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Research Article

Strongly ⊕-Locally Artinian Supplemented Modules

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Abstract

The aim of this paper is to investigate strong notion of strongly ⊕-supplemented modules in module theory, namely strongly ⊕-locally artinian supplemented modules. We call a module M *strongly ⊕-locally artinian supplemented* if it is locally artinian supplemented and its locally artinian supplement submodules are direct summand. In this study, we provide the basic properties of strongly ⊕-locally artinian supplemented modules. In particular, we show that every direct summand of a strongly ⊕-locally artinian supplemented module is strongly ⊕-locally artinian supplemented. Moreover, we prove that a ring R is semiperfect with locally artinian radical if and only if every projective R-module is strongly ⊕-locally artinian supplemented.

Keywords: Strongly ⊕-supplemented modules, Strongly ⊕-locally artinian supplemented modules, Locally artinian modules, Locally artinian supplemented modules, Semiperfect rings.

Güçlü ⊕-Yerel Artin Tümlenmiş Modüller

Öz

Bu makalenin amacı, modül teorisinde güçlü ⊕-tümlenmiş modüllerin kuvvetlenişi olarak güçlü ⊕-yerel artin tümlenmiş modül kavramını araştırmaktır. Yerel artin tümlenmiş olan ve yerel artin tümleyen altmodülleri direkt toplam terimi olan M modülü güçlü ⊕-yerel artin tümlenmiş modül olarak adlandırırız. Bu çalışmada, güçlü ⊕-yerel artin tümlenmiş modüllerin temel özelliklerini sunuyoruz. Özellikle, güçlü ⊕-yerel artin tümlenmiş bir modülün her direkt toplam teriminin güçlü ⊕-yerel artin tümlenmiş modül olduğunu gösteriyoruz. Ayrıca, yerel artin radikale sahip bir R halkasının yarı mükemmel olması için gerek ve yeter koşulun her projektif R-modülün güçlü ⊕-yerel artin tümlenmiş modül olması olduğunu kanıtlıyoruz.

Anahtar Kelimeler: Güçlü ⊕-tümlenmiş modüller, Güçlü ⊕-yerel artin tümlenmiş modüller, Yerel artin modüller, Yerel artin tümlenmiş modüller, Yarımükemmel halkalar.

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1. Introduction

Throughout this paper, R will always denote an associative ring with identity element and modules will be left unital. Our terminology and notation adheres to that of the major references in the theory of rings and modules such as (Kasch,1982) and (Wisbauer,1991). $Rad(R)$ will be denote the Jacobson radical of the ring R. A submodule N of M will be shown that $N \leq M$. We will use the notation $U \ll M$ to stress that U is a *small submodule* of M. $Rad(M)$ and $Soc(M)$ will indicate radical and socle of M which are sum of all small submodule of M, and are sum of all semisimple submodule of M, respectively. A non-zero module M is called *hollow* if every proper submodule of M is small in M, and M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. A ring R is called *local* if ${}_RR$ is a local module. A module M is called *locally artinian* if every finitely generated submodule of M is artinian (Wisbauer, 1991) in chapter 31. A submodule V of M is called a *supplement* of U in M if $M = U + V$ and $U \cap V \ll V$. The module M is called *supplemented* if every submodule of M has a supplement in M. A submodule U of M has *ample supplements* in M if every submodule V of M such that $M = U + V$ contains a supplement supplemented and every supplement submodule of M is direct summand. We say N *lies above* K *in* M if $\frac{N}{K} \ll \frac{M}{K}$. Detailed studies about lifting modules can be found at the (Clark et al, 2012). As a proper generalization of lifting modules, Nebiyev and Pancar defined this way strongly ⊕-supplemented modules in (Nebiyev & Pancar, 2011). A supplemented module M is called strongly ⊕-supplemented if every supplement submodule of M is a direct summand in M. Then this definition is available in (Clark et al., 2012) as *weak lifting modules* and there exists basic properties of these modules.

2. Materials and Methods

In (Zhou & Zhang, 2011), Zhou and Zhang generalized the concept of socle of a module M to that of $Soc_s(M)$ by considering the class of all simple submodules of M that are small in M in place of the class of all simple submodules of M and $Soc_s(M)$ is a generalization of concept of socle as a submodule of M , that is $Soc_s(M) = \sum\{U \ll M | U \text{ is simple}\}$. It is clear that the inclusions $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$ are hold. In (Kaynar et al, 2020), a module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. A submodule U of M is called *ss-supplement* of U in M if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. The module M is called *ss-supplemented* if every submodule of M has an ss-supplement in M. A submodule U of M has *ample ss-supplements* in M if every submodule V of M such that $M = U + V$ contains an ss-supplement V' of U in M. The module M is called *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. In (Şahin & Nişancı Türkmen, 2020), strongly local and (amply) ss-supplemented modules are generalized as RLA-local and (amply) locally artinian supplemented modules, respectively. A local module M is called *RLA-local* if $Rad(M)$ is a locally artinian submodule of M. A module M is called *locally artinian supplemented* if every submodule U of M has a locally artinian supplement in M, that is, V is a supplement of U in M such that $U \cap V$ is locally artinian. A module M is called *amply locally artinian supplemented* if every submodule U of M has ample locally artinian supplements in M. Here a submodule U of M has ample locally artinian supplements in M if every submodule V of M such that $M = U + V$ contains a locally artinian supplement V' of U in M.

In section 3, as a strong notion of strongly ⊕-supplemented modules, we define strongly ⊕-locally artinian supplemented modules and provide the basic properties of strongly ⊕-locally artinian supplemented modules. Especially, we show that every direct summand of a strongly ⊕-locally artinian supplemented module is strongly ⊕-locally artinian supplemented in Proposition 3.4. In Theorem 3.7, we characterize the rings whose projective modules are strongly ⊕-locally artinian supplemented.

3. Strongly ⊕-Locally Artinian Supplemented Modules

In this section we explain the notion of strongly ⊕-locally artinian supplemented modules which we have introduced as a new concept by emphasizing its useful features.

Definition 3.1 A locally artinian supplemented module M is called *strongly ⊕-locally artinian supplemented* if every locally artinian supplement submodule of M is a direct summand.

It is clear that the following implications hold:

Strongly ⊕-locally artinian supplemented ⟹strongly ⊕-supplemented

Lemma 3.2 Let M be a strongly ⊕-supplemented module and $Rad(M)$ is locally artinian. Then M is strongly ⊕-locally artinian supplemented.

Proof.

Clearly, M is supplemented. By (Şahin & Nişancı Türkmen, 2020) in Lemma 2.8, M is locally artinian supplemented because $Rad(M)$ is locally artinian. Let V be a locally artinian supplement in M. Therefore, we can write $M = U + V$ and $U \cap V \ll V$ *for* some submodule $U \leq M$. Since M is strongly ⊕-supplemented, V is a direct summand of M. It means that M is strongly ⊕-locally artinian supplemented. ∎

Recall from [8, 41.13] that a module M is called *π-projective* if, for every submodules N, K of M and identity homomorphism. $I_M : M \longrightarrow M$ with $M = N + K$, there exists a $\delta \in End(M)$ such that $Im(\delta) \leq N$ and $Im(I_M - \delta) \leq K$.

Lemma 3.3 Let M be a locally artinian supplemented and π-projective module. Then M is a strongly ⊕-locally artinian supplemented module.

Proof.

Since M is locally artinian supplemented and π-projective, M is amply locally artinian supplemented by (Şahin & Nişancı Türkmen, 2020) in Proposition 2.20. Let K be a locally artinian supplement of some submodule $N \leq M$. Since M is amply locally artinian supplemented, there exists a submodule N' of N such that $M = N + K = N' + K$, $N' \cap K \ll N'$ and $N' \cap K$ is locally artinian. It follows that N' and K are mutual supplements in M. By (Wisbauer, 1991) in chapter 41.14(2), we obtain that $K \cap N' = 0$. Therefore $M = N' \oplus K$. Thus M is strongly ⊕-locally artinian supplemented. ∎

Proposition 3.4 Let M be a strongly $\oplus$-locally artinian supplemented module. Then every direct summand of M is strongly $\oplus$-locally artinian supplemented.

Proof.
Given $M = U \oplus V$. By (Şahin & Nişancı Türkmen, 2020) in Proposition 2.16, we deduce that U is locally artinian supplemented as a factor module of M. Let K be a locally artinian supplement of N in U. It is clear that K is a locally artinian supplement of $N \oplus V$ in M. Since M is strongly $\oplus$-locally artinian supplemented, we can write $M = K \oplus W$ for some submodule W of M. Then applying modular law, we obtain that $U = U \cap M = U \cap (K \oplus W) = K \oplus (U \cap W)$. Therefore K is a direct summand of U. Hence U is a strongly $\oplus$-locally artinian supplemented. ∎

Theorem 3.5 Let M_i be projective module for every $1 < i < n$. Then $M = \oplus_{i \in I} M_i$ is strongly $\oplus$-locally artinian supplemented if and only if every M_i is strongly $\oplus$-locally artinian supplemented.

Proof.
(⇒) Since every M_i is a direct summand of M for $1 < i < n$, the proof follows from Proposition 3.4.
(⇐) Since every M_i is locally artinian supplemented for $1 < i < n$, M is locally artinian supplemented by [7, Corollary 2.15]. It follows from [8] that M is projective. Then M is π-projective by [1, 4.13]. So M is strongly $\oplus$-locally artinian supplemented by Lemma 3.3. ∎

The next proposition explains which concepts in the literature coincide in π-projective modules whose radical is locally artinian.

Proposition 3.6 Let M be a π-projective module with $Rad(M)$ is locally artinian. Then the following statements are equivalent.
(i) M is locally artinian supplemented;
(ii) M is strongly $\oplus$-locally artinian supplemented;
(iii) M is supplemented;
(iv) M is strongly $\oplus$-supplemented.

Proof.
(**i**) ⇒ (**ii**) Since M is π-projective, the proof follows from Lemma 3.3.
(**ii**) ⇒ (**i**) Clear.
(**i**) ⇔ (**iii**) ⇔ (**iv**) Follows from [7, Lemma 2.22]. ∎

Recall that dualizing the concept of an injective hull of a module, a *projective cover* of a module M is defined as to be an epimorphism $\alpha: P \longrightarrow M$ such that P is a projective module and $Ker(\alpha) \ll P$. Thus modules having projective covers are, up to isomorphism, of the form $\frac{P}{K}$, where P is a projective module and K its small submodule. An R-module M is *semiperfect* if every factor module of M has a projective cover. If the ring R as an R-module is semiperfect then the ring R is semiperfect.

Theorem 3.7 For any ring R, the following statements are equivalent.
(**i**) R is semiperfect and $Rad(R)$ is locally artinian;
(**ii**) ${}_RR$ is locally artinian supplemented;
(**iii**) ${}_RR$ is strongly $\oplus$-locally artinian supplemented;
(**iv**) every projective R-module is strongly $\oplus$-locally artinian supplemented;
(**v**) every projective R-module is locally artinian supplemented.

Proof.
By Lemma 3.2 and [7, Corollary 2.10]. ∎

By categorizing a locally artinian supplemented module with the help of locally artinian supplement submodule, the folowing theorem to the strongly $\oplus$-locally artinian supplemented module.

Theorem 3.8 For a locally artinian supplemented module M, the following statements are equivalent.
(**i**) M is strongly $\oplus$-locally artinian supplemented;
(**ii**) every locally artinian supplement submodule of M lies above a direct summand;
(**iii**) **(a)** every non-zero locally artinian supplement submodule of M contains a non-zero direct summand of M;
(b) every locally artinian supplement submodule of M contains a maximal direct summand of M.

Proof.
(**i**) ⟹ (**ii**) Obvious.
(**ii**) ⟹ (**i**) Let U be a submodule of M and V be a locally artinian supplement of U in M. By the hypothesis, there exist submodules $K, L \leq M$ such that $M = K \oplus L$, $K \leq V$ and $V \cap L \ll L$. Then $V = V \cap M = V \cap (K \oplus L) = K \oplus (V \cap L)$, we have $V \cap L \ll M$ and $M = U + V = U + K + (V \cap L) = U + K$. Since V is a locally artinian supplement of U, we obtain that $V = K$. Thus $M = V \oplus L$. Hence M is strongly $\oplus$-locally artinian supplemented.
(**i**) ⟹ (**iii**) Clear.
(**iii**) ⟹ (**i**) Let V be a locally artinian supplement of U in M. Suppose that $W \leq V$ and $M = W \oplus S$. Then $V = W \oplus (V \cap S)$ and $V \cap S$ is a locally artinian supplement of $U + W$ in M. If $V \cap S = 0$, then by (iii)-(a) there exists a non-zero direct summand T of M such that $T \leq V \oplus S$. Then we have $W \oplus T$ is a direct summand of M and $W \oplus T \leq V$. This contradicts the choice of W. Thus $V \cap S = 0$ and $V = W$. So V is a direct summand of M. Therefore M is strongly $\oplus$-locally artinian supplemented. ∎

Recall that an integral domain R is a *Dedekind ring* (or *Dedekind domain*) if every non-zero ideal of R is invertible. Let R be a principal ideal domain. If R has the unique prime element, then R is called a *discrete valuation ring.* A discrete valuation ring, or DVR, is a local Dedekind ring. Let R be a discrete valuation ring and p be the unique prime element of R. Then every ideal of R is of the form Rp^k which $k \in \mathbb{Z}$. If we take these ideals to be neighborhoods of 0 in R, we define a topology in R, making R a topological ring. If R is complete in this topology, R is called a *complete discrete valuation ring*.

Finally, let's give an example of a strongly $\oplus$-supplemented module and but not strongly $\oplus$-locally artinian supplemented.

Example 3.9 *Let R be a discrete valuation ring which not complete and K be a quotient field of R. Then $M = K^2$ is strongly $\oplus$-supplemented by [5, Example 2.1]. But M is not a strongly $\oplus$-locally artinian supplemented module because K is not locally artinian supplemented in [7, Example 2.7].*

4. Conclusions and Recommendations

In this paper, we define new concept of the module class to specialize notion of strongly $\oplus$-supplemented modules and we give fundamental algebraic properties of these modules. Also, we characterize projective R-module over a semiperfect ring R by using notion of strongly ⊕-locally artinian supplemented modules. Then we give a counter example which is a strongly ⊕-supplemented module but not strongly ⊕-locally artinian supplemented.

References

Clark, J., Lomp, C., Vanaja, N., & Wisbauer, R.. (2012). *Lifting Modules Supplements and Projectivity in Module Theory*. Basel. Boston. Berlin, Birkhauser Verlag.

Kasch. F. (1982). *Modules and Rings*. London New York, Academic Press.

Keskin Tütüncü, D., & Tribak, R. (2005). On lifting modules and weak lifting modules. *Kyungpook Math.J.*, 45:445-453.

Kaynar, E. , Çalışıcı, H., Türkmen, E. (2020). SS-supplemented modules. *Communications Faculty of Science University of Ankara Series A1 Mathematics and statistics*, 69(1):473-485.

Nebiyev, C., & Pancar, A. (2011). On strongly ⊕-supplemented modules. *Ukrainian Math.J.*, 63(5): 662-667.

Sharpe, D.W., & Vamos, P. (1972). *Injective Modules.* Cambridge University Press, Cambridge.

Şahin, Y., & Nişancı Türkmen, B. (2020). Locally artinian supplemented modules. *9th International Eurasian Conference On Mathematical Sciences and Applications Abstract Book*, 26.

Wisbauer, R. (1991). *Foundations of Modules and Rings*. Gordon and Breach, Springer-Verlag.

Zhou, D.X, & Zhang, X.R. (2011). Small-essential submodules and morita duality. *Southeast Asian Bulletin of Mathematics*, 35: 1051-1062.

Zöschinger, H. (1974). Komplemente als direkte Summanden, *Arch. Math.*, 25:241-243.