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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 3, Pages 846–855 (2022)
DOI:10.31801/cfsuasmas.1009068
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: October 13, 2021; Accepted: April 18, 2022

ON THE SOLUTIONS OF THE q-ANALOGUE OF THE TELEGRAPH DIFFERENTIAL EQUATION

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Abstract. In this work, q-analogue of the telegraph differential equation is investigated. The approximation solution of q-analogue of the telegraph differential equation is founded by using the Laplace transform collocation method (LTCM). Then, the exact solution is compared with the approximation solution for q-analogue of the telegraph differential equation. The results showed that the method is useful and effective for q-analogue of the telegraph differential equation.

1. Introduction

Quantum calculus (q-calculus) was initiated at the beginning of the 18th century by Euler [1]. The q-calculus is often called calculus without limits. It allows the substitution of the classical derivative with the q-derivative operator to deal with sets of non-differentiable functions. The q-calculus has an unexpected role in several mathematical areas such as fractal geometry, quantum theory, hypergeometric functions, orthogonal polynomials, the calculus of variation and theory of relativity. The works [2], [3] can be cited for some results related to the history of quantum calculus, its basic concepts and q-differential equations. In [4], [5], a q-analogue of Sturm-Liouville problems are investigated.

Partial differential equations are ubiquitous in mathematically-oriented scientific fields, such as physics and engineering. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics. In [6], an expansion theorem was proved for the analytic function in several variables which satisfies a system of q-partial differential equations by using the theory of functions of several variables and q-calculus. In [7], using the theory of functions of

2020 *Mathematics Subject Classification.* 05A30, 47F05, 44A10, 74G15.

Keywords. q-analogue of the telegraph differential equation, Laplace transform collocation method, exact solution, approximation solution.

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several complex variables, it was proved that if an analytic function in several variables satisfies a system of q-partial differential equations then, it can be expanded in terms of the product of the Rogers-Szegő polynomials. In [8], identities and evaluate integrals by expanding functions in terms of products of the q-hypergeometric polynomials was proved by homogeneous q-partial difference equations.

In [9], with the use of Laplace transform technique, a new form of trial function from the original equation is obtained. The unknown coefficients in the trial functions are determined using collocation method. In [10], using the Laplace transform collocation method (LTCM) and Daftar-Gejii-Jafaris method (DGJM), the fractional order time-varying linear dynamical system was investigated.

In this paper, the following the telegraph differential equation defined by q-difference operator which we call the q-analogue of the telegraph differential equation is studied

$$\begin{cases} D^2_{q,\eta}\varphi(\eta,\xi) + D_{q,\eta}\varphi(\eta,\xi) + \varphi(\eta,\xi) = D^2_{q,\xi}\varphi(\eta,\xi) + f(\eta,\xi), \\ 0 < \eta < L \quad 0 < \xi < L \quad 0 < q \le 1, \\ \\ \varphi(0,\xi) = h(\xi), \quad D_{q,\eta}\varphi(0,\xi) = g(\xi) \\ \\ \varphi(\eta,0) = \varphi(\eta,L) = 0, \end{cases} \tag{1}$$

where h, g and f are known continuous functions and the function φ is unknown function. $D_{q,\eta}\varphi(\eta,\xi) = \frac{\partial_q \varphi(\eta,\xi)}{\partial_q \eta}$, $D_{q,\xi}\varphi(\eta,\xi) = \frac{\partial_q \varphi(\eta,\xi)}{\partial_q \xi}$ are q-difference of $\varphi(\eta,\xi)$ respect to η and ξ, respectively. If $\alpha = 1$, and $q = 1$ then the equation (1) is called telegraph partial differential equation.

LTCM method is used for numerical solution of the problem (1). Using the Laplace transform method, the exact solution of the problem (1) and a new form of trial function from the basic equation are obtained.

2. Preliminaries

We first recall some basic definition in q-calculus.

Let parameter q be a positive real number and n a non-negative integer. $[n]_q$ denotes a q integer, defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Let $q > 0$ be given. We define a q-factorial, $[n]_q!$ of $k \in \mathbb{N}$, as

$$[n]_q! = \begin{cases} [1]_q[2]_q...[n]_q, & n = 1,2,... \\ 1, & n = 0. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[n-r]_q![r]_q!}.$$

The q-shifted factorials (q-Pochhammer symbol) are defined for $a \in \mathbb{C}$ by

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{j=0}^{\infty}(1-aq^j).$$

The q-exponential function is given by

$$E_q(-z) = ((1-q)z;q)_\infty = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!}z^n.$$

For $t, x, y \in \mathbb{R}$ and $n \in \mathbb{Z} \geq 0$, the q-binomial formula is given by

$$(x+y)_q^n = \prod_{j=0}^{n-1}(x+q^j y) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q x^{n-j}q^{\frac{n(n-1)}{2}}y^j.$$

Let q be a positive number with $0 < q < 1$. Let f be a real or complex valued function on A (A is q-geometric set (see [4])). The q-difference operator D_q (the Jackson q-derivative) is defined as

$$D_q f(x) =: \frac{\partial_q f(x)}{\partial_q x} = \frac{f(x)-f(qx)}{x(1-q)}, \quad x \neq 0.$$

Let f and g are defined on a q-geometric set A such that the q-derivatives of f and g exist for all $x \in A$. Then, there is a non-symmetric formula for the q-differentiation of a product

$$D_q[f(x)g(x)] = f(qx)D_q g(x) + g(x)D_q f(x).$$

The q-integral usually associated with the name of Jackson is defined in the interval $(0, x)$, as

$$\int_0^x f(t)d_q t = (1-q)\sum_{n=0}^{\infty} f(xq^n)xq^n,$$

$$\int_0^x D_q f(t)d_q t = f(x) - f(0).$$

The q-integration for a function f over $[0,\infty)$ is defined as the following by Hahn (see [11])

$$\int_0^\infty f(t)d_q t = \sum_{n=-\infty}^{\infty}(1-q)q^n f(q^n).$$

The q-analogue of the Laplace transformed is defined by

$$F_q(s) = \pounds_q(f(t)) = \int_0^\infty E_q(-qst)f(t)d_qt \quad (s > 0). \tag{2}$$

From (2), we obtain

$$\pounds_q(\alpha f(t) + \beta g(t)) = \alpha \pounds_q(f(t)) + \beta \pounds_q(g(t)),$$

where α, β are constants. The q-analogue of the Gamma function is defined as the following in [13]

$$\Gamma_q(t) = \int_0^{\frac{1}{(1-q)}} x^{t-1}E_q(-qx)d_qx, \quad (t > 0) \tag{3}$$

From (2) and (3), we get

$$\pounds_q(1) = \frac{1}{s} \quad (s > 0), \quad \pounds_q(t) = \frac{1}{s^2} \quad (s > 0), ..., \pounds_q(t^n) = \frac{\Gamma_q(n+1)}{s^{n+1}} = \frac{[n]_q!}{s^{n+1}}.$$

3. LTCM for q-Analogue of the Telegraph Differential Equation

We shall obtain numerical solution of q-analogue of the telegraph differential equation using the method LTCM. Taking the Laplace transform of the problem (1), we get

$$D_{q,\eta}\varphi(0,\xi) - s\varphi(0,\xi) + s^2\varphi_q(s,\xi)$$

$$= -\pounds_q\{D_{q,\eta}\varphi(\eta,\xi)\} - \pounds_q\{\varphi(\eta,\xi)\} + \pounds_q\{D^2_{q,\xi}\varphi(\eta,\xi)\} + \pounds_q\{f(\eta,\xi)\} \tag{4}$$

After simple algebraic simplification and using initial condition of the problem (1), we have

$$\begin{aligned}\varphi_q(s,\xi) = \frac{1}{s^2}\big[&D_{q,\eta}\varphi(0,\xi) + s\varphi(0,\xi) - \pounds_q\{D_{q,\eta}\varphi(\eta,\xi)\} - \pounds_q\{\varphi(\eta,\xi)\} \\ &+\pounds_q\{D^2_{q,\xi}\varphi(\eta,\xi)\} + \pounds_q\{f(\eta,\xi)\}\big]\end{aligned} \tag{5}$$

The function $\varphi_q(\eta,\xi)$ and its derivative function in the equation (5) are replaced with a trial function of the form

$$\varphi_q = \varphi_q^0 + \sum_{i=1}^{n} c_i\varphi_q^i, \tag{6}$$

then we will obtain the following equation

$$\begin{aligned}\varphi_q(s,\xi) = \frac{1}{s^2}\Bigg[& D_{q,\eta}\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(0,\xi)\right) + s\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(0,\xi)\right) \\ & -\pounds_q\left\{D_{q,\eta}\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(\eta,\xi)\right)\right\} - \pounds_q\left\{\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(\eta,\xi)\right\} \\ & +\pounds_q\left\{D_{q,\xi}^2\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(\eta,\xi)\right)\right\} + \pounds_q\{f(\eta,\xi)\}\Bigg], \end{aligned} \tag{7}$$

where c_i are constants to be stated which satisfy the given conditions in the problem (1). Taking the inverse q-Laplace transform of the equation (7), we obtain

$$\begin{aligned}\varphi_q^{new}(\eta,\xi) = \pounds_q^{-1}\Bigg[\frac{1}{s^2}\Bigg[& D_{q,\eta}\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(0,\xi)\right) + s\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(0,\xi)\right) \\ & -\pounds_q\left\{D_{q,\eta}\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(\eta,\xi)\right)\right\} - \pounds_q\left\{\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(\eta,\xi)\right\} \\ & +\pounds_q\left\{D_{q,\xi}^2\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n} c_i\varphi_q^i(\eta,\xi)\right)\right\} + \pounds_q\{f(\eta,\xi)\}\Bigg]\Bigg]. \end{aligned} \tag{8}$$

Substituting the equality (8) into the problem (1), we get new collocating at points $\xi=\xi_k$ as following

$$D_{q,\eta}^2\varphi_q^{new}(\eta,\xi_k)+\varphi_q^{new}(\eta,\xi_k)+D_{q,\eta}\varphi_q^{new}(\eta,\xi_k)-D_{q,\xi}^2\varphi_q^{new}(\eta,\xi_k)=f(\eta,\xi_k) \tag{9}$$

where $\xi_k=\frac{L-0}{n+1}$, $k=1,2,\ldots,n$.

Now, we shall define the residual function by the following formula

$$R_n(\eta,\xi)=L[\varphi_q^{new}(\eta,\xi)]-f(\eta,\xi). \tag{10}$$

Here $\varphi_q^{new}(\eta,\xi)$ demonstrates the approximate solution, $\varphi(\eta,\xi)$ demonstrates the exact solution and

$$L[\varphi_q^{new}(\eta,\xi)]=D_{q,\eta}^2\varphi_q^{new}(\eta,\xi)+D_{q,\eta}\varphi_q^{new}(\eta,\xi)+\varphi_q^{new}(\eta,\xi)-D_{q,\xi}^2\varphi_q^{new}(\eta,\xi). \tag{11}$$

From the equality (11), we write

$$D_{q,\eta}^2\varphi_q^{new}(\eta,\xi)+D_{q,\eta}\varphi_q^{new}(\eta,\xi)+\varphi_q^{new}(\eta,\xi)-D_{q,\xi}^2\varphi_q^{new}(\eta,\xi)=f(\eta,\xi)+R_n(\eta,\xi), \tag{12}$$

Now since L is a linear operator, we obtain for the error function
$e_n=\varphi_q^{new}(\eta,\xi)-\varphi(\eta,\xi)$

$$D_{q,\eta}^2e_n(\eta,\xi)+D_{q,\eta}e_n(\eta,\xi)+e_n(\eta,\xi)-D_{q,\xi}^2e_n(\eta,\xi)=R_n(\eta,\xi). \tag{13}$$

From the conditions in the problem (1), we get

$$e_n(0,\xi) = D_{q,\eta} e_n(0,\xi) = D^2_{q,\eta} e_n(0,\xi) = 0, \tag{14}$$

$$e_n(\eta,0) = e_n(\eta,L) = 0. \tag{15}$$

By solving (13) subject to the homogeneous conditions (14) and (15), we obtain the error function $e_n(\eta,\xi)$. This allows us to calculate $\varphi(\eta,\xi)$=$u_n(\eta,\xi) + e_n(\eta,\xi)$ even for problems without known exact solutions.

4. Numerical Applications

In this section, we shall present one test example for implementation of the LTCM. In the following example, the numerical solution calculated by the this method will be compared with the exact solution.

Example 1. *Consider the following initial-boundary value problem for q-analogue of the telegraph differential equation*

$$\begin{cases} D^2_{q,\eta}\varphi(\eta,\xi) + D_{q,\eta}\varphi(\eta,\xi) + \varphi(\eta,\xi) = D^2_{q,\xi}\varphi(\eta,\xi) + [3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 \\ \qquad\qquad + \eta^3\xi^3 - [3]_q!\xi\eta^3 \\ 0<\eta<L \quad 0<\xi<L \quad 0<q\le 1, \\ \varphi(0,\xi) = h(\xi), \quad D_{q,\eta}\varphi(0,\xi) = g(\xi) \\ \qquad \varphi(\eta,0) = 0 \quad \varphi(\eta,1) = \eta^3 \end{cases} \tag{16}$$

First, we shall calculate the example problem (16) by LTCM.

We assume that the trial function is the following form:

$$\varphi(\eta,\xi) = c_1\xi^2(\xi-1)\eta^3 + c_2\xi(\xi-1)^2\eta^3. \tag{17}$$

Taking the Laplace transform of the equation (16) and using the formula (7), we obtain

$$\begin{aligned} & -D_{q,\eta}\varphi_q(0,\xi) - s\varphi_q(0,\xi) + s^2\varphi_q(s,\xi) \\ = \; & -\pounds_q\left\{D_{q,\eta}\varphi(\eta,\xi)\right\} - \pounds_q\left\{\varphi(\eta,\xi)\right\} + \pounds\left\{D^2_{q,\xi}\varphi(\eta,\xi)\right\} \\ & +\pounds\left\{[3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 + \eta^3\xi^3 - [3]_q!\xi\eta^3\right\}. \end{aligned} \tag{18}$$

Using the initial condition of the problem (16), the formula (18) is obtained as:

$$\begin{aligned} \varphi_q(s,\xi) = \frac{1}{s^2}&\left[-\pounds_q\{D_{q,\eta}\varphi(\eta,\xi)\} - \pounds_q\{\varphi(\eta,\xi)\}\right. \\ &\left. +\pounds_q\{D^2_{q,\xi}\varphi(\eta,\xi)\} + \pounds_q\{[3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 + \eta^3\xi^3 - [3]_q!\xi\eta^3\}\right] \end{aligned} \tag{19}$$

Using the formulas (17) and (19), we obtain

$$\begin{aligned} \varphi_q(s,\xi) \;=\; & \frac{1}{s^2}\pounds_q\left\{\left(-[3]_q\xi^2(\xi-1)\eta^2 - \xi^2(\xi-1)\eta^3 + [3]_q!\xi\eta^3 - [2]_q!\eta^3\right)c_1\right. \\ & + \left(-[3]_q\xi(\xi-1)^2\eta^2 - \xi(\xi-1)^2\eta^3 + ([3]_q!\xi - [4]_q\eta^3\right)c_2 \\ & +\pounds_q\left\{[3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 + \eta^3\xi^3 - [3]_q!\xi\eta^3\right\}\right\}. \end{aligned} \tag{20}$$

From the formula (20), we can get

$$\begin{aligned}\varphi_q(s,\xi) &= \left(-[3]_q!\xi^2(\xi-1)\frac{1}{s^5}-[3]_q!\xi^2(\xi-1)\frac{1}{s^6}+[3]_q!([3]_q!\xi-[2]_q!)\frac{1}{s^6}\right)c_1\\ &+\left(-[3]_q!\xi(\xi-1)^2\frac{1}{s^5}-[3]_q!\xi(\xi-1)^2\frac{1}{s^6}+[3]_q!([3]_q!\xi-[4]_q)\frac{1}{s^6}\right)c_2\\ &+\left(\frac{[3]_q!}{s^4}+\frac{[3]_q!}{s^5}+\frac{[3]_q!}{s^6}\right)\xi^3-\frac{[3]_q!^2}{s^6}\xi \end{aligned} \tag{21}$$

Taking the inverse Laplace transform of (21), we get the following new trial solution:

$$\begin{aligned}\varphi_q^{new}(\eta,\xi) &= \left[\left(-\frac{\eta^4}{[4]_q}-\frac{\eta^5}{[4]_q[5]_q}\right)(c_1+c_2)+\eta^3+\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}\right]\xi^3\\ &+\left[\left(\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}\right)(c_1+2c_2)\right]\xi^2\\ &+\left[\frac{[3]_q!}{[4]_q[5]_q}\eta^5c_1-\left(\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}-\frac{[3]_q!}{[4]_q[5]_q}\eta^5\right)c_2-\frac{[3]_q!}{[4]_q[5]_q}\eta^5\right]\xi\\ &-\frac{\eta^5}{[5]_q}(c_1+c_2) \end{aligned} \tag{22}$$

Substituting (22) into (16), we have the following residual formula:

$$\begin{aligned} R(\eta,\xi,c_1,c_2) &= D^2_{q,\eta}\varphi_q^{new}(\eta,\xi)+D_{q,\eta}\varphi_q^{new}(\eta,\xi)+\varphi_q^{new}(\eta,\xi)-D^2_{q,\xi}\varphi_q^{new}(\eta,\xi)\\ &-\left([3]_q!\eta+[3]_q\eta^2+\eta^3\right)\xi^3+[3]_q!\eta^3\xi \end{aligned} \tag{23}$$

Taking the derivatives of the equation (22) as to ξ and η, and writing in the formula (23), we obtain

$$\begin{aligned} R(\eta,\xi,c_1,c_2) &= (A\xi^3-A\xi^2+D\xi-B-C)c_1\\ &+ (A\xi^3-2A\xi^2+(A+D)\xi-B-2C)c_2-A-D\\ &= 0, \end{aligned} \tag{24}$$

where,

$$A=-\frac{\eta^5}{[4]_q[5]_q}-2\frac{\eta^4}{[4]_q}-2\eta^3-[3]_q\eta^2,$$

$$B=[4]_q\eta^3+\eta^4+\frac{\eta^5}{[5]_q},$$

$$C=[2]_q\left(\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}\right),$$

$$D=\frac{2[3]_q!}{[4]_q[5]_q}\eta^5+\frac{2[3]_q!}{[4]_q}\eta^4+[3]_q!\eta^3.$$

From (24), we have

$$c_1 = \frac{A}{A\xi^3 - A\xi^2 + D\xi - B - C}$$

$$c_2 = \frac{D}{A\xi^3 - 2A\xi^2 + (A+D)\xi - B - 2C}.$$

Errors calculate by the following formula

$$Error = |exact \quad solution - approximate \quad solution|,$$

$$\epsilon = max|\varphi_{exact} - \varphi_{app}|,$$

where $\varphi_{exact} = \eta^3\xi^3$ is exact solution and $\varphi_{app} = c_1\xi^2(\xi-1)\eta^3 + c_2\xi(\xi-1)^2\eta^3$ is numerical solution that is obtained by using LTCM for the problem (16). As shows

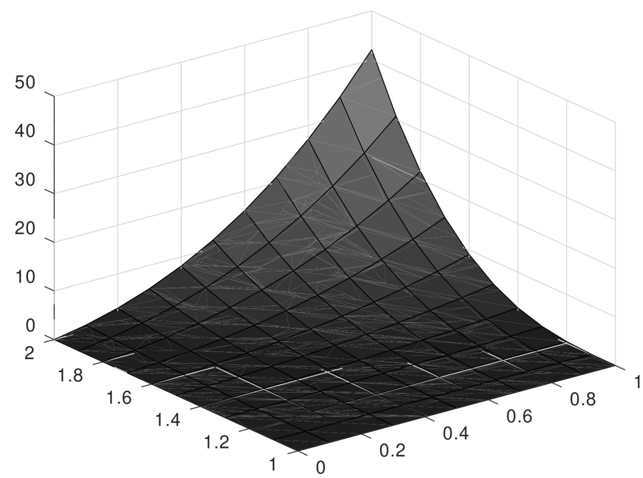

FIGURE 1. Gives the approximation solution of the example (16) for $1 \leq \xi \leq 2$, $0 \leq \eta \leq 1$ and $q = 0.01$.

from the figure of Figure 1 the difference better exact solution and approximation solutions is not clearly obvious. Therefore we present the numerical regents and error analysis in the following Table 1.

5. CONCLUSION

In this work, we adopted a combination of Laplace transform collocation method to develop numerical methods for the q-difference operator for the telegraph differential equation. Numerical example was considered to demonstrate the accuracy and efficiency of this method. The exact solution is compared with the approximate solution. Obtained results are given in the numerical error analysis Table 1 The simulations are showed for the exact and approximation solution.

$\xi = \eta$	α	*Exact Solution*	*LTCM method*	*Error Analysis*
0.99	0.01	0.941480149401000	0.769146969442938	0.172333179958063
0.5	0.01	0.015625000000000	0.032637283173177	0.017012283173177
0.5	0.5	0.015625000000000	0.037117402318906	0.021492402318906
0.5	0.99	0.015625000000000	0.027823738101408	0.012198738101408
0.1	0.01	1.0001×10^{-6}	2.3185×10^{-5}	2.2185×10^{-5}
0.1	0.5	1.0001×10^{-6}	1.6396×10^{-5}	1.5396×10^{-5}
0.1	0.99	1.0001×10^{-6}	1.1374×10^{-5}	1.0374×10^{-5}
0.01	0.01	1.000×10^{-12}	2.8934×10^{-10}	2.883410×10^{-10}
0.01	0.5	1.000×10^{-12}	1.7618×10^{-10}	1.7518×10^{-10}
0.01	0.99	1.000×10^{-12}	1.1629×10^{-10}	1.1529×10^{-10}

TABLE 1. Table error analysis of Example 1.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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