possible specific cases and the new Lorentz force density term, in detail.

Keywords: Weyl–Yang–Kaluza–Klein theory, Dimensional reduction, Conformal transformation, Field equations

Öz

Beş boyutlu teorinin, dönüştürülmüş Weyl–Yang–Kaluza–Klein kütleçekim olduğu durum incelendi. Boyutsal olarak indirgenmiş hareket denklemleri, teorinin ana denkleminin alternatif formu düşünülerek koordinat bazında türetildi. Değişmezler için konformal dönüşüm kuralları uygulandı. Ayrıca olası özel durumlar ve yeni Lorentz kuvvet yoğunluğu terimi ayrıntılı olarak tartışıldı.

Anahtar Kelimeler: Weyl–Yang–Kaluza–Klein kuramı, Boyutsal indirgenme, Konformal dönüşüm, Alan denklemleri

1. Introduction

Quadratic curvature Lagrangians have been used by many researchers to generalize or expand Einstein's theory of general relativity for over a century (for early works, see, e.g., Weyl 1918, Weyl 1919, Pauli 1919, Weyl 1921, Eddington 1924). The simple alternative forms of the scalar curvature, R, could be the square version of the scalar curvature, R^2, the Ricci tensor, $R_{\mu\nu}R^{\mu\nu}$, or of the Riemann tensor, $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, and a linear combination of those, such as the well-known Gauss–Bonnet invariant (Lanczos 1938, Lanczos 1949, Lanczos 1957), as one step forward. Among them, mathematically, the most similar to the Yang–Mills gauge theory (Yang and Mills 1954) is the following matter-free gravitational action quadratic equation in $R_{\mu\nu\lambda\sigma}$:

$$I=\frac{1}{2\varkappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \tag{1}$$

where $\varkappa$ and g represent the coupling constant and the determinant of $g_{\mu\nu}$, respectively, on a four-dimensional (4D) spacetime $\mathcal{M}$. By accepting that an affine connection is a Levi–Civita connection, i.e., torsion is absent, and considering a Palatini variational method, i.e., the metric $\{\mathfrak{g}\}$ and connection $\{\Gamma\}$ are assumed to be independent variables (Palatini 1919, Misner et al. 1973), the connection variation $\delta\{\Gamma\}$ of this curvature-squared action (1) provides the field equation without matter (Lichnerowicz 1958, Loos 1963, Loos and Treat 1967, Yang 1974)

$$D_\mu R^\mu{}_{\nu\lambda\sigma}=0 \tag{2}$$

which is known as Yang's gauge gravity equations in the literature. The third-order equations (2) are more complicated than well-established Einstein's field equations; however, they include Einstein's vacuum solutions in a natural manner. From the relation

$$D_\mu R^\mu{}_{\nu\lambda\sigma}\equiv D_\lambda R_{\nu\sigma}-D_\sigma R_{\nu\lambda}=0, \tag{3}$$

which is derived from the second Bianchi identity $D_{[\rho}R_{\mu\nu]\lambda\sigma}=0$, where D refers to the typical covariant derivative, the obvious vacuum solutions, $R_{\mu\nu}=0$ and $R_{\mu\nu}=\lambda g_{\mu\nu}$, for any constant λ, satisfy the equivalent equation (3). Conversely, the Palatini variation of the action (1) with respect to the metric $\{\mathfrak{g}\}$ without matter can be

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written as follows (Stephenson 1958, Fairchild 1976, Fairchild 1977):

$$R_{\mu\rho\lambda\sigma}R_{v}^{\ \rho\lambda\sigma}-\frac{1}{4}g_{\mu v}R_{\tau\rho\lambda\sigma}R^{\tau\rho\lambda\sigma}=0. \tag{4}$$

The field equations (4) are initially considered to eliminate nonphysical solutions (Pavelle 1975, Pavelle 1976, Fairchild 1976) even though numerous physical ones exist in the literature (Pavelle 1974, Thompson 1975, Thompson 1975a, Ni 1975, Pavelle 1978, Başkal 1999, Kuyrukcu 2013, Kuyrukcu 2021) (for more historical notes, see, e.g., Dean 1999, Kuyrukcu 2021). Besides, Baekler and Yasskin (Baekler and Yasskin 1984) noted that the standard variation of (1) produces the equations, which is fourth order for a metric

$$R_{\mu\rho\lambda\sigma}R_{v}^{\ \rho\lambda\sigma}-\frac{1}{4}g_{\mu v}R_{\tau\rho\lambda\sigma}R^{\tau\rho\lambda\sigma}+2D^{\lambda}D^{\sigma}R_{\sigma\mu\lambda v}=0, \tag{5}$$

which is Eddington's equation (Eddington 1924). They also noted that "if a metric (or tetrad) satisfies the torsion-free vacuum equations (2) and (4), it also satisfies the vacuum Eddington equation (5), but not vice versa". For the case in which torsion is zero, but the matter Lagrangian term exists and $\varkappa=1$, the field equations (2) and (4) become

$$D_{\mu}R^{\mu}_{\ v\lambda\sigma}=S_{\lambda\sigma v}, \tag{6}$$

$$R_{\mu\rho\lambda\sigma}R_{v}^{\ \rho\lambda\sigma}-\frac{1}{4}g_{\mu v}R_{\tau\rho\lambda\sigma}R^{\tau\rho\lambda\sigma}=T_{\mu v}, \tag{7}$$

where $S_{\lambda\sigma v}$ represents the canonical spin tensor and $T_{\mu v}$ represents the canonical energy-momentum tensor in the aspect of the quadratic Poincaré gauge theory of gravity (Baekler and Yasskin 1984). Camenzind and Fairchild also considered the equation (7) as an energy-momentum tensor of this alternative theory rather than the typical field equations (Camenzind 1975, Fairchild 1977). Kilmister et al. (1961) also introduced the current term $S_{\lambda\sigma v}$, which satisfies a covariant conservation property of the form $D_{v}S^{\lambda\sigma v}=0,$ but they could not clearly define what it was. Besides, the cyclic symmetry property of $S_{\lambda\sigma v}$, i.e., $S_{[\lambda\sigma v]}=0$, is proposed by Öktem (1985). Furthermore, by considering Einstein's field equations, $G_{\mu v}=T_{\mu v}$, along with the field equation (3), the source term can be written in the following forms

$$S_{\lambda\sigma v}=D_{\lambda}(T_{v\sigma}-\frac{1}{2}g_{v\sigma}T)-D_{\sigma}(T_{v\lambda}-\frac{1}{2}g_{v\lambda}T), \tag{8}$$

where $T_{v\sigma}$ represents the covariantly conservative, $D_{\mu}T^{\mu v}=0$, energy-momentum tensor, whose trace is $T=T_{v}^{\ v}$. This current density term $S_{\lambda\sigma v}$ in (8) was first used for Yang–Mills field equations with the SO(3,1) gauge group by Camenzind (Camenzind 1975, Camenzind 1975a, Camenzind 1977, Camenzind 1978, Camenzind 1978a). The metric is a nondynamic variable, i.e., a priori in this sense. Later, Cook used Camenzind's matter current term (8) by considering a formal analogy between Einstein's theory of relativity and classical electrodynamics, which means the connection $\{\Gamma\}$ and Riemann tensor $\{R\}$ correspond to the vector potential $\{A\}$ and electromagnetic field tensor $\{F\}$, respectively, to solve vacuum energy, cosmological constant, and dark energy problems (Cook 2008, see also, e.g., Chen et al. 2013). However, even if the term (8) is suggested to solve the source-term problem of this simple gravity model, it cannot be obtained from an action principle, meaning that it is inconsistent.

The main motivation in this work is to extend and generalize our previous results (Başkal and Kuyrukcu 2013) to the Einstein frame in which the Ricci scalar has its canonical form by considering that the five-dimensional (5D) theory is the Weyl–Yang–Kaluza–Klein (WYKK) theory of gravity (for non-Abelian WYKK theory, see also Kuyrukcu 2014). Weyl was the first to consider using the $R_{\mu v\lambda\sigma}R^{\mu v\lambda\sigma}$ invariant in the action to unify gravitation with electromagnetism by employing the principle of gauge invariance, the vacuum gravitational field equation (2) was proposed by Yang in an integral formalism for gauge fields without additional equations (4) and (5), and Kaluza and Klein (KK) assumed that gravitation and electromagnetism can be unified in the 5D spacetime (Kaluza 1921, Klein 1926, Mandel 1926, Klein 1926a). Then, we can prefer to call this higher-dimensional model the WYKK theory of modified gravity rather than just KK reduction of a quadratic gravity even though they are all different situations. Conversely, we can note that the field equations (2), (4), (6), (7), and (8) are also known as Stephenson–Kilmister–Yang–Camenzind equations in the literature. In this sense, we take advantage of the horizontal lift basis that is given by Misner et al. (1973) for faster calculations rather than the differential forms available in the literature (Pope, Perry 2009). Moreover, the dimensionally reduced vielbein components of the curvature tensor, which are important for the considered model, are explicitly presented in the opened-notation form rather than the compact-notation form. The possible special cases are also investigated and discussed, along with the new Lorentz force density term, in detail. Before we attempt to obtain the reduced conformal equations, we wish to revisit our previous work (Başkal and Kuyrukcu 2013) to explain how we can use an alternative equation (3) to derive the field equations for the reader's convenience.

The layout of this paper is as follows: In Section 2, we present a brief review of the 5D WYKK theory of gravity and

derive new accurate field equations, in particular. In Section 3, the transformed equations of motion are found for the generalized KK metric ansatz. Section 4 is concerned with obtaining the reduced form of the Kretschmann invariant, $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, (Kretschmann 1915) both by applying a circle reduction mechanism and performing the conformal scale transformations in the so-called Einstein frame. Section 5 is devoted to investigating special cases of the reduced equations. In Section 6, we finally present our findings and future work.

2. Revisit of the Reduced Equations of the WYKK Theory

As usual, the 5D standard KK action, $\hat{I}$, is given by

$$\hat{I}=\frac{1}{2\hat{\varkappa}^2}\int_{\hat{\mathcal{M}}} d^4x dy\sqrt{-\hat{g}}\hat{R}, \tag{9}$$

on the 5D manifold $\hat{\mathcal{M}}$ with compactification of the type $\hat{\mathcal{M}}=\mathcal{M}\times\mathcal{S}^1$, where $\mathcal{M}$ is the usual 4D spacetime as used previously, and $\mathcal{S}^1$ has the geometry of a small circle. Conversely, $\hat{R}$, $\hat{\varkappa}$, and $\hat{g}$ refer to the ordinary 5D Ricci curvature; the coupling constant, which is related to the 4D coupling constant, $\varkappa$ and the radius of the fifth dimension, r, as $\hat{\varkappa}^2=\varkappa^2\int dy=2\pi r\varkappa^2$; and the determinant of $\hat{g}_{MN}$, i.e., $\hat{g}=\det(\hat{g}_{MN})$, respectively. Here the classical metric ansatz of the theory, which is independent of the fifth dimension, becomes

$$\hat{g}_{MN}(x,y)=\begin{pmatrix} g_{\mu\nu}(x)+\varphi^2(x)A_\mu(x)A_\nu(x) & \varphi^2(x)A_\mu(x) \\ \varphi^2(x)A_\nu(x) & \varphi^2(x)\end{pmatrix}, \tag{10}$$

and hence $\sqrt{-\hat{g}}=\varphi\sqrt{-\det(g_{\mu\nu})}=\varphi\sqrt{-g}$. Moreover, $\varphi(x)$ is the dilaton field, $A_\mu(x)$ is the gauge potential, and $g_{\mu\nu}(x)$ is the 4D metric tensor in the usual manner. Kaluza also assumed that the derivatives of all the fields with respect to the new fifth coordinate, y, vanish (Kaluza 1921). In what follows, the hatted/unhatted fields are 5D/4D. We can also introduce 5D coordinates as $\hat{x}^A=(x^\mu,x^5)$ or, in short, $\hat{x}=(x,y)$, where the capital curved Latin indices are $A,B,...=0,1,2,3,5$, and the Greek curved indices are $\mu,\nu,...=0,1,2,3$.

For shortcut calculations, we can consider the Maurer–Cartan exterior forms rather than the coordinate basis as a type of strategy. In this fashion, the 5D line element, $\hat{G}(x,y)=\hat{g}_{MN}dx^M dx^N$, takes the form

$$\hat{G}(x,y)=\hat{\eta}_{\hat{A}\hat{B}}\hat{E}^{\hat{A}}\otimes\hat{E}^{\hat{B}}, \tag{11}$$

where the coframes one-form become

$$\begin{aligned}\hat{E}^{\hat{\mu}}(x,y)&=E^{\hat{\mu}}(x),\\ \hat{E}^{\hat{5}}(x,y)&=\varphi(x)[A(x)+dy],\end{aligned} \tag{12}$$

and the 5D Minkowski or orthonormal metric is $\hat{\eta}_{\hat{A}\hat{B}}=diag(\eta_{\hat{\mu}\hat{\nu}},+1)$. Hence, we can write the 4D line element as $G(x)=\eta_{\hat{\mu}\hat{\nu}}E^{\hat{\mu}}(x)\otimes E^{\hat{\nu}}(x)$ with $\eta_{\hat{\mu}\hat{\nu}}=diag(-1,+1,+1,+1)$, and the gauge fields as $A(x)=A_{\hat{\mu}}(x)E^{\hat{\mu}}(x)$, where all hatted indices refer to the orthonormal basis, i.e., the flat indices. We can now obtain the dimensionally reduced components of the spin connection one-form, $\hat{\Gamma}_{\hat{A}\hat{B}}$, and the curvature two-form $\hat{\mathcal{R}}^{\hat{A}}{}_{\hat{B}}$ by considering MaurerCartan structure equations (see, e.g., Dereli and Üçoluk 1990, Kuyrukcu 2013). However, we prefer to transform the desired components in opened-notation form rather than the compact-notation form to better understand the structure of these equations as follows:

$$\begin{aligned}\hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}}&=\Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}},\\ \hat{\Gamma}^{\hat{5}}{}_{\hat{\mu}\hat{\nu}}&=\frac{1}{2}\varphi F_{\hat{\mu}\hat{\nu}},\\ \hat{\Gamma}^{\hat{\mu}}{}_{\hat{5}\hat{\nu}}&=\hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{5}}=-\frac{1}{2}\varphi F^{\hat{\mu}}{}_{\hat{\nu}},\\ \hat{\Gamma}^{\hat{\mu}}{}_{\hat{5}\hat{5}}&=-\varphi^{-1}\varphi^{\hat{\mu}},\\ \hat{\Gamma}^{\hat{5}}{}_{\hat{\mu}\hat{5}}&=\varphi^{-1}\varphi_{\hat{\mu}},\end{aligned} \tag{13}$$

and

$$\begin{aligned}\hat{R}^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}&=R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}-\frac{1}{4}\varphi^2(2F^{\hat{\mu}}{}_{\hat{\nu}}F_{\hat{\lambda}\hat{\sigma}}+F^{\hat{\mu}}{}_{\hat{\lambda}}F_{\hat{\nu}\hat{\sigma}}-F^{\hat{\mu}}{}_{\hat{\sigma}}F_{\hat{\nu}\hat{\lambda}}),\\ \hat{R}^{\hat{\mu}}{}_{\hat{5}\hat{\lambda}\hat{\sigma}}&=-\frac{1}{2}\varphi D^{\hat{\mu}}F_{\hat{\lambda}\hat{\sigma}}-\frac{1}{2}(2\varphi^{\hat{\mu}}F_{\hat{\lambda}\hat{\sigma}}+\varphi_{\hat{\lambda}}F^{\hat{\mu}}{}_{\hat{\sigma}}-\varphi_{\hat{\sigma}}F^{\hat{\mu}}{}_{\hat{\lambda}}),\\ \hat{R}^{\hat{\mu}}{}_{\hat{5}\hat{\lambda}\hat{5}}&=-\varphi^{-1}D_{\hat{\lambda}}\varphi^{\hat{\mu}}-\frac{1}{4}\varphi^2F^{\hat{\mu}\hat{\rho}}F_{\hat{\rho}\hat{\lambda}}.\end{aligned} \tag{14}$$

Note that $\hat{R}^{\hat{\lambda}}{}_{\hat{\sigma}\hat{5}\hat{\mu}}=-\hat{R}^{\hat{\mu}}{}_{\hat{5}\hat{\lambda}\hat{\sigma}}$, and $D_{\hat{\mu}}\varphi_{\hat{\nu}}=D_{\hat{\nu}}\varphi_{\hat{\mu}}$ as well as D, $F_{\hat{\mu}\hat{\nu}}$, and $\varphi_{\hat{\mu}}$ refer to the 4D covariant derivative, the electromagnetic field tensor i.e., $F_{\hat{\mu}\hat{\nu}}=\partial_{\hat{\mu}}A_{\hat{\mu}}-\partial_{\hat{\nu}}A_{\hat{\mu}}$, and $\varphi_{\hat{\mu}}=D_{\hat{\mu}}\varphi=\partial_{\hat{\mu}}\varphi$. The same results, (13) and (14), have already been discussed in the language of horizontal lift basis (see, e.g., Başkal and Kuyrukcu 2013, as well as Lee 1983 for metric signature (+,-,-,-,-) and $\varphi(x)=1$). Conversely, for later convenience, we set

$$\hat{R}_{\hat{\mu}\hat{\nu}}\equiv\overline{\mathcal{P}}_{\hat{\mu}\hat{\nu}},\qquad \hat{R}_{\hat{5}\hat{\nu}}\equiv\overline{\mathcal{Q}}_{\hat{\nu}},\qquad \hat{R}_{\hat{5}\hat{5}}\equiv\overline{\mathcal{U}}, \tag{15}$$

and in terms of vielbeins, which are the orthonormal basis vectors, the 5D metric takes the form $\hat{g}_{MN}=\hat{h}^{\hat{A}}{}_{M}\hat{h}^{\hat{B}}{}_{N}\hat{\eta}_{\hat{A}\hat{B}}$, where the vielbeins and the inverse vielbeins are

$$\begin{aligned}\hat{h}^{\hat{A}}{}_{M}&=\begin{pmatrix}\hat{h}^{\hat{\mu}}{}_{\nu} & \hat{h}^{\hat{\mu}}{}_{5}\\ \hat{h}^{\hat{5}}{}_{\nu} & \hat{h}^{\hat{5}}{}_{5}\end{pmatrix}=\begin{pmatrix}h^{\hat{\mu}}{}_{\nu} & 0\\ \varphi A_\nu & \varphi\end{pmatrix},\\ \hat{h}^{M}{}_{\hat{A}}&=\begin{pmatrix}\hat{h}^{\nu}{}_{\hat{\mu}} & \hat{h}^{\nu}{}_{\hat{5}}\\ \hat{h}^{5}{}_{\hat{\mu}} & \hat{h}^{5}{}_{\hat{5}}\end{pmatrix}=\begin{pmatrix}h^{\nu}{}_{\hat{\mu}} & 0\\ -A_{\hat{\mu}} & \varphi^{-1}\end{pmatrix},\end{aligned} \tag{16}$$

which satisfy $\hat{h}^{\hat{A}}{}_{M}\hat{h}^{M}{}_{B}=\hat{\delta}^{\hat{A}}{}_{B}(h^{\hat{\mu}}{}_{\rho}h^{\rho}{}_{\hat{\nu}}=\delta^{\hat{\mu}}{}_{\hat{\nu}})$ and $\hat{h}^{M}{}_{B}\hat{h}^{B}{}_{N}=\hat{\delta}^{M}{}_{N}(h^{\mu}{}_{\hat{\rho}}h^{\hat{\rho}}{}_{\nu}=\delta^{\mu}{}_{\nu})$ in the five (four) dimensions.

Hence, the final results of the components of the 5D Ricci tensor are calculated by considering the vielbeins (16) in the coordinate basis as follows (for the metric signature (+,-,-,-,-), see, e.g., Liu and Wesson 1997):

$$\begin{aligned}&\hat{R}_{\mu\nu}=\overline{\mathcal{P}}_{\mu\nu}+\varphi A_{\mu}\overline{\mathcal{Q}}_{\nu}+\varphi A_{\nu}\overline{\mathcal{Q}}_{\mu}+\varphi^{2}A_{\mu}A_{\nu}\overline{\mathcal{U}},\\&\hat{R}_{5\nu}=\varphi\overline{\mathcal{Q}}_{\nu}+\varphi^{2}A_{\nu}\overline{\mathcal{U}},\\&\hat{R}_{55}=\varphi^{2}\overline{\mathcal{U}},\end{aligned}\tag{17}$$

where

$$\begin{aligned}&\overline{\mathcal{P}}_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}\varphi^{2}F_{\mu\rho}F_{\nu}{}^{\rho}-\varphi^{-1}D_{\mu}\varphi_{\nu},\\&\overline{Q}_{\nu}=-\frac{1}{2}\varphi\left(D_{\rho}F^{\rho}{}_{\nu}+3\varphi^{-1}\varphi_{\rho}F^{\rho}{}_{\nu}\right),\\&\overline{\mathcal{U}}=-\varphi^{-1}\left(D_{\rho}\varphi^{\rho}-\frac{1}{4}\varphi^{3}F_{\rho\tau}F^{\rho\tau}\right).\end{aligned}\tag{18}$$

The next step is to obtain the corresponding equations of motion. Thus, we consider the 5D Einstein equations $\hat{G}_{AB}=0$ or, equivalently, $\hat{R}_{AB}=0$; i.e., the Ricci curvature tensor vanishes, as usual. Hence, we have

$$\begin{aligned}&\hat{R}_{AB}=0,\Rightarrow\\&\hat{R}_{55}=0,\Rightarrow\overline{\mathcal{U}}=0,\\&\hat{R}_{5\nu}=0,\text{ and }\overline{\mathcal{U}}=0,\Rightarrow\overline{\mathcal{Q}}_{\nu}=0,\\&\hat{R}_{\mu\nu}=0,\text{ and }\overline{\mathcal{U}}=0,\text{ and }\overline{\mathcal{Q}}_{\nu}=0,\Rightarrow\overline{\mathcal{P}}_{\mu\nu}=0,\end{aligned}\tag{19}$$

which means that the solution set, S_{KK}, of the standard KK theory turns out to be

$$S_{KK}=\{\overline{\mathcal{P}}_{\mu\nu}=0,\quad\overline{\mathcal{Q}}_{\nu}=0,\quad\overline{\mathcal{U}}=0\}.\tag{20}$$

Let us now investigate the dimensionally reduced field equations of the WYKK theory, where the source-free field equations become $\hat{D}_{A}\hat{R}^{A}{}_{BCD}=0$ in the five dimensions. It is useful to consider the equivalent form (3) of field equations (2) to obtain the computational advantage in the orthonormal chart, as follows:

$$\hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{B}\hat{C}\hat{D}}\equiv\hat{D}_{\hat{C}}\hat{R}_{\hat{B}\hat{D}}-\hat{D}_{\hat{D}}\hat{R}_{\hat{B}\hat{C}}.\tag{21}$$

If we define

$$\begin{aligned}&\hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\equiv\mathbb{P}_{\hat{\nu}\hat{\lambda}\hat{\sigma}},\qquad\hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{5}\hat{\lambda}\hat{\sigma}}\equiv\mathbb{S}_{\hat{\lambda}\hat{\sigma}},\\&\hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{\nu}\hat{\lambda}\hat{5}}\equiv\mathbb{Q}_{\hat{\nu}\hat{\lambda}},\qquad\hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{5}\hat{\lambda}\hat{5}}\equiv\mathbb{U}_{\hat{\lambda}},\end{aligned}\tag{22}$$

and by substituting the expressions (15) into the equation (21) together with the connection terms (13), performing some manipulations, and then employing the vielbeins (16) to convert the resulting equations into the desired field equations that are written in coordinate basis, we find reduced equations in the following forms:

$$\begin{aligned}&\hat{D}_{A}\hat{R}^{A}{}_{\nu\lambda\sigma}=\mathbb{P}_{\nu\lambda\sigma}+\varphi A_{\nu}\mathbb{S}_{\lambda\sigma}+\varphi A_{\sigma}\mathbb{Q}_{\nu\lambda}\\&-\varphi A_{\lambda}\mathbb{Q}_{\nu\sigma}+\varphi^{2}A_{\nu}A_{\sigma}\mathbb{U}_{\lambda}-\varphi^{2}A_{\nu}A_{\lambda}\mathbb{U}_{\sigma},\\&\hat{D}_{A}\hat{R}^{A}{}_{5\lambda\sigma}=\varphi\mathbb{S}_{\lambda\sigma}+\varphi^{2}A_{\sigma}\mathbb{U}_{\lambda}-\varphi^{2}A_{\lambda}\mathbb{U}_{\sigma},\\&\hat{D}_{A}\hat{R}^{A}{}_{\nu\lambda5}=-\hat{D}_{A}\hat{R}^{A}{}_{\nu5\lambda}=\varphi\mathbb{Q}_{\nu\lambda}+\varphi^{2}A_{\nu}\mathbb{U}_{\lambda},\\&\hat{D}_{A}\hat{R}^{A}{}_{5\lambda5}=-\hat{D}_{A}\hat{R}^{A}{}_{55\lambda}=\varphi^{2}\mathbb{U}_{\lambda},\end{aligned}\tag{23}$$

where

$$\mathbb{P}_{\nu\lambda\sigma}=D_{\lambda}\overline{\mathcal{P}}_{\nu\sigma}-D_{\sigma}\overline{\mathcal{P}}_{\nu\lambda}+\frac{1}{2}\varphi(F_{\nu\sigma}\overline{\mathcal{Q}}_{\lambda}-F_{\nu\lambda}\overline{\mathcal{Q}}_{\sigma}+2F_{\lambda\sigma}\overline{\mathcal{Q}}_{\nu}),\tag{24}$$

$$\mathbb{S}_{\lambda\sigma}=D_{\lambda}\overline{\mathcal{Q}}_{\sigma}-D_{\sigma}\overline{\mathcal{Q}}_{\lambda}-\frac{1}{2}\varphi\left(F_{\lambda}{}^{\rho}\overline{\mathcal{P}}_{\rho\sigma}-F_{\sigma}{}^{\rho}\overline{\mathcal{P}}_{\rho\lambda}\right)+\varphi F_{\lambda\sigma}\overline{\mathcal{U}},\tag{25}$$

$$\mathbb{Q}_{\nu\lambda}=D_{\lambda}\overline{\mathcal{Q}}_{\nu}+\varphi^{-1}(\varphi_{\nu}\overline{\mathcal{Q}}_{\lambda}+\varphi_{\lambda}\overline{\mathcal{Q}}_{\nu})+\frac{1}{2}\varphi F_{\nu}{}^{\rho}\overline{\mathcal{P}}_{\rho\lambda}-\frac{1}{2}\varphi F_{\nu\lambda}\overline{\mathcal{U}},\tag{26}$$

$$\mathbb{U}_{\lambda}=D_{\lambda}\overline{\mathcal{U}}+\varphi^{-1}\varphi_{\lambda}\overline{\mathcal{U}}-\frac{1}{2}\varphi F_{\lambda}{}^{\rho}\overline{\mathcal{Q}}_{\rho}-\varphi^{-1}\varphi^{\rho}\overline{\mathcal{P}}_{\rho\lambda}.\tag{27}$$

There is no doubt that equations (24) – (27) naturally contain patterns of (18), as expected. As a result, we have

$$\begin{aligned}&\hat{D}_{A}\hat{R}^{A}{}_{BCD}=0,\Rightarrow\\&\hat{D}_{A}\hat{R}^{A}{}_{5\lambda5}=0,\Rightarrow\mathbb{U}_{\lambda}=0,\\&\hat{D}_{A}\hat{R}^{A}{}_{V\lambda5}=0,\text{ and }\mathbb{U}_{\lambda}=0,\Rightarrow\mathbb{Q}_{\nu\lambda}=0,\\&\hat{D}_{A}\hat{R}^{A}{}_{5\lambda\sigma}=0,\text{ and }\mathbb{U}_{\lambda}=0,\Rightarrow\mathbb{S}_{\lambda\sigma}=0,\\&\hat{D}_{A}\hat{R}^{A}{}_{V\lambda\sigma}=0,\text{ and }\mathbb{U}_{\lambda}=0,\text{ and }\mathbb{Q}_{\nu\lambda}=0,\text{ and}\\&\mathbb{S}_{\lambda\sigma}=0,\Rightarrow\mathbb{P}_{\nu\lambda\sigma}=0.\end{aligned}\tag{28}$$

In this case, the set of solution of the WYKK theory obviously becomes

$$S_{WYKK}=\{\mathbb{P}_{\nu\lambda\sigma}=0,\mathbb{S}_{\lambda\sigma}=0,\mathbb{Q}_{\nu\lambda}=0,\mathbb{U}_{\lambda}=0\}.\tag{29}$$

As can be easily seen, equation (25) can be obtained from (26) using $\mathbb{S}_{\lambda\sigma}=\mathbb{Q}_{\sigma\lambda}-\mathbb{Q}_{\lambda\sigma}$, meaning that equation (25) is not necessary. Hence, the final forms of dimensionally reduced field equations are given by

$$D_{\lambda}\overline{\mathcal{P}}_{\nu\sigma}-D_{\sigma}\overline{\mathcal{P}}_{\nu\lambda}+\frac{1}{2}\varphi(F_{\nu\sigma}\overline{\mathcal{Q}}_{\lambda}-F_{\nu\lambda}\overline{\mathcal{Q}}_{\sigma}+2F_{\lambda\sigma}\overline{\mathcal{Q}}_{\nu})=0,\tag{30}$$

$$D_{\nu}\overline{\mathcal{Q}}_{\lambda}+\varphi^{-1}(\varphi_{\nu}\overline{\mathcal{Q}}_{\lambda}+\varphi_{\lambda}\overline{\mathcal{Q}}_{\nu})+\frac{1}{2}\varphi F_{\nu}{}^{\rho}\overline{\mathcal{P}}_{\rho\lambda}-\frac{1}{2}\varphi F_{\nu\lambda}\overline{\mathcal{U}}=0,\tag{31}$$

$$D_{\lambda}\overline{\mathcal{U}}+\varphi^{-1}\varphi_{\lambda}\overline{\mathcal{U}}-\frac{1}{2}\varphi F_{\lambda}{}^{\rho}\overline{\mathcal{Q}}_{\rho}-\varphi^{-1}\varphi^{\rho}\overline{\mathcal{P}}_{\rho\lambda}=0,\tag{32}$$

and any solutions of the KK theory, (20), solve the reduced equations of the WYKK theory (30) – (32). It is essential here to note that the $\{\overline{\mathcal{P}}_{\mu\nu},\overline{\mathcal{Q}}_{\nu},\overline{\mathcal{U}}\}$ set (18) can turn into the $\{\mathcal{P}_{\mu\nu},\mathcal{Q}_{\nu},\mathcal{U}\}$ set via

$$\overline{\mathcal{P}}_{\mu\nu}=\mathcal{P}_{\mu\nu},\qquad\overline{\mathcal{Q}}_{\nu}=-\frac{1}{2}\varphi\mathcal{Q}_{\nu},\qquad\overline{\mathcal{U}}=-\varphi^{-1}\mathcal{U}.\tag{33}$$

The reduced field equations, which can be written by considering the $\{\mathcal{P}_{\mu\nu}, \mathcal{Q}_{\nu}, \mathcal{U}\}$ set, have already been given in our previous work (Başkal and Kuyrukcu 2013). However, the expressions (30) – (32) containing the $\{\overline{\mathcal{P}}_{\mu\nu}, \overline{\mathcal{Q}}_{\nu}, \overline{\mathcal{U}}\}$ set clearly seem to be more accurate than those containing the $\{\mathcal{P}_{\mu\nu}, \mathcal{Q}_{\nu}, \mathcal{U}\}$ set, and we now have the three equations corresponding to the three variables. Another way of obtaining the field equations is the 5D WYKK action, which is given by

$$\hat{I} = \frac{1}{2\hat{\varkappa}^2}\int_{\hat{\mathcal{M}}} d^4x\, dy \sqrt{-\hat{g}}\, \hat{R}_{ABCD}\hat{R}^{ABCD}. \tag{34}$$

By substituting the dimensionally reduced form of the Kretschmann invariant, $\hat{\mathcal{K}} = \hat{R}_{ABCD}\hat{R}^{ABCD}$, which is found to be in Başkal and Kuyrukcu (2013), into action (34) and dropping the total derivative terms that can come out with the help of Leibniz rule from the action gives

$$\begin{aligned} I = \frac{1}{2\varkappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\,\varphi\,[&R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - \frac{3}{2}\varphi^2 R_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} \\ &+\frac{3}{8}\varphi^4 F_{\mu\nu}F^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} + \frac{5}{8}\varphi^4 F_{\mu\nu}F^{\nu\lambda}F_{\lambda\sigma}F^{\sigma\mu} + 6\,(\varphi_\lambda\varphi^\lambda F_{\mu\nu}F^{\mu\nu} \\ &-\varphi_\mu\varphi^\lambda F_{\lambda\nu}F^{\nu\mu}) + 4\varphi(\varphi^\lambda F^{\mu\nu} + \varphi^\mu F^{\lambda\nu})D_\lambda F_{\mu\nu} \\ &-\varphi^2 F^{\mu\nu}D_\lambda D^\lambda F_{\mu\nu} - 2\varphi F^{\mu\lambda}F^{\nu}{}_{\lambda}D_\mu\varphi_\nu - 4\varphi^{-2}\varphi^\nu D^\mu D_\mu\varphi_\nu\,]. \end{aligned} \tag{35}$$

Finally, the independent variation of action, $I[\mathfrak{g}, \Gamma, F, \varphi_\mu]$, (35) with respect to the four variables gives the four field equations. The Palatini variation of the above action with respect to the metric $\{\mathfrak{g}\}$ should directly produce the dimensionally reduced equations of (4). However, we cannot directly obtain equations (30) – (32) from the variational principle, as expected. For instance, the connection variation $\delta\{\Gamma\}$ leads to an equation that can transform into (30) by using various identities (which are given by Başkal and Kuyrukcu 2013). Additionally, to obtain expressions that can be written in forms of (31) and (32), we should vary the action with respect to the field tensor $\{F\}$ and the partial derivative of the boson fields $\{\varphi_\mu\}$ rather than the usual variables $\{A\}$ and $\{\varphi\}$, respectively; otherwise we cannot obtain the proper equations (Çelik 2021).

3. The Reduced Field Equations from the Transformed WYKK theory

To obtain the field equations that are not only dimensionally reduced but also transformed from the conformal rescaling procedure, it is useful to first write the D-dimensional Weyl-rescaled Ricci scalar, $\tilde{R}$ (Hawking and Ellis 1999, Dabrowski et al. 2009), as

$$\tilde{R} = \xi^{-2}[R - 2(D-1)\xi^{-1}D_\mu\xi^\mu - (D-1)(D-4)\xi^{-2}\xi_\mu\xi^\mu], \tag{36}$$

where the tilde quantities denote fields in the Einstein frame, as usual, under the Weyl rescaling of the metric with a conformal factor, $\xi(x)$, as follows:

$$\tilde{g}_{\mu\nu}(x) = \xi^2(x)\, g_{\mu\nu}(x). \tag{37}$$

After that, for $D = 5$, the KK action (9) changes

$$\tilde{\hat{I}} = \frac{1}{2\hat{\varkappa}^2}\int_{\hat{\mathcal{M}}} d^4x\, dy \sqrt{-\tilde{\hat{g}}}\,\tilde{\hat{R}}, \tag{38}$$

$$= \frac{1}{2\hat{\varkappa}^2}\int_{\hat{\mathcal{M}}} d^4x\, dy\, \sqrt{-g}\,\xi^5\varphi\xi^{-2}[\,\hat{R} - 8\xi^{-1}\hat{D}_A\xi^A - 4\xi^{-2}\xi_A\xi^A], \tag{39}$$

by using $\sqrt{-\tilde{g}} = \xi^D\sqrt{-g}$. Hence, to obtain the correct coefficient of $\hat{R}$, meaning that $\xi^5\varphi\xi^{-2} = 1$, we must choose $\xi = \varphi^{-1/3}$. Therefore, the conformal transformation of the metric (10) becomes

$$\tilde{\hat{g}}_{MN} = \begin{pmatrix} \varphi^{-2/3}g_{\mu\nu} + \varphi^{4/3}A_\mu A_\nu & \varphi^{4/3}A_\mu \\ \varphi^{4/3}A_\nu & \varphi^{4/3} \end{pmatrix}. \tag{40}$$

Inspired by equation (40), one can write a generalized KK metric ansatz, which is the special case of the DeWitt ansatz (Cvetic et al. 2003), in terms of the actual 4D fields as follows:

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\psi}g_{\mu\nu} + e^{2\beta\psi}A_\mu A_\nu & e^{2\beta\psi}A_\mu \\ e^{2\beta\psi}A_\nu & e^{2\beta\psi} \end{pmatrix}, \tag{41}$$

where the $\psi = \psi(x)$ is a new scalar field, and α and β are arbitrary constants, which will be determined later. The obvious choice of the vielbein basis is inspired by coframes (12) as (see, e.g., Pope, Perry 2009).

$$\begin{aligned} \hat{E}^{\hat{\mu}}(x,y) &= e^{\alpha\psi(x)}E^{\hat{\mu}}(x), \\ \hat{E}^{\hat{5}}(x,y) &= e^{\beta\psi(x)}[A(x) + dy]. \end{aligned} \tag{42}$$

By using the relation between dual fields, $\hat{E}^{\hat{B}}(\hat{X}_{\hat{A}}) = \hat{\iota}_{\hat{X}_{\hat{A}}}\hat{E}^{\hat{B}} = \hat{\delta}_{\hat{A}}^{\hat{B}}$, the basis vectors are explicitly found to be

$$\begin{aligned} \hat{\iota}_{\hat{X}_{\hat{\mu}}}(x,y) &= e^{-\alpha\psi(x)}[\,\iota_{X_{\hat{\mu}}} - A_{\hat{\mu}}(x)\,\iota_{X_y}], \\ \hat{\iota}_{\hat{X}_{\hat{5}}}(x,y) &= e^{-\beta\psi(x)}\iota_{X_y}, \end{aligned} \tag{43}$$

where the $\iota_{X_{\hat{\mu}}}E^{\hat{\nu}} = \delta_{\hat{\mu}}^{\hat{\nu}}$ is also satisfied in four dimensions. We can employ the horizontal lift basis formalism, which is the easiest way to obtain not only field equations but also invariants. For this purpose, the 5D connection coefficients can be written as follows (Misner et al. 1973):

$$\hat{\Gamma}_{\hat{A}\hat{B}\hat{C}} = \frac{1}{2}[\,\hat{\iota}_{\hat{X}_{\hat{C}}}\hat{g}_{\hat{A}\hat{B}} + \hat{\iota}_{\hat{X}_{\hat{B}}}\hat{g}_{\hat{A}\hat{C}} - \hat{\iota}_{\hat{X}_{\hat{A}}}\hat{g}_{\hat{B}\hat{C}} + \hat{f}_{\hat{A}\hat{B}\hat{C}} + \hat{f}_{\hat{A}\hat{C}\hat{B}} + \hat{f}_{\hat{C}\hat{B}\hat{A}}\,]. \tag{44}$$

Here, the commutation coefficients, $\hat{f}_{\hat{A}\hat{B}}{}^{\hat{C}} = -\hat{f}_{\hat{B}\hat{A}}{}^{\hat{C}}$, are evaluated by $[\hat{\iota}_{\hat{X}_A}, \hat{\iota}_{\hat{X}_B}] \equiv \hat{f}_{\hat{A}\hat{B}}{}^{\hat{C}} \hat{\iota}_{\hat{X}_C}$, and the block diagonal metric becomes $\hat{g}_{\hat{A}\hat{B}} = diag(g_{\hat{\mu}\hat{\nu}}, +1)$. The nonzero commutators of the anholonomic basis vectors (43) now take the form

$$\begin{aligned}
&\hat{f}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}} = -\alpha e^{-\alpha\psi}(\psi_{\hat{\mu}}\delta_{\hat{\nu}}{}^{\hat{\lambda}} - \psi_{\hat{\nu}}\delta_{\hat{\mu}}{}^{\hat{\lambda}}),\\
&\hat{f}_{\hat{\mu}\hat{\nu}}{}^{\hat{5}} = -e^{(\beta-2\alpha)\psi}F_{\hat{\mu}\hat{\nu}},\\
&\hat{f}_{\hat{\mu}\hat{5}}{}^{\hat{5}} = -\hat{f}_{\hat{5}\hat{\mu}}{}^{\hat{5}} = -\beta e^{-\alpha\psi}\psi_{\hat{\mu}}.
\end{aligned} \tag{45}$$

Hence, the required higher-dimensional components of the connection are found by considering (44) and (45):

$$\begin{aligned}
&\hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}} = e^{-\alpha\psi}\Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}} - \alpha e^{-\alpha\psi}(\psi^{\mu}g_{\hat{\nu}\hat{\lambda}}) - \psi_{\hat{\nu}}\delta^{\hat{\mu}}{}_{\hat{\lambda}}),\\
&\hat{\Gamma}^{\hat{\mu}}{}_{\hat{5}\hat{\nu}} = \hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{5}} = -\frac{1}{2}e^{(\beta-2\alpha)\psi}F^{\hat{\mu}}{}_{\hat{\nu}},\\
&\hat{\Gamma}^{\hat{5}}{}_{\hat{\mu}\hat{\nu}} = \hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{5}} = \frac{1}{2}e^{(\beta-2\alpha)\psi}F_{\hat{\mu}\hat{\nu}},\\
&\hat{\Gamma}^{\hat{\mu}}{}_{\hat{5}\hat{5}} = -\beta e^{-\alpha\psi}\psi^{\hat{\mu}},\\
&\hat{\Gamma}^{\hat{5}}{}_{\hat{\mu}\hat{5}} = \beta e^{-\alpha\psi}\psi_{\hat{\mu}}.
\end{aligned} \tag{46}$$

Conversely, in the noncoordinate basis the Riemann tensor is defined by Misner et al. (1973)

$$\hat{R}^{\hat{A}}{}_{\hat{B}\hat{C}\hat{D}} = \hat{\iota}_{\hat{X}_C}\hat{\Gamma}^{\hat{A}}{}_{\hat{B}\hat{D}} - \hat{\iota}_{\hat{X}_D}\hat{\Gamma}^{\hat{A}}{}_{\hat{B}\hat{C}} + \hat{\Gamma}^{\hat{A}}{}_{\hat{E}\hat{C}}\hat{\Gamma}^{\hat{E}}{}_{\hat{B}\hat{D}} - \hat{\Gamma}^{\hat{A}}{}_{\hat{E}\hat{D}}\hat{\Gamma}^{\hat{E}}{}_{\hat{B}\hat{C}} - \hat{\Gamma}^{\hat{A}}{}_{\hat{B}\hat{E}}\hat{f}_{\hat{C}\hat{D}}{}^{\hat{E}}. \tag{47}$$

Now, the dimensionally reduced vielbein components of the curvature tensor can be obtained using (43), (45), and (46) in (47)

$$\begin{aligned}
&\hat{R}^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} = e^{-2\alpha\psi}R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} - \frac{1}{4}e^{2(\beta-2\alpha)\psi}(2F^{\hat{\mu}}{}_{\nu}F_{\hat{\lambda}\hat{\sigma}} + F^{\hat{\mu}}{}_{\hat{\lambda}}F_{\hat{\nu}\hat{\sigma}} - F^{\hat{\mu}}{}_{\hat{\sigma}}F_{\hat{\nu}\hat{\lambda}})\\
&+\alpha e^{-2\alpha\psi}(g_{\hat{\nu}\hat{\lambda}}D_{\hat{\sigma}}\psi^{\hat{\mu}} - g_{\hat{\nu}\hat{\sigma}}D_{\hat{\lambda}}\psi^{\hat{\mu}} + \delta^{\hat{\mu}}{}_{\hat{\sigma}}D_{\hat{\lambda}}\psi_{\hat{\nu}} - \delta^{\hat{\mu}}{}_{\hat{\lambda}}D_{\hat{\sigma}}\psi_{\hat{\nu}})\\
&+\alpha^2 e^{-2\alpha\psi}[\psi^{\hat{\mu}}(\psi_{\hat{\lambda}}g_{\hat{\nu}\hat{\sigma}} - \psi_{\hat{\sigma}}g_{\hat{\nu}\hat{\lambda}}) + \psi_{\hat{\rho}}\psi^{\hat{\rho}}(g_{\hat{\nu}\hat{\lambda}}\delta^{\hat{\mu}}{}_{\hat{\sigma}} - g_{\hat{\nu}\hat{\sigma}}\delta^{\hat{\mu}}{}_{\hat{\lambda}})\\
&+\psi_{\hat{\nu}}(\psi_{\hat{\sigma}}\delta^{\hat{\mu}}{}_{\hat{\lambda}} - \psi_{\hat{\lambda}}\delta^{\hat{\mu}}{}_{\hat{\sigma}})],
\end{aligned} \tag{48}$$

$$\begin{aligned}
&\hat{R}^{\hat{\mu}}{}_{\hat{5}\hat{\lambda}\hat{\sigma}} = -\frac{1}{2}e^{(\beta-3\alpha)\psi}D^{\hat{\mu}}F_{\hat{\lambda}\hat{\sigma}}\\
&+\frac{1}{2}(\alpha-\beta)e^{(\beta-3\alpha)\psi}(2\varphi^{\hat{\mu}}F_{\hat{\lambda}\hat{\sigma}} + \varphi_{\hat{\lambda}}F^{\hat{\mu}}{}_{\sigma} - \varphi_{\hat{\sigma}}F^{\hat{\mu}}{}_{\hat{\lambda}})\\
&+\frac{1}{2}\alpha e^{(\beta-3\alpha)\psi}\psi_{\hat{\rho}}(F^{\hat{\rho}}{}_{\hat{\lambda}}\delta^{\hat{\mu}}{}_{\hat{\sigma}} - F^{\hat{\rho}}{}_{\hat{\sigma}}\delta^{\hat{\mu}}{}_{\hat{\lambda}}),
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\hat{R}^{\hat{\mu}}{}_{\hat{5}\hat{\lambda}\hat{5}} = -\beta e^{-2\alpha\psi}D_{\hat{\lambda}}\varphi^{\hat{\mu}} - \frac{1}{4}e^{2(\beta-2\alpha)\psi}F^{\hat{\mu}\hat{\rho}}F_{\hat{\rho}\hat{\lambda}}\\
&+e^{-2\alpha\psi}[(2\alpha-\beta)\beta\psi^{\hat{\mu}}\psi_{\hat{\lambda}} - \alpha\beta\psi^{\rho}\psi_{\rho}\delta^{\hat{\mu}}{}_{\hat{\lambda}}].
\end{aligned} \tag{50}$$

After introducing $\hat{R}_{\hat{\mu}\hat{\nu}} \equiv \tilde{\mathcal{P}}_{\hat{\mu}\hat{\nu}}$ and $\hat{R}_{\hat{5}\hat{\nu}} \equiv \tilde{\mathcal{Q}}_{\hat{\nu}}$ together with $\hat{R}_{\hat{5}\hat{5}} \equiv \tilde{\mathcal{U}}$ and finding the vielbeins and the inverse vielbeins as

$$\begin{aligned}
&\hat{h}^{\hat{A}}{}_{M} = \begin{pmatrix} e^{\alpha\psi}h^{\hat{\mu}}{}_{\nu} & 0 \\ e^{\beta\psi}A_{\nu} & e^{\beta\psi} \end{pmatrix},\\
&\hat{h}^{M}{}_{\hat{A}} = \begin{pmatrix} e^{-\alpha\psi}h^{\nu}{}_{\hat{\mu}} & 0 \\ -e^{-\alpha\psi}A_{\hat{\mu}} & e^{-\beta\psi} \end{pmatrix},
\end{aligned} \tag{51}$$

we obtain the following components of the Ricci tensor in the coordinate basis

$$\begin{aligned}
&\hat{R}_{\mu\nu} = e^{2\alpha\psi}\tilde{\mathcal{P}}_{\mu\nu} + e^{(\alpha+\beta)\psi}A_{\mu}\tilde{\mathcal{Q}}_{\nu} + e^{(\alpha+\beta)\psi}A_{\nu}\tilde{\mathcal{Q}}_{\mu} + e^{2\beta\psi}A_{\mu}A_{\nu}\tilde{\mathcal{U}},\\
&\hat{R}_{5\nu} = e^{(\alpha+\beta)\psi}\tilde{\mathcal{Q}}_{\nu} + e^{2\beta\psi}A_{\nu}\tilde{\mathcal{U}},\\
&\hat{R}_{55} = e^{2\beta\psi}\tilde{\mathcal{U}},
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
&\tilde{\mathcal{P}}_{\mu\nu} = e^{-2\alpha\psi}R_{\mu\nu} - \frac{1}{2}e^{2(\beta-2\alpha)\psi}F_{\mu\rho}F_{\nu}{}^{\rho} - e^{-2\alpha\psi}[\alpha g_{\mu\nu}D_{\rho}\psi^{\rho}\\
&+(2\alpha+\beta)D_{\mu}\psi_{\nu}] - e^{-2\alpha\psi}[(2\alpha+\beta)\alpha g_{\mu\nu}\psi_{\rho}\psi^{\rho}\\
&-(2\alpha^2+2\alpha\beta-\beta^2)\psi_{\mu}\psi_{\nu}],\\
&\tilde{\mathcal{Q}}_{\nu} = -\frac{1}{2}e^{(\beta-3\alpha)\psi}D_{\rho}F^{\rho}{}_{\nu} - \frac{3}{2}\beta e^{(\beta-3\alpha)\psi}\psi_{\rho}F^{\rho}{}_{\nu},\\
&\tilde{\mathcal{U}} = -\beta e^{-2\alpha\psi}D_{\rho}\psi^{\rho} + \frac{1}{4}e^{2(\beta-2\alpha)\psi}F_{\rho\tau}F^{\rho\tau} - (2\alpha+\beta)\beta e^{-2\alpha\psi}\psi_{\rho}\psi^{\rho}.
\end{aligned} \tag{53}$$

Finally, one easily finds the solution set of the conformal KK theory as $\tilde{S}_{KK} = \{\tilde{\mathcal{P}}_{\mu\nu} = 0, \tilde{\mathcal{Q}}_{\nu} = 0, \tilde{\mathcal{U}} = 0\}$. Next, we can introduce that

$$\begin{aligned}
&\hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} \equiv \tilde{\mathbb{P}}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}, \quad \hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{5}\hat{\lambda}\hat{\sigma}} \equiv \tilde{\mathbb{S}}_{\hat{\lambda}\hat{\sigma}},\\
&\hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{\nu}\hat{\lambda}\hat{5}} \equiv \tilde{\mathbb{Q}}_{\hat{\nu}\hat{\lambda}}, \quad \hat{D}_{\hat{A}}\hat{R}^{\hat{A}}{}_{\hat{5}\hat{\lambda}\hat{5}} \equiv \tilde{\mathbb{U}}_{\hat{\lambda}}.
\end{aligned} \tag{54}$$

Then, we can derive the desired field equations in coordinate basis, as mentioned before, by substituting the connections (46) together with the $\{\hat{\mathcal{P}}_{\hat{\mu}\hat{\nu}}, \tilde{\mathcal{Q}}_{\hat{\nu}}, \tilde{\mathcal{U}}\}$ set in (53), and use the vielbeins fields (51) in the following forms:

$$\begin{aligned}
&\hat{D}_{A}\hat{R}^{A}{}_{\nu\lambda\sigma} = e^{3\alpha\psi}\tilde{\mathbb{P}}_{\nu\lambda\sigma} + e^{(2\alpha+\beta)\psi}A_{\nu}\tilde{\mathbb{S}}_{\lambda\sigma} + e^{(2\alpha+\beta)\psi}A_{\sigma}\tilde{\mathbb{Q}}_{\nu\lambda}\\
&-e^{(2\alpha+\beta)\psi}A_{\lambda}\tilde{\mathbb{Q}}_{\nu\sigma} + e^{(\alpha+2\beta)\psi}A_{\nu}A_{\sigma}\tilde{\mathbb{U}}_{\lambda} - e^{(\alpha+2\beta)\psi}A_{\nu}A_{\lambda}\tilde{\mathbb{U}}_{\sigma},\\
&\hat{D}_{A}\hat{R}^{A}{}_{5\lambda\sigma} = e^{(2\alpha+\beta)\psi}\tilde{\mathbb{S}}_{\lambda\sigma} + e^{(2\alpha+\beta)\psi}A_{\sigma}\tilde{\mathbb{U}}_{\lambda} - e^{(\alpha+2\beta)\psi}A_{\lambda}\tilde{\mathbb{U}}_{\sigma},\\
&\hat{D}_{A}\hat{R}^{A}{}_{\nu\lambda5} = -\hat{D}_{A}\hat{R}^{A}{}_{\nu5\lambda} = e^{(2\alpha+\beta)\psi}\tilde{\mathbb{Q}}_{\nu\lambda} + e^{(\alpha+2\beta)\psi}A_{\nu}\tilde{\mathbb{U}}_{\lambda},\\
&\hat{D}_{A}\hat{R}^{A}{}_{5\lambda5} = -\hat{D}_{A}\hat{R}^{A}{}_{55\lambda} = e^{(\alpha+2\beta)\psi}\tilde{\mathbb{U}}_{\lambda},
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
&\tilde{\mathbb{P}}_{\nu\lambda\sigma} = e^{-\alpha\psi}\{D_{\lambda}\tilde{\mathcal{P}}_{\nu\sigma} - D_{\sigma}\tilde{\mathcal{P}}_{\nu\lambda}\\
&+\frac{1}{2}e^{(\beta-\alpha)\psi}(F_{\nu\sigma}\tilde{\mathcal{Q}}_{\lambda} - F_{\nu\lambda}\tilde{\mathcal{Q}}_{\sigma} + 2F_{\lambda\sigma}\tilde{\mathcal{Q}}_{\nu})\\
&+\alpha[\psi^{\rho}(g_{\nu\lambda}\tilde{\mathcal{P}}_{\rho\sigma} - g_{\nu\sigma}\tilde{\mathcal{P}}_{\rho\lambda}) + \psi_{\lambda}\tilde{\mathcal{P}}_{\nu\sigma} - \psi_{\sigma}\tilde{\mathcal{P}}_{\nu\lambda}]\},
\end{aligned} \tag{56}$$

$$\begin{aligned}
&\tilde{\mathbb{S}}_{\lambda\sigma} = e^{-\alpha\psi}[D_{\lambda}\tilde{\mathcal{Q}}_{\sigma} - D_{\sigma}\tilde{\mathcal{Q}}_{\lambda} - \frac{1}{2}e^{(\beta-\alpha)\psi}(F_{\lambda}{}^{\rho}\tilde{\mathcal{P}}_{\rho\sigma} - F_{\sigma}{}^{\rho}\tilde{\mathcal{P}}_{\rho\lambda})\\
&+e^{(\beta-\alpha)\psi}F_{\lambda\sigma}\tilde{\mathcal{U}} + \alpha(\psi_{\lambda}\tilde{\mathcal{Q}}_{\sigma} - \psi_{\sigma}\tilde{\mathcal{Q}}_{\lambda})],
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\tilde{\mathcal{Q}}_{\nu\lambda} = e^{-\alpha\psi}[D_{\lambda}\tilde{\mathcal{Q}}_{\nu} + \alpha g_{\nu\lambda}\psi^{\rho}\tilde{\mathcal{Q}}_{\rho} + (\beta-\alpha)\psi_{\nu}\tilde{\mathcal{Q}}_{\lambda}\\
&+\beta\psi_{\lambda}\tilde{\mathcal{Q}}_{\nu} + \frac{1}{2}e^{(\beta-\alpha)\psi}F_{\nu}{}^{\rho}\tilde{\mathcal{P}}_{\rho\lambda} - \frac{1}{2}e^{(\beta-\alpha)\psi}F_{\nu\lambda}\tilde{\mathcal{U}}],
\end{aligned} \tag{58}$$

$$\tilde{\mathbb{U}}_{\lambda} = e^{-\alpha\psi}(D_{\lambda}\tilde{\mathcal{U}} + \beta\psi_{\lambda}\tilde{\mathcal{U}} - \frac{1}{2}e^{(\beta-\alpha)\psi}F_{\lambda}{}^{\rho}\tilde{\mathcal{Q}}_{\rho} - \beta\psi^{\rho}\tilde{\mathcal{P}}_{\rho\lambda}). \tag{59}$$

In this regard, we have the solution set: $\tilde{S}_{WYKK}=\{\tilde{\mathbb{P}}_{\nu\lambda\sigma}=0,\ \tilde{\mathbb{S}}_{\lambda\sigma}=0,\tilde{\mathbb{Q}}_{\nu\lambda}=0,\tilde{\mathbb{U}}_{\lambda}=0\}$. Again, it is easy to see that $\tilde{\mathbb{S}}_{\lambda\sigma}=\tilde{\mathbb{Q}}_{\sigma\lambda}-\tilde{\mathbb{Q}}_{\lambda\sigma}$, so that we can ignore equation (57) without loss of generality. Hence, the final forms of the reduced field equations of the transformed WYKK theory become

$$\begin{aligned}
&D_{\lambda}\tilde{\mathcal{P}}_{\nu\sigma}-D_{\sigma}\tilde{\mathcal{P}}_{\nu\lambda}+\frac{1}{2}e^{(\beta-\alpha)\psi}(F_{\nu\sigma}\tilde{\mathcal{Q}}_{\lambda}-F_{\nu\lambda}\tilde{\mathcal{Q}}_{\sigma}+2F_{\lambda\sigma}\tilde{\mathcal{Q}}_{\nu})\\
&+\alpha[\psi^{\rho}(g_{\nu\lambda}\tilde{\mathcal{P}}_{\rho\sigma}-g_{\nu\sigma}\tilde{\mathcal{P}}_{\rho\lambda})+\psi_{\lambda}\tilde{\mathcal{P}}_{\nu\sigma}-\psi_{\sigma}\tilde{\mathcal{P}}_{\nu\lambda}]=0,\\
&D_{\lambda}\tilde{\mathcal{Q}}_{\nu}+\alpha g_{\nu\lambda}\psi^{\rho}\tilde{\mathcal{Q}}_{\rho}+(\beta-\alpha)\psi_{\nu}\tilde{\mathcal{Q}}_{\lambda}+\frac{1}{2}e^{(\beta-\alpha)\psi}F_{\nu}{}^{\rho}\tilde{\mathcal{P}}_{\rho\lambda}\\
&-\frac{1}{2}e^{(\beta-\alpha)\psi}F_{\nu\lambda}\tilde{\mathcal{U}}=0,\\
&D_{\lambda}\tilde{\mathcal{U}}+\beta\psi_{\lambda}\tilde{\mathcal{U}}-\frac{1}{2}e^{(\beta-\alpha)\psi}F_{\lambda}{}^{\rho}\tilde{\mathcal{Q}}_{\rho}-\beta\psi^{\rho}\tilde{\mathcal{P}}_{\rho\lambda}=0.
\end{aligned}\tag{60}$$

As expected, the corresponding equations of motion are more complicated than those given in (30) – (32).

4. The Reduced Actions from the Transformed WYKK Theory

By considering a popular dimensional reduction method, the quadratic curvature term can be expanded as

$$\hat{\mathcal{K}}=\hat{R}_{ABCD}\hat{R}^{ABCD}=\hat{R}_{\mu\hat{\nu}\lambda\hat{\sigma}}\hat{R}^{\mu\hat{\nu}\lambda\hat{\sigma}}+4\hat{R}_{\mu\hat{5}\lambda\hat{\sigma}}\hat{R}^{\mu\hat{5}\lambda\hat{\sigma}}+4\hat{R}_{\mu\hat{5}\hat{\lambda}\hat{5}}\hat{R}^{\mu\hat{5}\hat{\lambda}\hat{5}}.\tag{61}$$

As is well-known, the invariant does not depend on the choice of basis. Thus, by using directly (48) – (50) and the relation $2R_{\mu\nu\lambda\sigma}F^{\mu\lambda}F^{\nu\sigma}=R_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}$ and after long but careful manipulations and reorganization of the terms, we have the following reduced expression:

$$\begin{aligned}
\hat{\mathcal{K}}&=e^{-4\alpha\psi}[R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}-\frac{3}{2}e^{2(\beta-\alpha)\psi}R_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}\\
&+\frac{3}{8}e^{4(\beta-\alpha)\psi}F_{\mu\nu}F^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}+\frac{5}{8}e^{4(\beta-\alpha)\psi}F_{\mu\nu}F^{\nu\lambda}F_{\lambda\sigma}F^{\sigma\mu}\\
&+(9\alpha^2-14\alpha\beta+6\beta^2)e^{2(\beta-\alpha)\psi}\psi_{\lambda}\psi^{\lambda}F_{\mu\nu}F^{\mu\nu}\\
&+2(3\alpha^2-2\alpha\beta-2\beta^2)e^{2(\beta-\alpha)\psi}\psi_{\mu}\psi^{\lambda}F_{\lambda\nu}F^{\nu\mu}\\
&+4(\beta-\alpha)e^{2(\beta-\alpha)\psi}(\psi^{\lambda}F^{\mu\nu}+\psi^{\mu}F^{\lambda\nu})D_{\lambda}F_{\mu\nu}\\
&+e^{2(\beta-\alpha)\psi}D_{\lambda}F_{\mu\nu}D^{\lambda}F^{\mu\nu}+2(3\alpha-\beta)e^{2(\beta-\alpha)\psi}F^{\mu\lambda}F^{\nu}{}_{\lambda}D_{\mu}\psi_{\nu}\\
&+4(2\alpha^2+\beta^2)D_{\mu}\psi_{\nu}D^{\mu}\psi^{\nu}\\
&+4\alpha e^{2(\beta-\alpha)\psi}\psi^{\mu}F_{\mu\nu}D_{\lambda}F^{\lambda\nu}+8\alpha(\alpha\psi_{\mu}\psi_{\nu}-D_{\mu}\psi_{\nu})R^{\mu\nu}\\
&-4\alpha^2\psi_{\mu}\psi^{\mu}R+4\alpha^2D_{\mu}\psi^{\mu}D_{\nu}\psi^{\nu}\\
&+8\alpha(2\alpha^2+\beta^2)\psi_{\mu}\psi^{\mu}D_{\nu}\psi^{\nu}-8(2\alpha^3+2\alpha\beta^2-\beta^3)\psi^{\mu}\psi^{\nu}D_{\mu}\psi_{\nu}\\
&+4(3\alpha^4+4\alpha^2\beta^2-2\alpha\beta^3+\beta^4)\psi_{\mu}\psi^{\mu}\psi_{\nu}\psi^{\nu}],
\end{aligned}\tag{62}$$

where the 4D Ricci tensor and Ricci scalar appear in the above conformal equation. Furthermore, the comparison between the invariants (35) and (62) causes us to introduce new interaction terms such as $\psi^2R,\psi^2D\psi,$ and ψ^4.

Another way of obtaining the reduced Kretschmann scalar (62) is directly using the conformal transformation rule of the squaring curvature $\tilde{R}_{\mu\nu\lambda\sigma}\tilde{R}^{\mu\nu\lambda\sigma}$, which is expressed as (see, e.g., Carneiro et al. 2004, Bao et al. 2008)

$$\begin{aligned}
\tilde{R}_{\mu\nu\lambda\sigma}\tilde{R}^{\mu\nu\lambda\sigma}&=e^{-4\xi}\{R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}+8(\xi_{\mu}\xi_{\nu}-D_{\mu}\xi_{\nu})R^{\mu\nu}\\
&+4[D_{\mu}\xi^{\mu}D_{\nu}\xi^{\nu}+(D-2)D_{\mu}\xi_{\nu}D^{\mu}\xi^{\nu}]\\
&-4\xi_{\mu}\xi^{\mu}R+8(D-2)[\xi_{\mu}\xi^{\mu}D_{\nu}\xi^{\nu}-\xi^{\mu}\xi^{\nu}D_{\mu}\xi_{\nu}]\\
&+2(D-1)(D-2)\xi_{\mu}\xi^{\mu}\xi_{\nu}\xi^{\nu}\},
\end{aligned}\tag{63}$$

if the metric changes as

$$\tilde{g}_{\mu\nu}(x)=e^{2\xi(x)}g_{\mu\nu}(x).\tag{64}$$

Conversely, the 5D metric (41) can be rewritten in terms of 4D fields as follows:

$$\tilde{g}_{MN}=e^{2\alpha\psi}\begin{pmatrix}g_{\mu\nu}+e^{2(\beta-\alpha)\psi}A_{\mu}A_{\nu} & e^{2(\beta-\alpha)\psi}A_{\mu}\\ e^{2(\beta-\alpha)\psi}A_{\nu} & e^{2(\beta-\alpha)\psi}\end{pmatrix},\tag{65}$$

which means that the new conformal factor is equal to $\xi(x)=\alpha\psi(x)$. Hence, equation (63) changes the following result for $D=5$

$$\begin{aligned}
\tilde{\hat{R}}_{ABCD}\tilde{\hat{R}}^{ABCD}&=e^{-4\alpha\psi}\{\hat{R}_{ABCD}\hat{R}^{ABCD}+8\alpha(\alpha\hat{\psi}_{A}\hat{\psi}_{B}-\hat{D}_{A}\hat{\psi}_{B})\hat{R}^{AB}\\
&-4\alpha^2\hat{\psi}_{A}\hat{\psi}^{A}\hat{R}+4\alpha^2[\hat{D}_{A}\hat{\psi}^{A}\hat{D}_{B}\hat{\psi}^{B}+3\hat{D}_{A}\hat{\psi}_{B}\hat{D}^{A}\hat{\psi}^{B}]\\
&+24\alpha^3[\hat{\psi}_{A}\hat{\psi}^{A}D_{B}\hat{\psi}^{B}-\hat{\psi}^{A}\hat{\psi}^{B}\hat{D}_{A}\hat{\psi}_{B}]+24\alpha^4\hat{\psi}_{A}\hat{\psi}^{A}\hat{\psi}_{B}\hat{\psi}^{B}\}.
\end{aligned}\tag{66}$$

Moreover, a comparison between metrics (10) and (65) leads us to obtain a new potential term, $e^{(\beta-\alpha)\psi(x)}$. As a result, terms $\hat{R}_{ABCD}\hat{R}^{ABCD}$ in (35), the anholonomic components of $\hat{R}^{AB}$, and $\hat{R}$, which comes from equation (14), rescale under the redefined scalar field, $\varphi(x)=e^{(\beta-\alpha)\psi(x)}$, transformation. For example, the curvature invariant changes accordingly in the following form:

$$\hat{R}=R-2(\beta-\alpha)^2\psi_{\mu}\psi^{\mu}-\frac{1}{4}e^{2(\beta-\alpha)\psi}F_{\mu\nu}F^{\mu\nu}-2(\beta-\alpha)D_{\mu}\psi^{\mu}.\tag{67}$$

We should also calculate the following necessary relations of the function $\psi(x)$ as

$$\begin{aligned}
&\hat{D}_{\hat{\mu}}\psi_{\hat{\nu}}=D_{\hat{\mu}}\psi_{\hat{\nu}},\\
&\hat{D}_{\hat{\mu}}\psi_{\hat{5}}=\hat{D}_{\hat{5}}\psi_{\hat{\mu}}=\frac{1}{2}e^{(\beta-\alpha)\psi}\psi_{\hat{\lambda}}F^{\hat{\lambda}}{}_{\hat{\mu}},\\
&\hat{D}_{\hat{5}}\psi_{\hat{5}}=(\beta-\alpha)\psi_{\hat{\lambda}}\psi^{\hat{\lambda}},
\end{aligned}\tag{68}$$

and it is clear that $\psi_{\hat{5}} = \psi^{\hat{5}} = 0$, by considering the new versions of the connections (13), which are obtained the same way

$$\begin{aligned}
&\hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}} = \Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}},\\
&\hat{\Gamma}^{\hat{5}}{}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}e^{(\beta-\alpha)\psi}F_{\hat{\mu}\hat{\nu}},\\
&\hat{\Gamma}^{\hat{\mu}}{}_{\hat{5}\hat{\nu}} = \hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{5}} = -\frac{1}{2}e^{(\beta-\alpha)\psi}F^{\hat{\mu}}{}_{\hat{\nu}},\\
&\hat{\Gamma}^{\hat{\mu}}{}_{\hat{5}\hat{5}} = -(\beta-\alpha)\psi^{\hat{\mu}},\\
&\hat{\Gamma}^{\hat{5}}{}_{\hat{\mu}\hat{5}} = (\beta-\alpha)\psi_{\hat{\mu}}.
\end{aligned} \tag{69}$$

As a mathematical challenge, if we substitute all results into equation (66), after careful calculations, we exactly obtain equation (62). Conversely, in the lower dimension, the modified KK action (38) is transformed into

$$\begin{aligned}
\tilde{\hat{I}} = &\frac{1}{2\hat{\chi}^2}\int_{\hat{\mathcal{M}}} d^4x\, dy\sqrt{-g}\, e^{(4\alpha+\beta)\psi}e^{-2\alpha\psi}\\
&\times[R - 2(3\alpha^2+2\alpha\beta+\beta^2)\psi_\mu\psi^\mu - \frac{1}{4}e^{2(\beta-\alpha)\psi}F_{\mu\nu}F^{\mu\nu}\\
&-2(3\alpha+\beta)D_\mu\psi^\mu],
\end{aligned} \tag{70}$$

where we use $\hat{R}$ that is obtained by (48)–(50), and $\sqrt{-\hat{g}} = e^{\xi D}\varphi\sqrt{-g} = e^{(4\alpha+\beta)\psi}\sqrt{-g}$ for $D=5$. We can find exactly the same result by considering the rescaled curvature scalar in Einstein frame, which is once again given by Carneiro et al. (2004) and Bao et al. (2008)

$$\tilde{R} = e^{-2\xi}[R - (D-1)(D-2)\xi_\mu\xi^\mu - 2(D-1)D_\mu\xi^\mu], \tag{71}$$

together with the equations (67) and (68). This is the easy way, if one wants to obtain field equations from implementing the least action principle to the action after computing the Ricci scalar by only considering (71).

5. The Special Cases

It is easy to read off from the action (70) that to rid of the coefficient of the Einstein-Hilbert term, i.e., to obtain minimally coupled gravity, we must choose $\beta = -2\alpha$. Besides, the term $\psi_\mu\psi^\mu$ can be a canonical normalized term if $\alpha^2 = 1/12$, i.e., $(\alpha,\beta) = (\pm 1/2\sqrt{3}, \mp 1/\sqrt{3})$. Note that, $(\alpha,\beta) = (-1/3, 2/3)$ in (40) also satisfies the condition $\beta = -2\alpha$, and see Gibbons and Wiltshire (1986) for $(\alpha,\beta) = (-1/\sqrt{3}, 2/\sqrt{3})$. Conversely, for a $D = N+1$ dimensional spacetime, the proper values of the constants become $\alpha^2 = 1/[2(N-1)(N-2)]$ and $\beta = -(N-2)\alpha$ in Pope, meaning that $(\alpha,\beta) = (\pm 1/2\sqrt{3}, \mp 1/\sqrt{3})$ is obviously correct for $N=4$. The $\hat{R}$ term in (70) is also compatible with the result of Pope for $N=4$, except for the coefficient of the term $D_\mu\psi^\mu$, which is a total divergence and does not contribute to the field equations, as usual. However, we can say that it seems that there is a typographical error in Pope because the same coefficient, i.e., $-2(3\alpha+\beta)$, was also obtained in Perry (2009). Finally, if we consider the modified WYKK action from (62), then we have

$$\begin{aligned}
\tilde{\hat{I}} = &\frac{1}{2\hat{\chi}^2}\int_{\hat{\mathcal{M}}} d^4x\, dy\sqrt{-g}\, e^{(4\alpha+\beta)\psi}e^{-4\alpha\psi}[R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\\
&-\frac{3}{2}e^{2(\beta-\alpha)\psi}R_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} + \frac{3}{8}e^{4(\beta-\alpha)\psi}F_{\mu\nu}F^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\\
&+\frac{5}{8}e^{4(\beta-\alpha)\psi}F_{\mu\nu}F^{\nu\lambda}F_{\lambda\sigma}F^{\sigma\mu}\\
&+(9\alpha^2-14\alpha\beta+6\beta^2)e^{2(\beta-\alpha)\psi}\psi_\lambda\psi^\lambda F_{\mu\nu}F^{\mu\nu}\\
&+2(3\alpha^2-2\alpha\beta-2\beta^2)e^{2(\beta-\alpha)\psi}\psi_\mu\psi^\lambda F_{\lambda\nu}F^{\nu\mu}\\
&+4(\beta-\alpha)e^{2(\beta-\alpha)\psi}(\psi^\lambda F^{\mu\nu}+\psi^\mu F^{\lambda\nu})D_\lambda F_{\mu\nu}\\
&+e^{2(\beta-\alpha)\psi}D_\lambda F_{\mu\nu}D^\lambda F^{\mu\nu} + 2(3\alpha-\beta)e^{2(\beta-\alpha)\psi}F^{\mu\lambda}F^\nu{}_\lambda D_\mu\psi_\nu\\
&+4(2\alpha^2+\beta^2)D_\mu\psi_\nu D^\mu\psi^\nu + 4\alpha e^{2(\beta-\alpha)\psi}\psi^\mu F_{\mu\nu}D_\lambda F^{\lambda\nu}\\
&+8\alpha(\alpha\psi_\mu\psi_\nu - D_\mu\psi_\nu)R^{\mu\nu} - 4\alpha^2\psi_\mu\psi^\mu R\\
&+4\alpha^2 D_\mu\psi^\mu D_\nu\psi^\nu + 8\alpha(2\alpha^2+\beta^2)\psi_\mu\psi^\mu D_\nu\psi^\nu\\
&-8(2\alpha^3+2\alpha\beta^2-\beta^3)\psi^\mu\psi^\nu D_\mu\psi_\nu\\
&+4(3\alpha^4+4\alpha^2\beta^2-2\alpha\beta^3+\beta^4)\psi_\mu\psi^\mu\psi_\nu\psi^\nu].
\end{aligned} \tag{72}$$

Now, we have to set $\beta = 0$ to obtain a standard form of $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$. Hence, the two special cases are investigated to find the new reduced equations in the following subsections:

5.1 The $\beta = -2\alpha$ case

For $\beta = -2\alpha$ case, the field equations (60) are simplified as follows:

$$\begin{aligned}
&D_\lambda\tilde{\mathcal{P}}_{\nu\sigma} - D_\sigma\tilde{\mathcal{P}}_{\nu\lambda} + \frac{1}{2}e^{-3\alpha\psi}(F_{\nu\sigma}\tilde{\mathcal{Q}}_\lambda - F_{\nu\lambda}\tilde{\mathcal{Q}}_\sigma + 2F_{\lambda\sigma}\tilde{\mathcal{Q}}_\nu)\\
&+\alpha[\psi^\rho(g_{\nu\lambda}\tilde{\mathcal{P}}_{\rho\sigma} - g_{\nu\sigma}\tilde{\mathcal{P}}_{\rho\lambda}) + \psi_\lambda\tilde{\mathcal{P}}_{\nu\sigma} - \psi_\sigma\tilde{\mathcal{P}}_{\nu\lambda}] = 0,\\
&D_\lambda\tilde{\mathcal{Q}}_\nu + \alpha g_{\nu\lambda}\psi^\rho\tilde{\mathcal{Q}}_\rho - 3\alpha\psi_\nu\tilde{\mathcal{Q}}_\lambda - 2\alpha\psi_\lambda\tilde{\mathcal{Q}}_\nu\\
&+\frac{1}{2}e^{-3\alpha\psi}F_\nu{}^\rho\tilde{\mathcal{P}}_{\rho\lambda} - \frac{1}{2}e^{-3\alpha\psi}F_{\nu\lambda}\tilde{\mathcal{U}} = 0,\\
&D_\lambda\tilde{\mathcal{U}} - 2\alpha\psi_\lambda\tilde{\mathcal{U}} - \frac{1}{2}e^{-3\alpha\psi}F_\lambda{}^\rho\tilde{\mathcal{Q}}_\rho + 2\alpha\psi^\rho\tilde{\mathcal{P}}_{\rho\lambda} = 0.
\end{aligned} \tag{73}$$

Here, the conformal KK equations $\{\tilde{\mathcal{P}}_{\mu\nu}, \tilde{\mathcal{Q}}_\nu, \tilde{\mathcal{U}}\}$ become

$$\begin{aligned}
&\tilde{\mathcal{P}}_{\mu\nu} = e^{-2\alpha\psi}R_{\mu\nu} - \frac{1}{2}e^{-8\alpha\psi}F_{\mu\rho}F_\nu{}^\rho - \alpha e^{-2\alpha\psi}g_{\mu\nu}D_\rho\psi^\rho\\
&-6\alpha^2 e^{-2\alpha\psi}\psi_\mu\psi_\nu,\\
&\tilde{\mathcal{Q}}_\nu = -\frac{1}{2}e^{-5\alpha\psi}D_\rho F^\rho{}_\nu + 3\alpha e^{-5\alpha\psi}\psi_\rho F^\rho{}_\nu,\\
&\tilde{\mathcal{U}} = 2\alpha e^{-2\alpha\psi}D_\rho\psi^\rho + \frac{1}{4}e^{-8\alpha\psi}F_{\rho\tau}F^{\rho\tau},
\end{aligned} \tag{74}$$

from the 4D point of view. There is full agreement between the above equations (74) and Pope for $N = 4$. Conversely, for action (72) one finds that

$$\tilde{\tilde{I}}=\frac{1}{2\hat{\chi}^2}\int_{\mathcal{M}} d^4x\,dy\,\sqrt{-g}\,e^{-2\alpha\psi}[R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}-\frac{3}{2}e^{-6\alpha\psi}R_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}$$
$$+\frac{3}{8}e^{-12\alpha\psi}F_{\mu\nu}F^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}+\frac{5}{8}e^{-12\alpha\psi}F_{\mu\nu}F^{\nu\lambda}F_{\lambda\sigma}F^{\sigma\mu}$$
$$+61\alpha^2e^{-6\alpha\psi}\psi_\lambda\psi^\lambda F_{\mu\nu}F^{\mu\nu}-2\alpha^2e^{-6\alpha\psi}\psi_\mu\psi^\lambda F_{\lambda\nu}F^{\nu\mu}$$
$$-12\alpha e^{-6\alpha\psi}(\psi^\lambda F^{\mu\nu}+\psi^\mu F^{\lambda\nu})D_\lambda F_{\mu\nu}+e^{-6\alpha\psi}D_\lambda F_{\mu\nu}D^\lambda F^{\mu\nu}$$
$$+10\alpha e^{-6\alpha\psi}F^{\mu\lambda}F^{\nu}{}_{\lambda}D_\mu\psi_\nu+24\alpha^2D_\mu\psi_\nu D^\mu\psi^\nu$$
$$+4\alpha e^{-6\alpha\psi}\psi^\mu F_{\mu\nu}D_\lambda F^{\lambda\nu}+8\alpha(\alpha\psi_\mu\psi_\nu-D_\mu\psi_\nu)R^{\mu\nu}$$
$$-4\alpha^2\psi_\mu\psi^\mu R+4\alpha^2D_\mu\psi^\mu D_\nu\psi^\nu+48\alpha^3\psi_\mu\psi^\mu D_\nu\psi^\nu$$
$$-144\alpha^3\psi^\mu\psi^\nu D_\mu\psi_\nu+204\alpha^4\psi_\mu\psi^\mu\psi_\nu\psi^\nu]. \tag{75}$$

Finally, we can say that the quadratic term $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ does not have its canonical form in this case, as expected. However, to avoid this problem, we can set $\psi(x)=0$ which is not an acceptable condition in the KK theories, so that a nonphysical constraint, $F_{\rho\tau}F^{\rho\tau}=0$, arises from the last equation of (74). In fact, this is the well-known inconsistency problem of the KK theories. Conversely, in the considered model, $\tilde{\boldsymbol{Q}}_\nu$ and $\tilde{\mathcal{U}}$ in (74) are not necessarily zero, and even if $\psi(x)=0$, we do not have $F_{\rho\tau}F^{\rho\tau}=0$, but the Lorentz force density term, $f_\lambda=F_\lambda{}^\rho D_\tau F^\tau{}_\rho$, appears in the last equation of (73) as

$$f_\lambda=-D_\lambda(F_{\rho\tau}F^{\rho\tau}), \tag{76}$$

which can also be obtained from equations (18) and (32) if we recall that $\varphi(x)=1$ in Başkal and Kuyrukcu (2013).

5.2 The $\beta=0$ case

Let us investigate the reduced equations by assuming that $\beta=0$. This strong restriction actually breaks the high-dimensional structure of the metric (41). Nonetheless, this case still deserves attention from the viewpoint of the considered model, and α is a free parameter. In this regard, we have from equation (60) that

$$D_\lambda\tilde{\mathcal{P}}_{\nu\sigma}-D_\sigma\tilde{\mathcal{P}}_{\nu\lambda}+\frac{1}{2}e^{-\alpha\psi}(F_{\nu\sigma}\tilde{\boldsymbol{Q}}_\lambda-F_{\nu\lambda}\tilde{\boldsymbol{Q}}_\sigma+2F_{\lambda\sigma}\tilde{\boldsymbol{Q}}_\nu)$$
$$+\alpha[\psi^\rho(g_{\nu\lambda}\tilde{\mathcal{P}}_{\rho\sigma}-g_{\nu\sigma}\tilde{\mathcal{P}}_{\rho\lambda})+\psi_\lambda\tilde{\mathcal{P}}_{\nu\sigma}-\psi_\sigma\tilde{\mathcal{P}}_{\nu\lambda}]=0,$$
$$D_\lambda\tilde{\boldsymbol{Q}}_\nu+\alpha g_{\nu\lambda}\psi^\rho\tilde{\boldsymbol{Q}}_\rho-\alpha\psi_\nu\tilde{\boldsymbol{Q}}_\lambda+\frac{1}{2}e^{-\alpha\psi}F_\nu{}^\rho\tilde{\mathcal{P}}_{\rho\lambda}$$
$$-\frac{1}{2}e^{-\alpha\psi}F_{\nu\lambda}\tilde{\mathcal{U}}=0,$$
$$D_\lambda\tilde{\mathcal{U}}-\frac{1}{2}e^{-\alpha\psi}F_\lambda{}^\rho\tilde{\boldsymbol{Q}}_\rho=0, \tag{77}$$

where the corresponding equations of motion of the unified theory are equal to

$$\tilde{\mathcal{P}}_{\mu\nu}=e^{-2\alpha\psi}R_{\mu\nu}-\frac{1}{2}e^{-4\alpha\psi}F_{\mu\rho}F_\nu{}^\rho-\alpha e^{-2\alpha\psi}[g_{\mu\nu}D_\rho\psi^\rho+2D_\mu\psi_\nu]$$
$$-2\alpha^2e^{-2\alpha\psi}[g_{\mu\nu}\psi_\rho\psi^\rho-\psi_\mu\psi_\nu],$$
$$\tilde{\boldsymbol{Q}}_\nu=-\frac{1}{2}e^{-3\alpha\psi}D_\rho F^\rho{}_\nu,$$
$$\tilde{\mathcal{U}}=\frac{1}{4}e^{-4\alpha\psi}F_{\rho\tau}F^{\rho\tau}, \tag{78}$$

in the four dimensions. Moreover, the final form of the action (72) is similarly reorganized as:

$$\tilde{\tilde{I}}=\frac{1}{2\hat{\chi}^2}\int_{\hat{\mathcal{M}}} d^4x\,dy\sqrt{-g}\,[R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}-\frac{3}{2}e^{-2\alpha\psi}R_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma}$$
$$+\frac{3}{8}e^{-4\alpha\psi}F_{\mu\nu}F^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}+\frac{5}{8}e^{-4\alpha\psi}F_{\mu\nu}F^{\nu\lambda}F_{\lambda\sigma}F^{\sigma\mu}$$
$$+3\alpha^2e^{-2\alpha\psi}(3\psi_\lambda\psi^\lambda F_{\mu\nu}F^{\mu\nu}+2\psi_\mu\psi^\lambda F_{\lambda\nu}F^{\nu\mu})$$
$$-4\alpha e^{-2\alpha\psi}(\psi^\lambda F^{\mu\nu}+\psi^\mu F^{\lambda\nu})D_\lambda F_{\mu\nu}+e^{-2\alpha\psi}D_\lambda F_{\mu\nu}D^\lambda F^{\mu\nu}$$
$$+6\alpha e^{-2\alpha\psi}F^{\mu\lambda}F^{\nu}{}_{\lambda}D_\mu\psi_\nu+4\alpha^2(2D_\mu\psi_\nu D^\mu\psi^\nu+D_\mu\psi^\mu D_\nu\psi^\nu)$$
$$+4\alpha e^{-2\alpha\psi}\psi^\mu F_{\mu\nu}D_\lambda F^{\lambda\nu}+8\alpha(\alpha\psi_\mu\psi_\nu-D_\mu\psi_\nu)R^{\mu\nu}$$
$$-4\alpha^2\psi_\mu\psi^\mu R+16\alpha^3(\psi_\mu\psi^\mu D_\nu\psi^\nu-\psi^\mu\psi^\nu D_\mu\psi_\nu)$$
$$+12\alpha^4\psi_\mu\psi^\mu\psi_\nu\psi^\nu]. \tag{79}$$

In this case, the coefficient of the gravitational part R, which does not have its canonical form, becomes $e^{2\alpha\psi}$ in (70), but the $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ term is canonical in (79). The equation (78) also gives the $D_\rho F^\rho{}_\nu=0$ and $F_{\rho\tau}F^{\rho\tau}=0$, if $\tilde{\boldsymbol{Q}}=0$ and $\tilde{\mathcal{U}}=0$, respectively, despite $\psi(x)\neq0$. As mentioned before, this is not a problem for the WYKK theory due to the structure of general field equations (77). Actually, $\tilde{\mathcal{U}}\neq0$ is the necessary condition in this case; otherwise all equations in (77) fall into triviality. Besides, we can now determine the following new Lorentz force density term including boson fields as follows:

$$f_\lambda=-D_\lambda(\psi F_{\rho\tau}F^{\rho\tau}), \tag{80}$$

from the last equations of (77) and (78) if we choose the parameter $\alpha=-1/4$, and without considering, certainly, $\psi(x)\neq0$.

6. Conclusion

In this work, by employing an alternative form of the basis equation of the considered model in the review section, we have recomputed the reduced field equations in terms of the new $\{\overline{\mathcal{P}}_{\mu\nu},\overline{\boldsymbol{Q}}_\nu,\overline{\mathcal{U}}\}$ set, i.e., the equations of the typical KK theory. The new expressions (30)–(32) seem to be more accurate than others, including the $\{\mathcal{P}_{\mu\nu},\boldsymbol{Q}_\nu,\mathcal{U}\}$ set in Başkal and Kuyrukcu (2013). Then, using simplifications that result from the horizontal lift basis, we have generalized the resulting equations to the Einstein frame in which the conformal metric ansatz contains arbitrary parameters α

and β. After these reduction procedures, we have derived the desired set of field equations, which are governed by the new conformal $\{\tilde{\mathcal{P}}_{\mu\nu}, \tilde{\mathcal{Q}}_{\nu}, \tilde{\mathcal{U}}\}$ set, and the transformed quadratic action in the coordinate basis both by considering the vielbeins fields and the conformal transformation rules. Conversely, we have investigated the consequences of the two possible cases, $\beta = -2\alpha$ and $\beta = 0$, on this approach. As we mentioned in our previous work (Başkal and Kuyrukcu 2013), for the condition $\psi(x) = 0$, the standard KK theory renders the well-known nonphysical constraint, $F_{\rho\tau}F^{\rho\tau} = 0$, whereas this Lorentz invariant term does not have to be equal to zero in the WYKK theory because of the more general field equation (73). By contrast, the Lorentz force density, $f_{\lambda} = F_{\lambda}{}^{\rho} D_{\tau} F^{\tau}{}_{\rho}$, appears for the former case naturally. Additionally, even if $\psi(x) \neq 0$, we have demonstrated that the density term can only be equal to the negative gradient of the new invariant, i.e., $f_{\lambda} = -D_{\lambda}(\psi F_{\rho\tau}F^{\rho\tau})$ for the latter case with $\alpha = -1/4$.

Let us finally remark that we can extend our formalism to investigate the dimensionally reduced form of equation (7), whose second term had already been calculated in (62), in the 5D generalized KK theory for completeness. Furthermore, whether the first equation of (60) can be written in the form of the equation (8) can be studied. We have already shown that (Kuyrukcu 2016) the first field equations, i.e.,

$$D_{\lambda}\mathcal{P}_{\nu\sigma} - D_{\sigma}\mathcal{P}_{\nu\lambda} - \frac{1}{4}\varphi^{2}(F_{\nu\sigma}\mathcal{Q}_{\lambda} - F_{\nu\lambda}\mathcal{Q}_{\sigma} + 2F_{\lambda\sigma}\mathcal{Q}_{\nu}) = 0, \quad (81)$$

which is given by Başkal and Kuyrukcu (2013) or can be obtained from (30) using $\overline{\mathcal{P}}_{\mu\nu} = \mathcal{P}_{\mu\nu}$ and $\overline{\mathcal{Q}}_{\nu} = -(1/2)\varphi\mathcal{Q}_{\nu}$, can exactly be written in the following Camenzind's current density form

$$D_{\mu}R^{\mu}{}_{\nu\lambda\sigma} = D_{\lambda}(T_{\nu\sigma} - \frac{1}{2}g_{\nu\sigma}T) - D_{\sigma}(T_{\nu\lambda} - \frac{1}{2}g_{\nu\lambda}T), \quad (82)$$

for the case in which $\mathcal{Q}_{\lambda} = 0, \mathcal{U} = 0$, or more generally $\mathcal{Q}_{\lambda} = 0$, and $D_{\lambda}\mathcal{U} = 0$. Here $T_{\nu\sigma}$ represents the stress-energy tensor that comes from the KK theory of gravity (Liu and Wesson 1997). We are working on obtaining such relationships for the model under consideration.

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8. References

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