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Solution of Fractional Kinetic Equations Involving generalized $q-$Bessel function

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Abstract

In this paper, we pursue and investigate the solutions for fractional kinetic equations, involving $q-$Bessel function by means of their Sumudu transforms. In the process, one Important special case is then revealed. The results obtained in terms of $q-$Bessel function are rather general in nature and can easily construct various known and new fractional kinetic equations.

Keywords: Fractional kinetic equation Generalized Mittag-Leffler function $q-$Bessel function Sumudu Transform.
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1. Introduction

Fractional calculus (FC) is a useful mathematical method for studying fractional-order integrals and derivatives. Fractional calculus has developed and is now used in a variety of engineering and analysis fields. The theory of fractional differential equations and its applications has played a vital role in a variety of fields, including material science, applied research, chemistry, mathematical physics, and architecture. The theory and implementations of fractional differential equations have played a crucial role. The complicated conditions program is based on differential equations and depicts the amount of chemical composition modification a star undergoes as a result of each configuration in terms of generation and annihilation reaction levels.[16, 17, 18, 19, 25, 26] is a good example.
Because of their relevance in astronomy and scientific material science, there has recently been a surge in interest in learning about the solution of fractional kinetic equations. The fractional-order kinetic equations have been successfully used to determine various physical issues such as diffusion in permeable mediums and

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response and unwinding forms in complicated frameworks. As a result, there has been a considerable amount of study into the application of these equations. Look into it [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 24].
In 2000, Haubold and Mathai [13] created the fractional differential equation between the rate of change of the reaction, the production rate, and the destruction rates given as follows:

$$\frac{dN}{dt} = -\mathfrak{d}(N_t) + \mathfrak{p}(N_t),$$

where $\mathfrak{d} = \mathfrak{d}(N)$ the rate of destruction, $N = N(t)$, the rate of reaction, $\mathfrak{p} = \mathfrak{p}(N)$ the rate of production and N_t is the function identified by

$$N_t(t_1^*) = N(t - t_1^*), t_1^* > 0,$$

ignoring the inhomogeneity in the quantity $N(t)$ that is the equation

$$\frac{dN_i}{dt} = -c_i N_i(t), \tag{1}$$

is a part of the initial condition $N_i(t = 0) = N_0$ is the density number of the index $(\imath i \jmath)$ at time $(t = 0)$.

The solution of equation (1) can be referred to

$$N_i(t) = N_0 \ e^{-c_i t}.$$

Another alternative solution , we can take

$$N(t) - N_0 = -c_0 \ D_t^{-1} N(t), \tag{2}$$

where the ${}_0D_t^{-1}$ is the standard integral fractional operator. Furthermore, the fractional generalization defined by Haubold and Mathai[13] as the form for the standard kinetic equation (2)

$$N(t) - N_0 = -c^{\gamma} \ {}_0D_t^{-\gamma} \ N(t), \tag{3}$$

where ${}_0D_t^{-\gamma}$ is the Riemann-Liouville fractional integral operator expressed as

$${}_0D_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma - 1} f(\tau) d\tau, \qquad (t > 0, \Re(\gamma) > 0).$$

the equation solution (3) has been provided by Haubold and Mathai [13] in the form:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\gamma k + 1)} (ct)^{\gamma k}.$$

Besides that, Saxena and Kalla[23] stated the following fractional kinetic equation as the following form:

$$N(t) - N_0 f(t) = -c^{\gamma} \ ({}_0D_t^{-\gamma} \ N)(t), \ \ \Re(\gamma) > 0$$

where $N(t)$ refers to a species' density number at each time t, $N_0 = N(0)$ is a number density which species at a time t = 0, c is a constant and $f \in L(0, \infty)$.

The Sumudu Transform, defined by Watugala [27] over the set $'A'$ of functions as

$$G(\tau) = S[f(t);\tau] = \int_0^\infty e^{-t} f(\tau t)dt \quad ; \quad \tau \in (-\eta_1, \eta_2) \tag{4}$$

where $\quad A = f(t)\big|\exists \mathfrak{M}, \eta_1, \eta_2 > 0, \big|f(t)\big| < \mathfrak{M} e^{\frac{|t|}{\tau_j}}, t \in (-1)^j \times [0, \infty).$

In the proposed work, we find the results in terms of Mittag-Leffler function [21] defined as:

$$E_\xi(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\xi r + 1)} \qquad (\xi, z \in \mathbb{C}; |z| < 0, \Re(\xi) > 0).$$

2. Generalized $q-$Bessel function

Bessel functions have significant role with a wide range of issues in significant fields of mathematical physics, such as hydrodynamics, radiophysics, acoustics, and atomic and nuclear physics, and they play an essential role in analysing solutions of differential equations. They looked at various possible expansions of Bessel functions, as well as many other aspects of Bessel functions.
The $q-$analogues of Bessel functions given by Jackson [15] are as follows:

$$J_\eta^{(1)}(z;q) = \frac{(q^{\eta+1};q)_\infty}{(q;q)_\infty} (z/2)^\eta \; {}_2\phi_1\left(\begin{matrix} 0, 0 \\ q^{\eta+1} \end{matrix}; q, -\frac{z^2}{4}\right), \qquad |z| < 2$$

$$J_\eta^{(2)}(z;q) = \frac{(q^{\eta+1};q)_\infty}{(q;q)_\infty} (z/2)^\eta \; {}_0\phi_1\left(\begin{matrix} - \\ q^{\eta+1} \end{matrix}; q, -\frac{q^{\eta+1} z^2}{4}\right),$$

$$J_\eta^{(3)}(z;q) = \frac{(q^{\eta+1};q)_\infty}{(q;q)_\infty} (z/2)^\eta \; {}_1\phi_1\left(\begin{matrix} 0 \\ q^{\eta+1} \end{matrix}; q, -\frac{q z^2}{4}\right).$$

Mourad Ismail [14] proposed the following modified q$-$Bessel functions:

$$I_\eta^{(1)}(z;q) = \frac{(q^{\eta+1};q)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty} \frac{(z/2)^{\eta+2r}}{(q, q^{\eta+1}, q)_r}, \qquad |z| < 2$$

$$I_\eta^{(2)}(z;q) = \frac{(q^{\eta+1};q)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty} \frac{q^{r(r+\eta)}}{(q, q^{\eta+1}, q)_r} (z/2)^{\eta+2r},$$

$$I_\eta^{(3)}(z;q) = \frac{(q^{\eta+1};q)_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty} \frac{q^{\binom{r+1}{2}}}{(q, q^{\eta+1}, q)_r} (z/2)^{\eta+2r}.$$

It is obvious that

$$I_\eta^{(j)}(z;q) = e^{\frac{-i\pi\eta}{2}} J_\eta^{(j)}(e^{\frac{i\pi}{2}} z, q), \qquad j \in \{1, 2, 3\}.$$

The following q$-$Bessel function was introduced and investigated by Mansour and Al-Shomrani [22]:

$$I_\eta^{(4)}(z;q) = \frac{(q^{\eta+1};q)_\infty}{(q;q)_\infty} (z/2)^\eta \; {}_0\phi_2\left(\begin{matrix} - \\ q^{\eta+1} \end{matrix}; q, -\frac{q^{\frac{3(\eta+1)}{2}} z^2}{4}\right),$$

which is a q-analogy of the modified Bessel function.
Recently, Mahmoud [20] has defined the generalized $q-$Bessel function for $\xi \in Z$, $|z| < 2$ and $\xi = 0$ as follows.

$$J_\eta(z, \xi; q) = \frac{(z/2)^\eta}{(q;q)_\eta} \sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r} (z^2/4)^r, \tag{5}$$

where

$$(q;q)_r = \Pi_{i=1}^{r-1}(1-q^{r+1}).$$

By substituting $\xi = 0, 2, 1, (2.9)$ reduced to $q-$Bessel functions of the first, second and third kind respectively. $J_\eta(z,\xi;q)$ is a $q-$Bessel function $J_\eta(z)$ and modified Bessel function $I_\eta(z)$.

$$\lim_{q\to 1} J_\eta((1-q)z,\xi;q) = J_\eta(z), \xi = 0, 2, 4, ... \tag{6}$$

$$\lim_{q\to 1} J_\eta((1-q)z,\xi;q) = I_\eta(z), \xi = 1, 3, 5, ... \tag{7}$$

3. Solution of fractional Kinetic Equations including the generalized $q-$Bessel function

In this section, we solve the fractional kinetic equation associated with the generalized $q-$Bessel function using the method of Sumudu transform.

Theorem 3.1. *Let $\gamma > 0, d > 0, t \in C, \xi \in Z^+$ then the following equation:*

$$N(t) - N_0 J_\eta(t,\xi;q) = -d^\gamma \ {}_0D_t^{-\gamma} N(t), \tag{8}$$

has a solution given by

$$N(t) = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{t}{2}\right)^{2r+\eta} \times \frac{1}{t}$$

$$\times \Gamma(2r+\eta+1) \ E_{\gamma,2r+\eta} \ (-d^\gamma t^\gamma).$$

Proof. Sumudu transform of Riemann-Liouville fractional integral operator can be presented as

$$\mathcal{S} \ \{{}_0D_t^\gamma f(t);\tau\} = (\tau)^\gamma G(\tau), \tag{9}$$

where $G(\tau)$ is define in (4)
Now, after we apply the Sumudu transform to both sides of equation (8) and using (9) we have

$$\mathcal{S}\Big(N(t);\tau\Big) = N_0 \ \ \mathcal{S}\Big[J_\eta(t,\xi;q)\Big] - d^\gamma \ \ \mathcal{S}\Big({}_0D_t^{-\gamma} N(t);\tau\Big)$$

that is

$$N(\tau) = N_0 \int_0^\infty e^{-t} \sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta} (\tau t/2)^{2r+\eta} dt - d^\gamma (\tau)^\gamma N(\tau), \tag{10}$$

through we interchange the integration and summation order in the equation (10), we obtain

$$N(\tau)\big[1 + d^\gamma (\tau)^\gamma\big] = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right) \int_0^\infty e^{-t}\left(\frac{\tau t}{2}\right)^{2r+\eta} dt,$$

$$= N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{\tau}{2}\right)^{2r+\eta} \int_0^\infty e^{-t} (t)^{2r+\eta} dt,$$

that is

$$N(\tau)\big[1 + d^\gamma (\tau)^\gamma\big] = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{\tau}{2}\right)^{2r+\eta} \ \Gamma(2r+\eta+1), \tag{11}$$

equation (11) leads to

$$N(\tau) = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{\tau}{2}\right)^{2r+\eta} \ \Gamma(2r+\eta+1)\sum_{s=0}^{\infty}(-1)^s (d\tau)^{\gamma s}. \tag{12}$$

Now, we take inverse the Sumudu transform on both sides of the equation (12), and using

$$\mathcal{S}^{-1}\{\tau^{\gamma};t\} = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \quad (\mathcal{R}(\gamma) > 0), \tag{13}$$

we have

$$\mathcal{S}^{-1}N(\tau) = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{1}{2}\right)^{2r+\eta}$$

$$\times\Gamma(2r+\eta+1) \ \ \mathcal{S}^{-1}\left(\sum_{s=0}^{\infty}(-1)^s (d)^{\gamma s}(\tau)^{2r+\eta+\gamma s}\right).$$

that is

$$N(t) = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{1}{2}\right)^{2r+\eta}$$

$$\times\Gamma(2r+\eta+1) \ \ \left(\sum_{s=0}^{\infty}(-1)^s (d)^{\gamma s}\frac{(t)^{2r+\eta+\gamma s-1}}{\Gamma(2r+\eta+\gamma s)}\right),$$

$$N(t) = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{t}{2}\right)^{2r+\eta} \quad \times\frac{1}{t}$$

$$\times\Gamma(2r+\eta+1) \ \ \left(\sum_{s=0}^{\infty}(-1)^s \frac{(t^{\gamma}d^{\gamma})^s}{\Gamma(2r+\eta+\gamma s)}\right). \tag{14}$$

Now, we can write eq (14) as

$$N(t) = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta}\right)\left(\frac{t}{2}\right)^{2r+\eta} \quad \times\frac{1}{t}$$

$$\times\Gamma(2r+\eta+1) \ E_{\gamma,2r+\eta} \ (-d^{\gamma}t^{\gamma}).$$

□

Corollary 3.2. *let $\gamma > 0, d > 0, t \in C, \xi = 0, 2, 4, ...$ then the following equation:*

$$N(t) - N_0 \lim_{q\to 1} J_\eta[(1-q)t, \xi; q] = -d^{\gamma} \ {}_0D_t^{-\gamma}N(t).$$

OR

$$N(t) - N_0 J_\eta(t) = -d^{\gamma} \ {}_0D_t^{-\gamma}N(t),$$

(where $J_\eta(t)$ is defined by (6) has a solution given by

$$N(t) = N_0\left(\sum_{r=0}^{\infty} \frac{(-1)^r\Gamma(2r+\eta+1)}{\Gamma(r+1)\Gamma(\eta+r+1)}\right)\left(\frac{t}{2}\right)^{2r+\eta} \quad \times\frac{1}{t}$$

$$\times \ E_{\gamma,2r+\eta} \ (-d^{\gamma}t^{\gamma}).$$

Corollary 3.3. *let $\gamma > 0, d > 0, t \in C, \xi = 1, 3, 5, ...$ then the following equation:*

$$N(t) - N_0 \lim_{q\to 1} J_\eta[(1-q)t, \xi; q] = -d^\gamma \; {}_0D_t^{-\gamma} N(t).$$

OR

$$N(t) - N_0 I_\eta(t) = -d^\gamma \; {}_0D_t^{-\gamma} N(t),$$

(where $I_\eta(t)$ is defined by (7) has a solution given by

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{\Gamma(2r+\eta+1)}{\Gamma(r+1)\Gamma(\eta+r+1)} \right) \left(\frac{t}{2}\right)^{2r+\eta} \times \frac{1}{t}$$

$$\times \; E_{\gamma, 2r+\eta} \; (-d^\gamma t^\gamma).$$

Theorem 3.4. *Let $\gamma > 0, \delta > 0, d > 0, \delta \neq d, t \in C, \xi \in Z^+$ then the following equation:*

$$N(t) - N_0 J_\eta(d^\gamma t^\gamma, \xi; q) = -\delta^\gamma \; {}_0D_t^{-\gamma} N(t), \tag{15}$$

has a solution given by

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta} \right) \left(\frac{d^\gamma t^\gamma}{2}\right)^{2r+\eta} \times \frac{1}{t}$$

$$\times \Gamma(2r\gamma + \eta\gamma + 1) \; E_{\gamma, 2r\gamma+\eta\gamma} \; (-\delta^\gamma t^\gamma).$$

Proof. Sumudu transform of Riemann-Liouville fractional integral operator can be presented as (9) where $G(\tau)$ is define in (4)
Now, after we apply the Sumudu transform to both sides of equation (15) and using (9) we have

$$\mathcal{S}\Big(N(t);\tau\Big) = N_0 \;\; \mathcal{S}\Big[J_\eta(d^\gamma t^\gamma, \xi; q)\Big] - \delta^\gamma \;\; \mathcal{S}\Big({}_0D_t^{-\gamma} N(t); \tau\Big).$$

that is

$$N(\tau) = N_0 \int_0^\infty e^{-t} \sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta} (d^\gamma (\tau t)^\gamma/2)^{2r+\eta} dt - \delta^\gamma (\tau)^\gamma N(\tau), \tag{16}$$

through we interchange the integration and summation order in the equation (16), we obtain

$$N(\tau)\big[1 + \delta^\gamma(\tau)^\gamma\big] = N_0 \left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta} \right) \int_0^\infty e^{-t} \left(\frac{d^\gamma (\tau t)^\gamma}{2}\right)^{2r+\eta} dt,$$

$$= N_0 \left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta} \right) \left(\frac{d^\gamma \tau^\gamma}{2}\right)^{2r+\eta} \int_0^\infty e^{-t} (t)^{2r\gamma+\eta\gamma} dt,$$

$$N(\tau)\big[1 + \delta^\gamma(\tau)^\gamma\big] = N_0 \left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta} \right) \left(\frac{d^\gamma \tau^\gamma}{2}\right)^{2r+\eta} \; \Gamma(2r\gamma + \eta\gamma + 1), \tag{17}$$

equation (17) leads to

$$N(\tau) = N_0 \left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r (q;q)_r (q;q)_\eta} \right) \left(\frac{d^\gamma \tau^\gamma}{2}\right)^{2r+\eta} \; \Gamma(2r\gamma + \eta\gamma + 1) \sum_{s=0}^{\infty} (-1)^s (\delta\tau)^{\gamma s}. \tag{18}$$

Now, we take inverse the Sumudu transform on both sides of the equation (18), and using (13) we have

$$\mathcal{S}^{-1}N(\tau) = N_0\left(\sum_{r=0}^{\infty}\frac{(-1)^{r(\xi+1)}q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r(q;q)_r(q;q)_\eta}\right)\left(\frac{d^\gamma}{2}\right)^{2r+\eta}$$

$$\times\Gamma(2r\gamma+\eta\gamma+1)\ \ \mathcal{S}^{-1}\left(\sum_{s=0}^{\infty}(-1)^s(\delta)^{\gamma s}(\tau)^{2r\gamma+\eta\gamma+\gamma s}\right).$$

that is

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(-1)^{r(\xi+1)}q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r(q;q)_r(q;q)_\eta}\right)\left(\frac{d^\gamma}{2}\right)^{2r+\eta}$$

$$\times\Gamma(2r\gamma+\eta\gamma+1)\ \ \left(\sum_{s=0}^{\infty}(-1)^s(\delta)^{\gamma s}\frac{(t)^{2r\gamma+\eta\gamma+\gamma s-1}}{\Gamma(2r\gamma+\eta\gamma+\gamma s)}\right),$$

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(-1)^{r(\xi+1)}q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r(q;q)_r(q;q)_\eta}\right)\left(\frac{d^\gamma t^\gamma}{2}\right)^{2r+\eta}\quad\times\frac{1}{t}$$

$$\times\Gamma(2r\gamma+\eta\gamma+1)\ \ \left(\sum_{s=0}^{\infty}(-1)^s\frac{(t^\gamma\delta^\gamma)^s}{\Gamma(2r\gamma+\eta\gamma+\gamma s)}\right). \tag{19}$$

Now, we can write eq (19) as

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(-1)^{r(\xi+1)}q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1};q)_r(q;q)_r(q;q)_\eta}\right)\left(\frac{d^\gamma t^\gamma}{2}\right)^{2r+\eta}\quad\times\frac{1}{t}$$

$$\times\Gamma(2r\gamma+\eta\gamma+1)\ E_{\gamma,2r\gamma+\eta\gamma}\ (-\delta^\gamma t^\gamma).$$

□

Corollary 3.5. *let $\gamma>0,\delta>0,d>0,\delta\neq d,t\in C,\xi=0,2,4,...$ then the following equation:*

$$N(t)-N_0\lim_{q\to 1}J_\eta[(1-q)d^\gamma t^\gamma,\xi;q] = -\delta^\gamma\ {}_0D_t^{-\gamma}N(t).$$

OR

$$N(t)-N_0J_\eta(d^\gamma t^\gamma) = -\delta^\gamma\ {}_0D_t^{-\gamma}N(t),$$

has a solution given by

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(2r\gamma+\eta\gamma+1)}{\Gamma(r+1)\Gamma(\eta+r+1)}\right)\left(\frac{d^\gamma t^\gamma}{2}\right)^{2r+\eta}\quad\times\frac{1}{t}$$

$$\times\ E_{\gamma,2r\gamma+\eta\gamma}\ (-\delta^\gamma t^\gamma).$$

Corollary 3.6. *let $\gamma>0,\delta>0,d>0,\delta\neq d,t\in C,\xi=1,3,5,...$ then the following equation:*

$$N(t)-N_0\lim_{q\to 1}J_\eta[(1-q)d^\gamma t^\gamma,\xi;q] = -\delta^\gamma\ {}_0D_t^{-\gamma}N(t).$$

OR

$$N(t) - N_0 I_\eta(d^\gamma t^\gamma) = -\delta^\gamma \ {}_0D_t^{-\gamma} N(t),$$

has a solution given by

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{\Gamma(2r\gamma + \eta\gamma + 1)}{\Gamma(r+1)\Gamma(\eta + r + 1)} \right) \left(\frac{d^\gamma t^\gamma}{2} \right)^{2r+\eta} \quad \times \frac{1}{t}$$

$$\times \ E_{\gamma, 2r\gamma+\eta\gamma} \ (-\delta^\gamma t^\gamma).$$

Theorem 3.7. *Let $\gamma > 0, d > 0, t \in C, \xi \in Z^+$ then the following equation:*

$$N(t) - N_0 J_\eta(d^\gamma t^\gamma, \xi; q) = -d^\gamma \ {}_0D_t^{-\gamma} N(t),$$

has a solution given by

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{(\xi r(r+\eta)/2)}}{(q^{\eta+1}; q)_r (q; q)_r (q; q)_\eta} \right) \left(\frac{d^\gamma t^\gamma}{2} \right)^{2r+\eta} \quad \times \frac{1}{t}$$

$$\times \Gamma(2r\gamma + \eta\gamma + 1) \ E_{\gamma, 2r\gamma+\eta\gamma} \ (-d^\gamma t^\gamma).$$

Proof. Proof of Theorem 3.7 is similar to the proof of Theorems 3.1 and 3.4 so it is omitted here. □

Remark 3.8. *Similarly, one can develop the corollaries for the Theorem 3.7.*

4. Conclusion

In this paper we have studied a new fractional generalization of the standard kinetic equation and derived solutions for it. It is not difficult to obtain several further analogous fractional kinetic equations and their solutions as those exhibited here by main results.

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