Turk. J. Math. Comput. Sci.
15(2)(2023) 218–226
© MatDer
DOI : 10.47000/tjmcs.1009950

On Double Laplace-Shehu Transform and its Properties with Applications

Mona Hunaiber[1], Ali Al-aati[1,*]

[1]*Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Albaydha, Yemen.*

Received: 17-10-2021 • Accepted: 13-12-2022

Abstract. In this paper, we introduce a new method for solving some partial differential equations called double Laplace-Shehu transform, some useful properties for the transform are presented. In addition, we use this transform to solve the Laplace, Poisson, Wave and Heat equations and find Laplace-Shehu transform for some functions.

2010 AMS Classification: 44A10, 44A15, 44A20, 44A30

Keywords: Double Laplace-Shehu transform, Laplace transform, Shehu transform, partial differential equations.

1. Introduction

In the literature, there are many different types of integral transforms such as Fourier transform, Laplace transform, Sumudu transform, Ezaki transform, Shehu transform, and so on. These kinds of integral transforms have many applications in various fields of mathematical sciences and engineering such as physics, mechanics, chemistry, acoustic, etc, [10]. They play an important role in solving integral equations or partial differential equations describing physical phenomena [3, 4, 11]. Solving such equations using single transforms is more difficult than using double transforms [3, 4]. In recent years, great attention has been given to deal with the double integral transforms, see for example [2, 5, 7, 8]. Eltayeb and Kilicman [9], applied double Laplace transform (DLT) to solve wave, Laplace's and heat equations with convolution terms, general linear telegraph and partial integro-differential equations. Alfaqeih and Misirli in [5] dealt with double Shehu transform to get the solution of initial and boundary value problems in different areas of real life science and engineering.

Analogous to [2], we applied new double Laplace-Shehu transform to solve Laplace, Poisson, Wave and Heat equations, through the derivation of general formula for solutions of these equations, or by applying the double Laplace-Shehu transform directly to the given equation.

The main objective of this paper is to introduce a new method for solving some partial differential equations subject to the initial and boundary conditions called double Laplace-Shehu transform, the definition of double Laplace-Shehu transform and its inverse. We also discussed some theorems and properties about the double Laplace-Shehu transform and gave the double Laplace-Shehu transform of some elementary functions. Moreover, we implement the double Laplace-Shehu transform method to some equations.

*Corresponding Author
Email addresses: hunaiber2021@gmail.com (M. Hunaiber), alati2005@yahoo.com (A. Al-aati)

Definition 1.1 ([1]). The Laplace transform of the continuous function $\psi(\xi)$ is defined by

$$L[\psi(\xi)] = \int_0^{\infty} e^{-\gamma\xi}\psi(\xi)d\xi = \Psi(\gamma),$$

where the operator L is the Laplace operator. The inverse Laplace transform is defined by

$$L^{-1}[\Psi(\gamma)] = \psi(\xi) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\gamma\xi}d\gamma,$$

where κ is a real constant.

Definition 1.2 ([6]). The Shehu transform of the real function $\psi(\eta)$ of exponential order is defined over the set of functions,

$$\mathcal{M} = \Big\{\psi(\eta) : \exists N, \tau_1, \tau_2 > 0, |\psi(\eta)| < Ke^{\frac{|\eta|}{\tau_i}}, \quad for \quad \eta \in (-1)^i \times [0,\infty), \quad i = 1,2\Big\},$$

by the following integral

$$S[\psi(\eta)] = \int_0^{\infty} e^{-\frac{\delta}{\mu}\eta}\psi(\eta)d\eta = \Psi(\delta,\mu), \quad \delta,\mu > 0,$$

where $e^{-\frac{\delta}{\mu}}$ is the kernel function, and S is the operator of Shehu transform. The inverse Shehu transform is defined by

$$S^{-1}[\Psi(\delta,\mu)] = \psi(\eta) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{\mu}e^{\frac{\delta}{\mu}\eta}\Psi(\delta,\mu)d\delta,$$

where ω is a real constant.

In the next definition, we introduce the double Laplace-Shehu transform.

Definition 1.3. The double Laplace-Shehu transform of the function $\psi(\xi,\eta)$ of two variables $\xi > 0$ and $\eta > 0$ is denoted by $L_\xi S_\eta[\psi(\xi,\eta)] = \Psi(\gamma,\delta,\mu)$ and defined as

$$\begin{aligned} L_\xi S_\eta[\psi(\xi,\eta)] = \Psi(\gamma,\delta,\mu) &= \int_0^{\infty}\int_0^{\infty} e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\psi(\xi,\eta)d\xi d\eta \\ &= \lim_{\alpha\to\infty,\beta\to\infty}\int_0^{\alpha}\int_0^{\beta} e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\psi(\xi,\eta)d\xi d\eta. \end{aligned}$$

It converges if the limit of the integral exists, and diverges if not.

Definition 1.4. The inverse double Laplace-Shehu transform of a function $\Psi(\gamma,\delta,\mu)$ is given by

$$L_\xi^{-1}S_\eta^{-1}[\Psi(\gamma,\delta,\mu)] = \psi(\xi,\eta).$$

Equivalently,

$$\psi(\xi,\eta) = L_\xi^{-1}S_\eta^{-1}[\Psi(\gamma,\delta,\mu)] = \frac{1}{(2\pi i)^2}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\gamma\xi}d\gamma\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}\eta}\Psi(\gamma,\delta,\mu)d\delta,$$

where κ and ω are real constants.

2. Existence and Uniqueness of the Double Laplace-Shehu Transform

Definition 2.1 ([7]). A function $\psi(\xi,\eta)$ is said to be of exponential orders $\alpha > 0, \beta > 0$, on $0 \le \xi < \infty$, $0 \le \eta < \infty$, if there exists positive constants K, X and Y such that

$$|\psi(\xi,\eta)| \le Ke^{\alpha\xi+\beta\eta}, \qquad for \qquad all \quad \xi > X, \quad \eta > Y,$$

and we write

$$\psi(\xi,\eta) = o(e^{\alpha\xi+\beta\eta}) \qquad as \quad \xi\to\infty, \quad \eta\to\infty.$$

Or, equivalently,

$$\lim_{\xi\to\infty,\eta\to\infty} e^{-(\alpha\xi+\beta\eta)}|\psi(\xi,\eta)| = K\lim_{\xi\to\infty,\eta\to\infty} e^{-(\gamma-\alpha)\xi}e^{-(\frac{\delta}{\mu}-\beta)\eta} = 0, \qquad \gamma > \alpha, \qquad \frac{\delta}{\mu} > \beta.$$

Theorem 2.2 ([6]). *Let $\psi(\xi,\eta)$ be a continuous function in every finite intervals $(0,X)$ and $(0,Y)$ and of exponential order $exp(\alpha\xi+\beta\eta)$, then the double Laplace-Shehu transform of $\psi(\xi,\eta)$ exists for all $\gamma>\alpha$ and $\frac{\delta}{\mu}>\beta$.*

Proof. Let $\psi(\xi,\eta)$ be of exponential order $exp(\alpha\xi+\beta\eta)$ such that

$$\Big|\psi(\xi,\eta)\Big|\leq Ke^{(\alpha\xi+\beta\eta)}, \qquad \forall \qquad \xi>X, \qquad \eta>Y.$$

Then, we have

$$\begin{aligned}\Big|\Psi(\gamma,\delta,\mu)\Big| &= \Big|\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\psi(\xi,\eta)d\xi d\eta\Big| \\ &\leqq \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}|\psi(\xi,\eta)|d\xi d\eta \\ &\leqq K\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}e^{(\alpha\xi+\beta\eta)}d\xi d\eta \\ &= K\int_0^\infty e^{-(\gamma-\alpha)\xi}d\xi\int_0^\infty e^{-(\frac{\delta}{\mu}-\beta)\eta}d\eta \\ &= \frac{K\mu}{(\gamma-\alpha)(\delta-\beta\mu)}.\end{aligned}$$

Thus, the proof is complete. □

Theorem 2.3. *Let $\Psi_1(\gamma,\delta,\mu)$ and $\Psi_2(\gamma,\delta,\mu)$ be the double Laplace-Shehu transform of the continuous functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ defined for $\xi,\eta\geq 0$ respectively. If $\Psi_1(\gamma,\delta,\mu)=\Psi_2(\gamma,\delta,\mu)$, then $\psi_1(\xi,\eta)=\psi_2(\xi,\eta)$.*

Proof. Assume that κ and ω are sufficiently large, since

$$\psi(\xi,\eta)=L_\xi^{-1}S_\eta^{-1}[\Psi(\gamma,\delta,\mu)]=\Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}e^{\gamma\xi}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}\eta}\Psi(\gamma,\delta,\mu)d\delta\Big),$$

we deduce that

$$\begin{aligned}\psi_1(\xi,\eta) &= \Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}e^{\gamma\xi}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}\eta}\Psi_1(\gamma,\delta,\mu)d\delta\Big) \\ &= \Big(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}e^{\gamma\xi}d\gamma\Big)\Big(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{1}{\mu}e^{\frac{\delta}{\mu}\eta}\Psi_2(\gamma,\delta,\mu)d\delta\Big) \\ &= \psi_2(\xi,\eta).\end{aligned}$$

This ends the proof of the theorem. □

3. Some Useful Properties of the Double Laplace-Shehu Transform

3.1. **Linearity Property.** If the double Laplace-Shehu transform of functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ are $\Psi_1(\gamma,\delta,\mu)$ and $\Psi_2(\gamma,\delta,\mu)$ respectively, then the double Laplace-Shehu transform of $\alpha\psi_1(\xi,\eta)+\beta\psi_2(\xi,\eta)$ is given by $\alpha\Psi_1(\gamma,\delta,\mu)+\beta\Psi_2(\gamma,\delta,\mu)$, where α and β are arbitrary constants.

Proof.

$$\begin{aligned}L_\xi S_\eta[\alpha\psi_1(\xi,\eta)+\beta\psi_2(\xi,\eta)] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\Big(\alpha\psi_1(\xi,\eta)+\beta\psi_2(\xi,\eta)\Big)d\xi d\eta \\ &= \alpha\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\psi_1(\xi,\eta)d\xi d\eta \\ &+ \beta\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\psi_2(\xi,\eta)d\xi d\eta \\ &= \alpha\Psi_1(\gamma,\delta,\mu)+\beta\Psi_2(\gamma,\delta,\mu).\end{aligned}$$

□

3.2. **Shifting Property.** If the double Laplace-Shehu transform of $\psi(\xi,\eta)$ is $\Psi(\gamma,\delta,\mu)$, then double Laplace-Shehu transform of the function $e^{(\alpha\xi+\beta\eta)}\psi(\xi,\eta)$ is given by
$\Psi(\gamma-\alpha,\delta-\beta\mu,\mu)$.

Proof.

$$\begin{aligned} L_\xi S_\eta[e^{(\alpha\xi+\beta\eta)}\psi(\xi,\eta)] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}e^{(\alpha\xi+\beta\eta)}\psi(\xi,\eta)d\xi d\eta \\ &= \int_0^\infty\int_0^\infty e^{-\left((\gamma-\alpha)\xi+(\frac{\delta-\beta\mu}{\mu})\eta\right)}\psi(\xi,\eta)d\xi d\eta \\ &= \Psi(\gamma-\alpha,\delta-\beta\mu,\mu). \end{aligned}$$

□

3.3. **Change of Scale Property.** If the double Laplace-Shehu transform of the function $\psi(\xi,\eta)$ is $\Psi(\gamma,\delta,\mu)$ then the double Laplace-Shehu transform of $\psi(\alpha\xi,\beta\eta)$ is given by $\frac{1}{\alpha\beta}\Psi(\frac{\gamma}{\alpha},\frac{\delta}{\beta},\mu)$.

Proof.

$$L_\xi S_\eta[\psi(\alpha\xi,\beta\eta)] = \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\psi(\alpha\xi,\beta\eta)d\xi d\eta.$$

Let $\upsilon=\alpha\xi,\ \ \tau=\beta\eta$, then

$$\begin{aligned} L_\xi S_\eta[\psi(\alpha\xi,\beta\eta)] &= \frac{1}{\alpha\beta}\int_0^\infty\int_0^\infty e^{-\frac{\gamma}{\alpha}\upsilon}e^{-\frac{\delta}{\beta\mu}\tau}\psi(\upsilon,\tau)d\upsilon d\tau \\ &= \frac{1}{\alpha\beta}\Psi(\frac{\gamma}{\alpha},\frac{\delta}{\beta},\mu). \end{aligned}$$

□

3.4. **Derivatives Properties.** If $L_\xi S_\eta[\psi(\xi,\eta)]=\Psi(\gamma,\delta,\mu)$, then:

$$(1).\qquad L_\xi S_\eta\Big[\frac{\partial\psi(\xi,\eta)}{\partial\xi}\Big]=\gamma\Psi(\gamma,\delta,\mu)-S[\psi(0,\eta)].$$

Proof.

$$\begin{aligned} L_\xi S_\eta\Big[\frac{\partial\psi(\xi,\eta)}{\partial\xi}\Big] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\frac{\partial\psi(\xi,\eta)}{\partial\xi}d\xi d\eta \\ &= \int_0^\infty e^{-\frac{\delta}{\mu}\eta}d\eta\Big(\int_0^\infty e^{-\gamma\xi}\psi_\xi(\xi,\eta)d\xi\Big). \end{aligned}$$

Using integration by parts, let $u=e^{-\gamma\xi},\ \ dv=\psi_\xi(\xi,\eta)d\xi$, then we obtain

$$\begin{aligned} L_\xi S_\eta\Big[\frac{\partial\psi(\xi,\eta)}{\partial\xi}\Big] &= \int_0^\infty e^{-\frac{\delta}{\mu}\eta}d\eta\Big(-\psi(0,\eta)+\gamma\int_0^\infty e^{-\gamma\xi}\psi(\xi,\eta)d\xi\Big) \\ &= \gamma\Psi(\gamma,\delta,\mu)-S[\psi(0,\eta)]. \end{aligned}$$

□

$$(2).\qquad L_\xi S_\eta\Big[\frac{\partial\psi(\xi,\eta)}{\partial\eta}\Big]=\frac{\delta}{\mu}\Psi(\gamma,\delta,\mu)-L[\psi(\xi,0)].$$

Proof.

$$\begin{aligned} L_\xi S_\eta\Big[\frac{\partial\psi(\xi,\eta)}{\partial\eta}\Big] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\frac{\partial\psi(\xi,\eta)}{\partial\eta}d\xi d\eta \\ &= \int_0^\infty e^{-\gamma\xi}d\xi\Big(\int_0^\infty e^{-\frac{\delta}{\mu}\eta}\psi_\eta(\xi,\eta)d\eta\Big). \end{aligned}$$

Using integration by parts, let $u = e^{-\frac{\delta}{\mu}\eta}$, $dv = \psi_\eta(\xi,\eta)d\eta$, then we obtain

$$
\begin{aligned}
L_\xi S_\eta\Big[\frac{\partial\psi(\xi,\eta)}{\partial\eta}\Big] &= \int_0^\infty e^{-\gamma\xi}d\xi\Big(-\psi(\xi,0) + \frac{\delta}{\mu}\int_0^\infty e^{-\frac{\delta}{\mu}\eta}\psi(\xi,\eta)d\eta\Big) \\
&= \frac{\delta}{\mu}\Psi(\gamma,\delta,\mu) - L[\psi(\xi,0)].
\end{aligned}
$$

□

(3). $$L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\xi^2}\Big] = \gamma^2\Psi(\gamma,\delta,\mu) - \gamma S[\psi(0,\eta)] - S[\psi_\xi(0,\eta)].$$

Proof.

$$
\begin{aligned}
L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\xi^2}\Big] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\frac{\partial^2\psi(\xi,\eta)}{\partial\xi^2}d\xi d\eta \\
&= \int_0^\infty e^{-\frac{\delta}{\mu}\eta}d\eta\Big(\int_0^\infty e^{-\gamma\xi}\psi_{\xi\xi}(\xi,\eta)d\xi\Big).
\end{aligned}
$$

Integration by parts twice, we obtain

$$
\begin{aligned}
L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\xi^2}\Big] &= \int_0^\infty e^{-\frac{\delta}{\mu}\eta}d\eta\Big(-\psi_\xi(0,\eta) - \gamma\psi(0,\eta) + \gamma^2\int_0^\infty e^{-\gamma\xi}\psi(\xi,\eta)d\xi\Big) \\
&= \gamma^2\Psi(\gamma,\delta,\mu) - \gamma S[\psi(0,\eta)] - S[\psi_\xi(0,\eta)].
\end{aligned}
$$

□

(4). $$L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\eta^2}\Big] = \frac{\delta^2}{\mu^2}\Psi(\gamma,\delta,\mu) - \frac{\delta}{\mu}L[\psi(\xi,0)] - L[\psi_\eta(\xi,0)].$$

Proof.

$$
\begin{aligned}
L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\eta^2}\Big] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\frac{\partial^2\psi(\xi,\eta)}{\partial\eta^2}d\xi d\eta \\
&= \int_0^\infty e^{-\gamma\xi}d\xi\Big(\int_0^\infty e^{-\frac{\delta}{\mu}\eta}\psi_{\eta\eta}(\xi,\eta)d\eta\Big).
\end{aligned}
$$

Integration by parts twice, we obtain

$$
\begin{aligned}
L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\eta^2}\Big] &= \int_0^\infty e^{-\gamma\xi}d\xi\Big(-\psi_\eta(\xi,0) - \frac{\delta}{\mu}\psi(\xi,0) + \frac{\delta^2}{\mu^2}\int_0^\infty e^{-\frac{\delta}{\mu}\eta}\psi(\xi,\eta)d\eta\Big) \\
&= \frac{\delta^2}{\mu^2}\Psi(\gamma,\delta,\mu) - \frac{\delta}{\mu}L[\psi(\xi,0)] - L[\psi_\eta(\xi,0)].
\end{aligned}
$$

□

(5). $$L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\xi\partial\eta}\Big] = \frac{\gamma\delta}{\mu}\Psi(\gamma,\delta,\mu) - \gamma L[\psi(\xi,0)] - S[\psi_\eta(0,\eta)].$$

Proof.

$$
\begin{aligned}
L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\xi\partial\eta}\Big] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\frac{\partial^2\psi(\xi,\eta)}{\partial\xi\partial\eta}d\xi d\eta \\
&= \int_0^\infty e^{-\frac{\delta}{\mu}\eta}d\eta\Big(\int_0^\infty e^{-\gamma\xi}\psi_{\xi\eta}(\xi,\eta)d\xi\Big).
\end{aligned}
$$

Integration by parts twice, we obtain

$$\begin{aligned} L_\xi S_\eta\Big[\frac{\partial^2\psi(\xi,\eta)}{\partial\xi\partial\eta}\Big] &= \int_0^\infty e^{-\frac{\delta}{\mu}\eta}d\eta\Big(-\psi_\eta(0,\eta)+\gamma\int_0^\infty e^{-\gamma\xi}\psi_\eta(\xi,\eta)d\xi\Big)\\ &= -S[\psi_\eta(0,\eta)]+\gamma\int_0^\infty e^{-\gamma\xi}d\xi\int_0^\infty e^{-\frac{\delta}{\mu}\eta}\psi_\eta(\xi,\eta)d\eta\\ &= -S[\psi_\eta(0,\eta)]+\gamma\int_0^\infty e^{-\gamma\xi}d\xi\Big(-\psi(\xi,0)+\frac{\delta}{\mu}\int_0^\infty e^{-\frac{\delta}{\mu}\eta}\psi(\xi,\eta)d\eta\Big)\\ &= \frac{\gamma\delta}{\mu}\Psi(\gamma,\delta,\mu)-\gamma L[\psi(\xi,0)]-S[\psi_\eta(0,\eta)]. \end{aligned}$$

□

4. The Double Laplace-Shehu Transform of Some Elementary Functions

(1). If the function $\psi(\xi,\eta)=1$, then

$$L_\xi S_\eta[\psi(\xi,\eta)]=\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}d\xi d\eta=\frac{\mu}{\gamma\delta}.$$

(2). If the function $\psi(\xi,\eta)=\xi\eta$, then

$$L_\xi S_\eta[\psi(\xi,\eta)]=\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\xi\eta d\xi d\eta=\frac{\mu^2}{\gamma^2\delta^2}.$$

(3). If the function $\psi(\xi,\eta)=\xi^n\eta^m,\quad n,m=0,1,2,\ \dots$, then

$$L_\xi S_\eta[\psi(\xi,\eta)] = \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\xi^n\eta^m d\xi d\eta=\frac{n!m!}{\gamma^{n+1}}\Big(\frac{\mu}{\delta}\Big)^{m+1}.$$

(4). If the function $\psi(\xi,\eta)=\xi^\sigma\eta^\nu,\ \sigma\geq-1,\ \nu\geq-1$, then

$$L_\xi S_\eta[\psi(\xi,\eta)]=\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\xi^\sigma\eta^\nu d\xi d\eta = \int_0^\infty e^{-\gamma\xi}\xi^\sigma d\xi\int_0^\infty e^{-\frac{\delta}{\mu}\eta}\eta^\nu d\eta.$$

Let $x=\gamma\xi$ and $y=\frac{\delta}{\mu}\eta$, then we have

$$\begin{aligned} L_\xi S_\eta[\psi(\xi,\eta)] &= \frac{1}{\gamma^{\sigma+1}}\int_0^\infty e^{-x}x^\sigma dx(\frac{\mu}{\delta})^{\nu+1}\int_0^\infty e^{-y}y^\nu dy\\ &= \Gamma(\sigma+1)\Big(\frac{1}{\gamma^{\sigma+1}}\Big)\Gamma(\nu+1)\Big(\frac{\mu}{\delta}\Big)^{\nu+1}, \end{aligned}$$

where $\Gamma(.)$ is the Euler gamma function.

(5). If the function $\psi(\xi,\eta)=e^{n\xi+m\eta},\quad n,m=0,1,2,\ \dots$, then

$$L_\xi S_\eta[\psi(\xi,\eta)]=\int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}e^{n\xi+m\eta}d\xi d\eta=\frac{\mu}{(\gamma-n)(\delta-m\mu)}.$$

(6). If the function $\psi(\xi,\eta)=\cos(n\xi+m\eta),\quad n,m=0,1,2,\ \dots$, then

$$\begin{aligned} L_\xi S_\eta[\psi(\xi,\eta)] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\cos(n\xi+m\eta)d\xi d\eta\\ &= \frac{\mu(\gamma\delta-nm\mu)}{(\gamma^2+n^2)(\delta^2+m^2\mu^2)}. \end{aligned}$$

(7). If the function $\psi(\xi,\eta)=\sin(n\xi+m\eta),\quad n,m=0,1,2,\ \dots$, then

$$\begin{aligned} L_\xi S_\eta[\psi(\xi,\eta)] &= \int_0^\infty\int_0^\infty e^{-(\gamma\xi+\frac{\delta}{\mu}\eta)}\sin(n\xi+m\eta)d\xi d\eta\\ &= \frac{\mu(n\delta+m\gamma\mu)}{(\gamma^2+n^2)(\delta^2+m^2\mu^2)}. \end{aligned}$$

Consequently,

$$L_\xi S_\eta[\cosh(n\xi + m\eta)] = \frac{\mu(\gamma\delta + nm\mu)}{(\gamma^2 - n^2)(\delta^2 - m^2\mu^2)},$$

$$L_\xi S_\eta[\sinh(n\xi + m\eta)] = \frac{\mu(n\delta + m\gamma\mu)}{(\gamma^2 - n^2)(\delta^2 - m^2\mu^2)}.$$

5. Applications

In this section, we apply the double Laplace-Shehu transform method to linear partial differential equations. Let the second-order nonhomogeneous partial differential equation in two independent variables be in the form:

$$A\psi_{\xi\xi} + B\psi_{\eta\eta} + C\psi_\xi + D\psi_\eta + E\psi = f(\xi,\eta), \qquad (\xi,\eta) \in \mathbb{R}_+^2, \tag{5.1}$$

with the initial conditions:

$$\psi(\xi,0) = \hbar_1(\xi), \qquad \psi_\eta(\xi,0) = \hbar_2(\xi), \tag{5.2}$$

and the boundary conditions:

$$\psi(0,\eta) = \hbar_3(\eta), \qquad \psi_\xi(0,\eta) = \hbar_4(\eta), \tag{5.3}$$

where A, B, C, D and E are constants and $f(\xi,\eta)$ is the source term.
Using the property of partial derivative of the double Laplace-Shehu transform for equation (5.1), single Laplace transform for equation (5.2) and single Shehu transform for equation (5.3) and simplifying, we obtain that:

$$\Psi(\gamma,\delta,\mu) = \Big[\frac{(B\frac{\delta}{\mu} + D)\hbar_1(\gamma) + B\hbar_2(\gamma) + (A\gamma + C)\hbar_3(\delta,\mu) + A\hbar_4(\delta,\mu) + F(\gamma,\delta,\mu)}{(A\gamma^2 + B\frac{\delta^2}{\mu^2} + C\gamma + D\frac{\delta}{\mu} + E)}\Big], \tag{5.4}$$

where $F(\gamma,\delta,\mu) = L_\xi S_\eta[f(\xi,\eta)]$.
Finally, solving this algebraic equation in $\Psi(\gamma,\delta,\mu)$ and taking the inverse double Laplace-Shehu transform on both sides of equation (5.4), yields:

$$\psi(\xi,\eta) = L_\xi^{-1} S_\eta^{-1}\Big[\frac{(B\frac{\delta}{\mu} + D)\hbar_1(\gamma) + B\hbar_2(\gamma) + (A\gamma + C)\hbar_3(\delta,\mu) + A\hbar_4(\delta,\mu) + F(\gamma,\delta,\mu)}{(A\gamma^2 + B\frac{\delta^2}{\mu^2} + C\gamma + D\frac{\delta}{\mu} + E)}\Big], \tag{5.5}$$

which represent the general formula for the solution of equation (5.1) by the double Laplace-Shehu transform method.

Example 5.1. Consider the following boundary Laplace equation

$$\psi_{\xi\xi}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta) = 0, \tag{5.6}$$

with the conditions:

$$\psi(\xi,0) = \cos\xi = \hbar_1(\xi), \qquad \psi_\eta(\xi,0) = 0 = \hbar_2(\xi),$$
$$\psi(0,\eta) = \cosh\eta = \hbar_3(\eta), \qquad \psi_\xi(0,\eta) = 0 = \hbar_4(\eta).$$

Solution:
Substituting

$$\hbar_1(\gamma) = \frac{\gamma}{\gamma^2+1}, \qquad \hbar_2(\gamma) = 0, \qquad \hbar_3(\delta,\mu) = \frac{\delta\mu}{\delta^2-\mu^2}, \qquad \hbar_4(\delta,\mu) = 0,$$

in (5.5) and simplifying, we get a solution of (5.6)

$$\psi(\xi,\eta) = L_\xi^{-1} S_\eta^{-1}\Big[\frac{\gamma}{\gamma^2+1}\ \frac{\delta\mu}{\delta^2-\mu^2}\Big] = \cos\xi\cosh\eta.$$

Example 5.2. Consider the following boundary Poisson equation

$$\psi_{\xi\xi}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta) = \eta\sin\xi, \tag{5.7}$$

with the conditions:

$$\psi(\xi,0) = 0 = \hbar_1(\xi), \qquad \psi_\eta(\xi,0) = -\sin\xi = \hbar_2(\xi),$$
$$\psi(0,\eta) = 0 = \hbar_3(\eta), \qquad \psi_\xi(0,\eta) = -\eta = \hbar_4(\eta).$$

Solution:
Applying the double Laplace-Shehu transform on both sides of equation (5.7), we get

$$\begin{aligned}\gamma^2\Psi(\gamma,\delta,\mu) \quad &- \quad \gamma S[\psi(0,\eta)] - S[\psi_\xi(0,\eta)] + \frac{\delta^2}{\mu^2}\Psi(\gamma,\delta,\mu) - \frac{\delta}{\mu}L[\psi(\xi,0)] - L[\psi_\eta(\xi,0)] \\ &= \quad \frac{\mu^2}{\delta^2(\gamma^2+1)}. \end{aligned} \tag{5.8}$$

Substituting

$$L[\hbar_1(\xi)] = 0, \qquad L[\hbar_2(\xi)] = \frac{-1}{\gamma^2+1}, \qquad S[\hbar_3(\eta)] = 0, \qquad S[\hbar_4(\eta)] = -\frac{\mu^2}{\delta^2},$$

in (5.8) and simplifying, we get

$$\Psi(\gamma,\delta,\mu) = \frac{-\mu^2}{\delta^2(\gamma^2+1)}. \tag{5.9}$$

Taking the inverse double Laplace-Shehu transform of equation (5.9), we get a solution of (5.7)

$$\psi(\xi,\eta) = L_\xi^{-1}S_\eta^{-1}\Big[\frac{-\mu^2}{\delta^2(\gamma^2+1)}\Big] = -L_\xi^{-1}S_\eta^{-1}\Big[\frac{\mu^2}{\delta^2}\ \frac{1}{\gamma^2+1}\Big] = -\eta\sin\xi.$$

In the following two examples, we will replace the independent variable η with the time variable t. Therefore, $S_\eta \equiv S_t$ and $S_\eta^{-1} \equiv S_t^{-1}$ will be used.

Example 5.3. Consider the following nonhomogeneous Wave equation

$$\psi_{tt}(\xi,t) = \psi_{\xi\xi}(\xi,t) - 3\psi(\xi,t) + 3, \qquad t > 0, \tag{5.10}$$

with the conditions:

$$\begin{aligned}\psi(\xi,0) = 1 = \hbar_1(\xi), &\qquad \psi_t(\xi,0) = 2\sin\xi = \hbar_2(\xi),\\ \psi(0,t) = 1 = \hbar_3(t), &\qquad \psi_\xi(0,t) = \sin 2t = \hbar_4(t).\end{aligned}$$

Solution:
Substituting

$$\hbar_1(\gamma) = \frac{1}{\gamma}, \qquad \hbar_2(\gamma) = \frac{2}{\gamma^2+1}, \qquad \hbar_3(\delta,\mu) = \frac{\mu}{\delta}, \qquad \hbar_4(\delta,\mu) = \frac{2\mu^2}{\delta^2+4\mu^2}, \qquad F(\gamma,\delta,\mu) = \frac{-3\mu}{\gamma\delta},$$

in (5.5) and simplifying,we get a solution of (5.10)

$$\psi(\xi,t) = L_\xi^{-1}S_t^{-1}\Big[\frac{\mu}{\gamma\delta} + \frac{1}{\gamma^2+1}\ \frac{2\mu^2}{\delta^2+4\mu^2}\Big] = 1 + \sin\xi\sin 2t.$$

Example 5.4. Consider the following nonhomogeneous Heat equation

$$\psi_t(\xi,t) = \psi_{\xi\xi}(\xi,t) - 6\xi, \qquad t > 0, \tag{5.11}$$

with the conditions:

$$\begin{aligned}\psi(\xi,0) = \xi^3 + \sin\xi = \hbar_1(\xi), &\qquad \psi_t(\xi,0) = -\sin\xi = \hbar_2(\xi),\\ \psi(0,t) = 0 = \hbar_3(t), &\qquad \psi_\xi(0,t) = e^{-t} = \hbar_4(t).\end{aligned}$$

Solution:
Substituting

$$\hbar_1(\gamma) = \frac{6}{\gamma^4} + \frac{1}{\gamma^2+1}, \qquad \hbar_2(\gamma) = \frac{-1}{\gamma^2+1}, \qquad \hbar_3(\delta,\mu) = 0, \qquad \hbar_4(\delta,\mu) = \frac{\mu}{\delta+\mu}, \qquad F(\gamma,\delta,\mu) = \frac{6}{\gamma^2},$$

in (5.5) and simplifying, we get a solution of (5.11)

$$\psi(\xi,t) = L_\xi^{-1}S_t^{-1}\Big[\frac{6}{\gamma^4} + \frac{\mu}{\delta+\mu}\ \frac{1}{\gamma^2+1}\Big] = \xi^3 + e^{-t}\sin\xi.$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have read and agreed to the published version of the manuscript.

References

[1] Aggarwal, S., Singh, A., Kumar, A., Kumar, N., *Application of Laplace transform for solving improper integrals whose integrand consisting error function*, Journal of Advanced Research in Applied Mathematics and Statistics, **4**(2019), 1–7.

[2] Ahmed, S.A., Elzaki, T.M., Elbadri, M., Mohamed, M., *Solution of partial differential equations by new double integral transform (Laplace - Sumudu transform)*, Ain Shams Engineering Journal, **12**(2021), 4045–4049.

[3] Albukhuttar, A., Jubear, B., Neamah, M., *Solve the Laplace, Poisson and telegraph equations using the Shehu transform*, Turkish Journal of Computer and Mathematics Education, Turkey, **10**(2021), 1759–1768.

[4] Albukhuttar, A., Ridha, Z., Kadhim, H., *Applications of a Shehu transform to the heat and transport equations*, International Journal of Psychosocial Rehabilitation, **24**(2020), 4254–4263.

[5] Alfaqeih, S., Misirli, E., *On double Shehu transform and its properties with applications*, International Journal of Analysis and Applications, **3**(2020), 381–395.

[6] Alkaleeli, S., Mtawal, A., Hmad, M., *Triple Shehu transform and its properties with applications*, African Journal of Mathematics and Computer, **14**(2021), 4–12.

[7] Debnath, L., *The double Laplace transforms and their properties with applications to functional, integral and partial differential equations*, Int. J. Appl. Comput. Math, **2**(2016), 223–241.

[8] Dhunde, R., Waghmare, G., *Double Laplace transform method in mathematical physics*, International Journal of Theoretical and Mathematical Physics, **7**(2017), 14–20.

[9] Eltayeb, H., Kilicman, A., *A note on solutions of wave, Laplace's and heat equations with convolution terms by using a double Laplace transform*, Applied Mathematics Letters **21**(2008), 1324–1329.

[10] Lakmon, K., Mensah, Y., *The Plancherel formula for the Shehu transform*, Annals of Pure and Applied Mathematical Sciences, **1**(2021) 14–19.

[11] Maitama, S., Zhao, W., *New integral transfrom: Shehu transform a generalizattion of Sumudu and Laplace transfrom for solving differential equations*, International Journal of Analysis and Applications, **17**(2019), 167–190.