http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 2, Pages 456–480 (2022)
DOI:10.31801/cfsuasmas.1009988
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: October 15, 2021; Accepted: December 9, 2021

QUANTUM ANALOG OF SOME TRAPEZOID AND MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper a new quantum analog of Hermite-Hadamard inequality is presented, and based on it, two new quantum trapezoid and midpoint identities are obtained. Moreover, the quantum analog of some trapezoid and midpoint type inequalities are established.

1. Introduction

A function $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval J, if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in J$ and $t \in [0, 1]$.

One of the most useful inequalities for convex functions is Hermite-Hadamard's inequality, due to its geometrical importance and applications, which is described as follows:

Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of real numbers and $a, b \in J$ with $a < b$. Then

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Hermite-Hadamard's inequality is investigated for several classes of functions in a number of papers and different types of inequalities have been obtained from it. For more details, see [16, 18, 21, 22, 28] and references therein.

2020 *Mathematics Subject Classification.* 05C38, 15A15, 05A15, 15A18.
Keywords. Trapezoid inequality, convex function, q-integral, midpoint inequality.
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In recent years Tariboon and Ntouyas in [31], generalized the classic quantum derivative and integral. Also, in [23], the authors by using the notions of left and right quantum derivative and integral have derived a similar generalization of classic quantum derivative and integral. In many research papers, the quantum analogue of Hermite-Hadamard type inequalities has been obtained via the generalized form of quantum integral, which were given in [31]. For more information in this regard, the reader is refer to [1]- [6], [8]- [15], [17], [19], [24]- [27], [29]- [36]

In this paper, we use the notions of left and right quantum derivatives and integrals together to introduce a new quantum analogue of Hermite-Hadamard inequality and based on it we obtain two new quantum trapezoid and midpoint type identities. In addition by using these new identities, we establish quantum analogue of some trapezoid and midpoint inequalities. We get the results of the trapezoid and midpoint inequalities as a special case when $q \to 1$. The idea and techniques of this paper may help the interested reader for further research in this area.

2. Preliminaries

In this section, we recall some previously known concepts.

In [31], Tariboon and Ntouyas introduced the concepts of quantum derivative and definite quantum integral for the functions of defined on an arbitrary finite intervals as follows:

Definition 1. *[31] A function $f(t)$ defined on $[a,b]$ is called quantum differentiable on $(a,b]$ with the following expression:*

$$ {}_aD_qf(t) \ = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)} \in \mathbb{R}, \quad t \neq a, \tag{2}$$

and quantum differentiable on $t = a$, if the following limit exists:

$$ {}_aD_qf(a) \ = \lim_{t \to a^+} \ {}_aD_qf(t) \ ,$$

for any $a < b$.

Clearly, if $a = 0$ in (2), then ${}_0D_qf(t) = D_qf(t)$ where $D_qf(t)$ is familiar quantum derivative of the function f defined by

$$D_qf(t) := \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \ D_qf(0) = \lim_{t \to 0} D_qf(t). \ \ [7,20] \tag{3}$$

Definition 2. *[31] Let a function f be defined on $[a,b]$. Then the quantum integral of f on $[a,b]$ is defined by*

$$\int_a^b f(t) \ {}_ad_qt \ = (1-q)(b-a) \sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a). \tag{4}$$

If the series in the right-hand side of (4) converges, then f is said to be quantum integrable on $[a,b]$. Also, for any $c \in (a,b)$

$$\int_c^b f(t) \ {}_a d_q t = \int_a^b f(t) \ {}_a d_q t - \int_a^c f(t) \ {}_a d_q t \ . \quad (5)$$

Clearly, if $a=0$ in (5), then

$$\int_0^b f(t) \ {}_0 d_q t = \int_0^b f(t) \ d_q t \ ,$$

where $\int_0^b f(t) \ d_q t$ is well-known Jackson integral of f on $[0,b]$. For more details, see [7, 20].

In [23], the authors have denoted (2) and (4) respectively as left quantum derivative and definite left quantum integral and it has been written in this wise:

$${}_a D_q f(t) = {}_{a^+} D_q f(t),$$

$$\int_a^b f(t) \ {}_a d_q t = \int_a^b f(t) \ {}_{a^+} d_q t \ .$$

We use these notations in the rest of the paper.

Lemma 1. *[31] Let $f:[a,b] \to \mathbb{R}$ be a differentiable function. Then we have*

$$\lim_{q \to 1^-} {}_{a^+} D_q f(t) = \frac{df(t)}{dt}. \quad (6)$$

Lemma 2. *[31] Let $f:[a,b] \to \mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\ dt$ is exist, then we have*

$$\lim_{q \to 1^-} \int_a^b f(t) \ {}_{a^+} d_q t = \int_a^b f(t) \ dt. \quad (7)$$

Recently, Kunt et al. [23], presented the notions of right quantum derivative and right definite quantum integral as follows:

Definition 3. *[23] A function $f(t)$ defined on $[a,b]$ is called the right quantum differentiable on $[a,b)$ with the following expression:*

$${}_{b^-} D_q f(t) = \frac{f(t) - f(qt + (1-q)b)}{(1-q)(t-b)} \in \mathbb{R}, \quad t \neq b, \quad (8)$$

and quantum differentiable on $t=b$, if the following limit exists:

$${}_{b^-} D_q f(b) = \lim_{t \to b^-} {}_{b^-} D_q f(t) \ ,$$

for any $a < b$.

Definition 4. *[23] Let a function f be defined on $[a,b]$. Then the right quantum integral of f on $[a,b]$ is defined by*

$$\int_a^b f(t) \ {}_{b^-}d_q t = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n) b). \tag{9}$$

If the series in right hand side of (9) converges, then f is said to be right quantum integrable on $[a,b]$. Also, for any $c \in (a,b)$

$$\int_a^c f(t) \ {}_{b^-}d_q t = \int_a^b f(t) \ {}_{b^-}d_q t - \int_c^b f(t) \ {}_{b^-}d_q t . \tag{10}$$

Lemma 3. *[23] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then we have*

$$\lim_{q\to 1^-} {}_{b^-}D_q f(t) = \frac{df(t)}{dt}. \tag{11}$$

Lemma 4. *[23] Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\ dt$ is exist, then we have*

$$\lim_{q\to 1^-} \int_a^b f(t) \ {}_{b^-}d_q t = \int_a^b f(t) \ dt. \tag{12}$$

3. Auxiliary Results

In this section, we describe some auxiliary lemmas which are used in the obtaining of main results.

Lemma 5. *Let $0 < q < 1$ be a constant, then the following equality holds:*

$$\int_0^1 |1-qt|\, t \ {}_{0^+}d_q t = \frac{1}{(1+q)(1+q+q^2)}. \tag{13}$$

Proof. By using the definition of q-integral, we have

$$\begin{aligned}
\int_0^1 |1-qt|\, t \ {}_{0^+}d_q t &= \int_0^1 (1-qt)\, t \ {}_{0^+}d_q t \\
&= \int_0^1 t \ {}_{0^+}d_q t - \int_0^1 qt^2 \ {}_{0^+}d_q t \\
&= \frac{1}{1+q} - \frac{q}{1+q+q^2} \\
&= \frac{1}{(1+q)(1+q+q^2)}.
\end{aligned}$$

The proof is completed. □

Lemma 6. *Let $0 < q < 1$ be a constant, then the following equality holds:*

$$\int_0^1 |1-qt|\,(1-t) \ {}_{0^+}d_q t = \frac{q}{1+q+q^2}. \tag{14}$$

Proof. By using Lemma (5) and the definition of q-integral, we have

$$\begin{aligned}\int_0^1 |1-qt|\,(1-t)\ \ {}_{0^+}d_qt &= \int_0^1 |1-qt|\ \ {}_{0^+}d_qt\ -\int_0^1 |1-qt|\,t\ \ {}_{0^+}d_qt \\ &= \int_0^1 (1-qt)\ \ {}_{0^+}d_qt\ -\frac{1}{(1+q)(1+q+q^2)} \\ &= \frac{1}{1+q}-\frac{1}{(1+q)(1+q+q^2)} \\ &= \frac{q}{1+q+q^2}.\end{aligned}$$

The proof is completed. □

Lemma 7. *Let $0<q<1$ be a constant, then the following equality holds:*

$$\int_0^1 t\,(1-t)\ \ {}_{0^+}d_qt\ =\frac{q^2}{(1+q)(1+q+q^2)}. \tag{15}$$

Proof. By using the definition of q-integral, we have

$$\begin{aligned}\int_0^1 t\,(1-t)\ \ {}_{0^+}d_qt &= \int_0^1 t\ \ {}_{0^+}d_qt\ -\int_0^1 t^2\ \ {}_{0^+}d_qt \\ &= \frac{1}{1+q}-\frac{1}{1+q+q^2} \\ &= \frac{q^2}{(1+q)(1+q+q^2)}.\end{aligned}$$

The proof is completed. □

Lemma 8. *Let $0<q<1$ be a constant, then the following equality holds:*

$$\int_0^1 t^p\ \ {}_{0^+}d_qt\ =\frac{1-q}{1-q^{p+1}}. \tag{16}$$

Proof. By using the definition of q-integral, we have

$$\begin{aligned}\int_0^1 t^p\ \ {}_{0^+}d_qt &= (1-q)\sum_{n=0}^{\infty} q^n\,(q^n)^p \\ &= (1-q)\sum_{n=0}^{\infty}\left(q^{p+1}\right)^n \\ &= \frac{1-q}{1-q^{p+1}}.\end{aligned}$$

The proof is completed. □

4. Main Results

In this section, we use Definition 2 together with Definition 4 of the quantum integrals to present a new quantum form of the Hermite-Hadamard inequality.

Let a function f be defined on $[a,b]\subset\mathbb{R}$, then from (4) and (9), we can write

$$\begin{aligned}&\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\\ =&\ \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n\left[f\left(q^n\left(\frac{b-a}{2}\right)+a\right)+f\left(q^n\left(\frac{a-b}{2}\right)+b\right)\right]\end{aligned}\tag{17}$$

For shortness we write the left-hand side of (17), as follows:

$$\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ :=\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\ .$$

If the series in the right-hand side of (17) converges or f is left quantum integrable on $\left[a,\frac{a+b}{2}\right]$ and right quantum integrable on $\left[\frac{a+b}{2},b\right]$, then $\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt$ is exist.

Lemma 9. *Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\ dt$ converges, then we have*

$$\lim_{q\to1^-}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ =\int_a^b f(t)\ dt.\tag{18}$$

Proof. By using (17) and lemma 2 and lemma 4, we get

$$\begin{aligned}\lim_{q\to1^-}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ &=\lim_{q\to1^-}\left[\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\ \right]\\ &=\lim_{q\to1^-}\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\lim_{q\to1^-}\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\\ &=\int_a^{\frac{a+b}{2}} f(t)\,dt+\int_{\frac{a+b}{2}}^b f(t)\,dt\\ &=\int_a^b f(t)\,dt.\end{aligned}$$

The proof is completed. □

Lemma 10. *Let $\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt$ be exist. Then we have*

$$\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ =\int_a^b f(a+b-t)\ {}_{\frac{a+b}{2}}d_qt\ .\tag{19}$$

Proof. By direct computing from (17), we get

$$\int_a^b f(a+b-t)\ {}_{\frac{a+b}{2}}d_qt$$

$$= \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n \left[\begin{array}{c} f\left(a+b-\left(q^n\left(\frac{b-a}{2}\right)+a\right)\right) \\ +f\left(a+b-\left(q^n\left(\frac{a-b}{2}\right)+b\right)\right) \end{array} \right]$$

$$= \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n \left[f\left(b-q^n\left(\frac{b-a}{2}\right)\right)+f\left(a-q^n\left(\frac{a-b}{2}\right)\right)\right]$$

$$= \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n \left[f\left(q^n\left(\frac{a-b}{2}\right)+b\right)+f\left(q^n\left(\frac{b-a}{2}\right)+a\right)\right]$$

$$= \int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ .$$

This complete the proof. □

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $0<q<1$. Then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ \le \frac{f(a)+f(b)}{2}. \tag{20}$$

Proof. Clearly $\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t$ is exist. By using (17), we have

$$\int_0^1 f(ta+(1-t)b)\ {}_{\frac{0+1}{2}}d_q t$$

$$= \frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty} q^n \left[\begin{array}{c} f\left(\left[q^n\left(\frac{1-0}{2}\right)+0\right]a+\left[1-\left(q^n\left(\frac{1-0}{2}\right)+0\right)\right]b\right) \\ +f\left(\left[q^n\left(\frac{0-1}{2}\right)+1\right]a+\left[1-\left(q^n\left(\frac{0-1}{2}\right)+1\right)\right]b\right) \end{array} \right]$$

$$= \frac{1}{b-a}\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n \left[f\left(\frac{q^n(a-b)}{2}+b\right)+f\left(\frac{q^n(b-a)}{2}+a\right)\right]$$

$$= \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ , \tag{21}$$

and

$$\int_0^1 f(tb+(1-t)a)\ {}_{\frac{0+1}{2}}d_q t$$

$$= \frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty} q^n \left[\begin{array}{c} f\left(\left[q^n\left(\frac{1-0}{2}\right)+0\right]b+\left[1-\left(q^n\left(\frac{1-0}{2}\right)+0\right)\right]a\right)+ \\ f\left(\left[q^n\left(\frac{0-1}{2}\right)+1\right]b+\left[1-\left(q^n\left(\frac{0-1}{2}\right)+1\right)\right]a\right) \end{array} \right]$$

$$= \frac{1}{b-a}\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n \left[f\left(\frac{q^n(b-a)}{2}+a\right)+f\left(\frac{q^n(a-b)}{2}+b\right)\right]$$

$$= \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ . \tag{22}$$

Again by applying (17), we get

$$\int_0^1 f\left(\frac{a+b}{2}\right)\ {}_{\frac{0+1}{2}}d_q t \quad = \quad f\left(\frac{a+b}{2}\right)\int_0^1 1\ {}_{\frac{0+1}{2}}d_q t$$

$$= f\left(\frac{a+b}{2}\right)\frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty}q^n[1+1]$$

$$= f\left(\frac{a+b}{2}\right), \tag{23}$$

and

$$\int_0^1 \frac{f(a)+f(b)}{2}\ {}_{\frac{0+1}{2}}d_qt = \frac{f(a)+f(b)}{2}\int_0^1 1\ {}_{\frac{0+1}{2}}d_qt$$

$$= \frac{f(a)+f(b)}{2}\frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty}q^n[1+1]$$

$$= \frac{f(a)+f(b)}{2}. \tag{24}$$

Since f is convex on $[a,b]$, then we can write

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f(ta+(1-t)b)+f(tb+(1-t)a)\right] \le \frac{f(a)+f(b)}{2},$$

for all $t\in[0,1]$, and

$$\int_0^1 f\left(\frac{a+b}{2}\right)\ {}_{\frac{0+1}{2}}d_qt$$

$$\le \int_0^1 \left(\frac{1}{2}\left[f(ta+(1-t)b)+f(tb+(1-t)a)\right]\right)\ {}_{\frac{0+1}{2}}d_qt$$

$$\le \int_0^1 \frac{f(a)+f(b)}{2}\ {}_{\frac{0+1}{2}}d_qt\ .$$

Therefore by using (21),(22) , (23) and (24), we get

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \le \frac{f(a)+f(b)}{2}.$$

This complete the proof. □

Remark 1. *If $q\to 1$, then by using Lemma 9, the inequality (20)reduce to (1).*

Lemma 11. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function. If ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, then the following identity holds:*

$$\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ - \frac{f(a)+f(b)}{2}$$

$$= \frac{b-a}{4}\int_0^1 (1-qt)\begin{pmatrix} {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ -\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{pmatrix}\ {}_{0^+}d_qt\ . \tag{25}$$

Proof. Since ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, using the linearity of left quantum integral, then we have

$$\begin{aligned}
&\frac{b-a}{4}\int_0^1 (1-qt)\begin{pmatrix} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{pmatrix}\ {}_{0^+}d_q t \\
&= \frac{b-a}{4}\left[\int_0^1 (1-qt)\ {}_{a^+}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)a\right)\ {}_{0^+}d_q t\right. \\
&\left. - \int_0^1 (1-qt)_{b^-} D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\ {}_{0^+}d_q t\ \right] \\
&= \frac{b-a}{4}\left[M_1 - M_2\right]. \qquad (26)
\end{aligned}$$

Since f is continuous on $[a,b]$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right) - \sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right) \\
= \quad & f\left(\frac{a+b}{2}\right) - f(a), \qquad (27)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right) - \sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right) \\
= \quad & f\left(\frac{a+b}{2}\right) - f(b). \qquad (28)
\end{aligned}$$

Using (27), we achieve

$$\begin{aligned}
M_1 &= \int_0^1 (1-qt)\ {}_{a^+}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)a\right)\ {}_{0^+}d_q t \\
&= \int_0^1 {}_{a^+}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)a\right)\ {}_{0^+}d_q t \\
&- q\int_0^1 t\ {}_{a^+}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)a\right)\ {}_{0^+}d_q t \\
&= \int_0^1 \left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) - f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)a\right)}{(1-q)\left(\frac{a+b}{2}-a\right)t}\right]\ {}_{0^+}d_q t \\
&- q\int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) - f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)a\right)}{(1-q)\left(\frac{a+b}{2}-a\right)t}\right]\ {}_{0^+}d_q t \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right)\Bigg]\\
&-\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+\left(1-q^n\right)a\right)\right.\\
&\left.-\sum_{n=0}^{\infty} q^n f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right)\right]\\
&=\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\\
&-\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+\left(1-q^n\right)a\right)\right.\\
&\left.-\frac{1}{q}\sum_{n=1}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+\left(1-q^n\right)a\right)\right]\\
&=\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\\
&-\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+\left(1-q^n\right)a\right)\right.\\
&\left.-\frac{1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+\left(1-q^n\right)a\right)+\frac{1}{q}f\left(\frac{a+b}{2}\right)\right]\\
&=\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(a)\right]-\frac{2}{b-a}f\left(\frac{a+b}{2}\right)\\
&-\frac{2q}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+\left(1-q^n\right)a\right)\right]\\
&=\frac{4}{(b-a)^2}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+\left(1-q^n\right)a\right)\right]\\
&-\frac{4}{b-a}\frac{f(a)}{2}\\
&=\frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_q t\ -\frac{4}{b-a}\frac{f(a)}{2}. \qquad (29)
\end{aligned}
$$

Similarly, using (28) 3, we get

$$
M_2 = \int_0^1 (1-qt)\ {}_{b^-}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)b\right)\ {}_{0^+}d_q t
$$

$$
\begin{aligned}
&= \int_0^1 {}_{b^-}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,b\right) \quad {}_{0^+}d_q t \\
&\quad -q\int_0^1 t \;{}_{b^-}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,b\right) \quad {}_{0^+}d_q t \\
&= \int_0^1 \left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,b\right)}{(1-q)\left(\frac{a+b}{2}-b\right)t}\right] \;{}_{0^+}d_q t \\
&\quad -q\int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,b\right)}{(1-q)\left(\frac{a+b}{2}-b\right)t}\right] \;{}_{0^+}d_q t \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right)\right] \\
&\quad +\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\sum_{n=0}^{\infty} q^n f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right)\right] \\
&= -\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(b)\right] \\
&\quad +\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\frac{1}{q}\sum_{n=1}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right] \\
&= -\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(b)\right] \\
&\quad +\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\frac{1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)+\frac{1}{q}f\left(\frac{a+b}{2}\right)\right] \\
&= -\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(b)\right]+\frac{2}{b-a}f\left(\frac{a+b}{2}\right) \\
&\quad +\frac{2q}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right]
\end{aligned}
$$

$$\begin{aligned} &= \frac{4}{b-a}\frac{f(b)}{2} \\ &\quad -\frac{4}{(b-a)^2}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right] \\ &= \frac{4}{b-a}\frac{f(b)}{2}-\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^{b} f(t)\ {}_{b^-}d_q t\ . \end{aligned} \tag{30}$$

Combining (26), (29) and (30) , we obtain

$$\begin{aligned} &\frac{b-a}{4}[M_1-M_2] \\ &= \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_q t\ +\int_{\frac{a+b}{2}}^{b} f(t)\ {}_{b^-}d_q t\right)-\frac{f(a)+f(b)}{2} \\ &= \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ -\frac{f(a)+f(b)}{2}, \end{aligned}$$

which gives (25). This complete the proof. □

Remark 2. *If f is differentiable on $[a,b]$ and $q\to 1$, then the identity (25) reduce to*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2} \\ &= \frac{b-a}{4}\int_0^1 (1-t)\left(\begin{array}{c} f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ -f'\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{array}\right)dt. \end{aligned}$$

See also [21, Lemma 1, for $x=\frac{a+b}{2}$].

Next, we present quantum analogue of some trapezoid type inequalities as follows:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$. If $|{}_{a^+}D_q f|$ and $|{}_{b^-}D_q f|$ are convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ -\frac{f(a)+f(b)}{2}\right| \\ &\leq \frac{b-a}{4}\left[\begin{array}{c}\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|+\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} \\ +\frac{q\left(\left|{}_{a^+}D_q f(a)\right|+\left|{}_{b^-}D_q f(b)\right|\right)}{1+q+q^2}\end{array}\right]. \end{aligned} \tag{31}$$

Proof. By using Lemma (11) and convexity of $|{}_{a^+}D_q f|$ and $|{}_{b^-}D_q f|$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ -\frac{f(a)+f(b)}{2}\right|$$

$$\begin{aligned}
&= \left| \frac{b-a}{4} \int_0^1 (1-qt) \begin{pmatrix} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right) \end{pmatrix} {}_{0^+}d_q t \right| \\
&\le \frac{b-a}{4} \begin{bmatrix} \int_0^1 |(1-qt)| \left|{}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right| {}_{0^+}d_q t \\ +\int_0^1 |(1-qt)| \left|{}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right| {}_{0^+}d_q t \end{bmatrix} \\
&\le \frac{b-a}{4} \begin{bmatrix} \left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right| \int_0^1 |(1-qt)|\, t\ {}_{0^+}d_q t \\ +\left|{}_{a^+}D_q f(a)\right| \int_0^1 |(1-qt)|\,(1-t){}_{0^+}d_q t \\ +\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right| \int_0^1 |(1-qt)|\, t\ {}_{0^+}d_q t \\ +\left|{}_{b^-}D_q f(b)\right| \int_0^1 |(1-qt)|\,(1-t){}_{0^+}d_q t \end{bmatrix}.
\end{aligned}$$

Applying Lemma 5 and Lemma 6, we have

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ - \frac{f(a)+f(b)}{2} \right| \\
&\le \frac{b-a}{4} \begin{bmatrix} \frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} + \frac{q\left|{}_{a^+}D_q f(a)\right|}{1+q+q^2} \\ +\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} + \frac{\left|{}_{b^-}D_q f(b)\right|}{1+q+q^2} \end{bmatrix} \\
&= \frac{b-a}{4} \begin{bmatrix} \frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right| + \left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} \\ +\frac{q\left(\left|{}_{a^+}D_q f(a)\right| + \left|{}_{b^-}D_q f(b)\right|\right)}{1+q+q^2} \end{bmatrix}.
\end{aligned}$$

This complete the proof. □

Remark 3. *If f is differentiable on $[a,b]$ and $q \to 1$, then the inequality (31) reduce to*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{b-a}{12}\left(|f'(a)| + \left|f'(\frac{a+b}{2})\right| + |f'(b)| \right).$$

See also [21, Corollary 2].

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$. If $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex on $[a,b]$ where $r>0$, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ - \frac{f(a)+f(b)}{2} \right| \\
&\le \frac{b-a}{4}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}} \begin{bmatrix} \left(\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)} + \frac{q\left|{}_{a^+}D_q f(a)\right|^r}{1+q+q^2} \right)^{\frac{1}{r}} + \\ \left(\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)} + \frac{q\left|{}_{b^-}D_q f(a)\right|^r}{1+q+q^2} \right)^{\frac{1}{r}} \end{bmatrix}. \quad (32)
\end{aligned}$$

Proof. Since $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex functions, so from Lemma 11 and using the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ -\frac{f(a)+f(b)}{2}\right| \\
=\ &\left|\frac{b-a}{4}\int_0^1 (1-qt)\begin{pmatrix} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{pmatrix}\ {}_{0^+}d_q t\right| \\
\leq\ &\frac{b-a}{4}\begin{bmatrix} \int_0^1 |(1-qt)|\left|{}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|\ {}_{0^+}d_q t \\ +\int_0^1 |(1-qt)|\left|\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\ \right|\ {}_{0^+}d_q t\end{bmatrix} \\
\leq\ &\frac{b-a}{4}\begin{bmatrix} \left(\int_0^1 |(1-qt)|\ \ {}_{0^+}d_q t\ \right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 |(1-qt)|\left|{}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^r\ {}_{0^+}d_q t\ \right)^{\frac{1}{r}} \\ +\left(\int_0^1 |(1-qt)|\ \ {}_{0^+}d_q t\ \right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 |(1-qt)|\left|\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\ \right|\ {}_{0^+}d_q t\ \right)^{\frac{1}{r}}\end{bmatrix} \\
\leq\ &\frac{b-a}{4}\left(\int_0^1 |(1-qt)|\ \ {}_{0^+}d_q t\ \right)^{1-\frac{1}{r}} \\
&\times\begin{bmatrix} \begin{pmatrix} \left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 |(1-qt)|\,t\ \ {}_{0^+}d_q t \\ +\left|{}_{a^+}D_q f(a)\right|^r\int_0^1 |(1-qt)|\,(1-t)_{0^+}d_q t\end{pmatrix}^{\frac{1}{r}} \\ +\begin{pmatrix} \left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 |(1-qt)|\,t\ \ {}_{0^+}d_q t \\ +\left|{}_{b^-}D_q f(b)\right|^r\int_0^1 |(1-qt)|\,(1-t)_{0^+}d_q t\end{pmatrix}^{\frac{1}{r}}\end{bmatrix}.
\end{aligned}$$

Applying Lemma 5 and Lemma 6, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ -\frac{f(a)+f(b)}{2}\right| \\
\leq\ &\frac{b-a}{4}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\begin{bmatrix} \left(\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)}+\frac{q\left|{}_{a^+}D_q f(a)\right|^r}{1+q+q^2}\right)^{\frac{1}{r}} \\ +\left(\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)}+\frac{q\left|{}_{b^-}D_q f(a)\right|^r}{1+q+q^2}\right)^{\frac{1}{r}}\end{bmatrix}.
\end{aligned}$$

This complete the proof. ☐

Remark 4. *If f is differentiable on $[a,b]$ and $q\to 1$, then the inequality* (32) *reduce to*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \left(\frac{b-a}{8}\right)\left(\frac{1}{3}\right)^{\frac{1}{r}}\left[\begin{array}{c}\left(\left|f'(\frac{a+b}{2})\right|^r+2\left|f'(a)\right|^r\right)^{\frac{1}{r}}\\+\left(\left|f'(\frac{a+b}{2})\right|^r+2\left|f'(b)\right|^r\right)^{\frac{1}{r}}\end{array}\right].$$

See also [21, Corollary 4].

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$. If $\left|{}_{a^+}D_qf\right|^r$ and $\left|{}_{b^-}D_qf\right|^r$ are convex on $[a,b]$ where $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt-\frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{b-a}{4}\left(S_q(p)\right)^{\frac{1}{p}}\left[\left(\begin{array}{c}\left(\frac{\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{a^+}D_qf(a)\right|^r}{(1+q)}\right)^{\frac{1}{r}}\\+\left(\frac{\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{b^-}D_qf(a)\right|^r}{(1+q)}\right)^{\frac{1}{r}}\end{array}\right)\right], \tag{33}$$

where

$$S_q(p)=\int_0^1(1-qt)^p\ {}_{0^+}d_qt\ ,$$

is fulfilled.

Proof. From Lemma 11 and using Holder's inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt-\frac{f(a)+f(b)}{2}\right|$$

$$=\left|\frac{b-a}{4}\int_0^1(1-qt)\left(\begin{array}{c}{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\\-{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{array}\right){}_{0^+}d_qt\right|$$

$$\leq\frac{b-a}{4}\left[\int_0^1|(1-qt)|\left|{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|{}_{0^+}d_qt\right.$$

$$\left.\int_0^1|(1-qt)|\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|{}_{0^+}d_qt\right]$$

$$\leq\frac{b-a}{4}\left(\int_0^1(1-qt)^p\ {}_{0^+}d_qt\right)^{\frac{1}{p}}$$

$$\times\left[\begin{array}{c}\left(\int_0^1\left|{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^r{}_{0^+}d_qt\right)^{\frac{1}{r}}\\+\left(\int_0^1\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|^r{}_{0^+}d_qt\right)^{\frac{1}{r}}\end{array}\right].$$

By applying the convexity of $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ -\frac{f(a)+f(b)}{2}\right|\\ \leq\ & \frac{b-a}{4}\left(S(p)\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\begin{array}{c}\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_q t\\ +\left|{}_{a^+}D_q f(a)\right|^r\int_0^1(1-t)\ {}_{0^+}d_q t\end{array}\right)^{\frac{1}{r}}\\ +\left(\begin{array}{c}\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_q t\\ +\left|{}_{b^-}D_q f(b)\right|^r\int_0^1(1-t)\ {}_{0^+}d_q t\end{array}\right)^{\frac{1}{r}}\end{array}\right],\end{aligned}$$

Also,

$$\int_0^1 t\ {}_{0^+}d_q t\ =\frac{1}{1+q},\qquad \int_0^1(1-t)\ {}_{0^+}d_q t\ =\frac{q}{1+q}.$$

Therefore

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ -\frac{f(a)+f(b)}{2}\right|\\ \leq\ & \frac{b-a}{4}\left(S_q(p)\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{a^+}D_q f(a)\right|^r}{1+q}\right)^{\frac{1}{r}}\\ +\left(\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{b^-}D_q f(b)\right|^r}{1+q}\right)^{\frac{1}{r}}\end{array}\right].\end{aligned}$$

This complete the proof. □

Remark 5. *If f is differentiable on $[a,b]$ and $q\to 1$, then $S_1(p)=\frac{1}{p+1}$ and the inequality (33) reduce to*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ & \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{r}}\left[\begin{array}{c}\left[\left|f'(a)\right|^r+\left|f'(\frac{a+b}{2})\right|^r\right]^{\frac{1}{r}}\\ +\left[\left|f'(b)\right|^r+\left|f'(\frac{a+b}{2})\right|^r\right]^{\frac{1}{r}}\end{array}\right].\end{aligned}$$

See also [21, Corollary 3].

Lemma 12. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function. If ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and*
${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, then the following identity holds:

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t$$

$$= \frac{b-a}{4}\int_0^1 qt\left(\begin{array}{c} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right) \end{array}\right) {}_{0^+}d_q t\ . \tag{34}$$

Proof. By using the similar proving argument as in Lemma 11, we have

$$\begin{aligned}
&\frac{b-a}{4}\int_0^1 qt\left(\begin{array}{c} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right) \end{array}\right) {}_{0^+}d_q t \\
&= \frac{q\,(b-a)}{4}\left[\begin{array}{c} \int_0^1 t\ {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_q t \\ -\int_0^1 t\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\ {}_{0^+}d_q t \end{array}\right] \\
&= \frac{q\,(b-a)}{4}\left[K_1 - K_2\right].
\end{aligned} \tag{35}$$

Also,

$$\begin{aligned}
K_1 &= \int_0^1 t\ {}_{a^+}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_q t \\
&= \int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,a\right)}{(1-q)\left(\frac{a+b}{2}-a\right)t}\right] {}_{0^+}d_q t \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right. \\
&\left. -\sum_{n=0}^{\infty} q^n f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right)\right] \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right. \\
&\left. -\frac{1}{q}\sum_{n=1}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right] \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right. \\
&\left. -\frac{1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)+\frac{1}{q}f\left(\frac{a+b}{2}\right)\right] \\
&= \frac{2}{q\,(b-a)}f\left(\frac{a+b}{2}\right) \\
&- \frac{2}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right] \\
&= \frac{2}{q\,(b-a)}f\left(\frac{a+b}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{4}{q(b-a)^2}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right)\right] \\
&=\frac{2}{q(b-a)}f\left(\frac{a+b}{2}\right)-\frac{4}{q(b-a)^2}\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ , \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
K_2 &= \int_0^1 t\ {}_{b^-}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)b\right)\ {}_{0^+}d_qt \\
&= \int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)b\right)}{(1-q)\left(\frac{a+b}{2}-b\right)t}\right]\ {}_{0^+}d_qt \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right. \\
&\left.-\sum_{n=0}^{\infty}q^n f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right)\right] \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right. \\
&\left.-\frac{1}{q}\sum_{n=1}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right] \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right)\right. \\
&\left.-\frac{1}{q}\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right)+\frac{1}{q}f\left(\frac{a+b}{2}\right)\right] \\
&= -\frac{2}{q(b-a)}f\left(\frac{a+b}{2}\right) \\
&-\frac{2}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right] \\
&= -\frac{2}{q(b-a)}f\left(\frac{a+b}{2}\right) \\
&+\frac{4}{q(b-a)^2}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right] \\
&= -\frac{2}{q(b-a)}f\left(\frac{a+b}{2}\right)+\frac{4}{q(b-a)^2}\int_{\frac{a+b}{2}}^{b} f(t)\ {}_{b^-}d_qt\ . \qquad (37)
\end{aligned}
$$

Combining (35), (36) and (37), we get

$$\begin{aligned}&\frac{q\left(b-a\right)}{4}\left[K_1-K_2\right]\\ &= f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f\left(t\right)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^{b} f\left(t\right)\ {}_{b^-}d_qt\ \right)\\ &= f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f\left(t\right)\ {}_{\frac{a+b}{2}}d_qt\ ,\end{aligned}$$

which leads to the (34).This complete the proof. □

Remark 6. *If f is differentiable on $[a,b]$ and $q\to 1$, then the identity* (34) *reduce to*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f\left(t\right)dt\\ &= \frac{b-a}{4}\int_0^1 t\left(\begin{array}{c} f'\left(t\left(\frac{a+b}{2}\right)+\left(1-t\right)a\right)\\ -f'\left(t\left(\frac{a+b}{2}\right)+\left(1-t\right)b\right)\end{array}\right)dt.\end{aligned}$$

See also [18, Lemma 2.1].

Next, we establish quantum analogue of some midpoint type inequalities as follows:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+\left(1-t\right)a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+\left(1-t\right)b\right)$ are left quantum integrable on $[0,1]$. If $\left|{}_{a^+}D_qf\right|$ and $\left|{}_{b^-}D_qf\right|$ are convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f\left(t\right)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\ \leq\ &\frac{b-a}{4}\left(\frac{q}{1+q+q^2}\right)\left[\begin{array}{c}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|+\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|\\ +\frac{q^2}{1+q}\left(\left|{}_{a^+}D_qf\left(a\right)\right|+\left|{}_{b^-}D_qf\left(b\right)\right|\right)\end{array}\right].\end{aligned}\tag{38}$$

Proof. From Lemma 12 and using convexity of $\left|{}_{a^+}D_qf\right|$ and $\left|{}_{b^-}D_qf\right|$, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f\left(t\right)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\ =\ &\left|\frac{b-a}{4}\int_0^1 qt\left(\begin{array}{c}{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+\left(1-t\right)a\right)\\ -\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+\left(1-t\right)b\right)\end{array}\right)\ {}_{0^+}d_qt\ \right|\\ \leq\ &\frac{q(b-a)}{4}\left[\begin{array}{c}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|\int_0^1 t^2\ {}_{0^+}d_qt\\ +\left|{}_{a^+}D_qf\left(a\right)\right|\int_0^1(t-t^2)\ {}_{0^+}d_qt\\ +\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|\int_0^1 t^2\ {}_{0^+}d_qt\\ +\left|{}_{b^-}D_qf\left(b\right)\right|\int_0^1(t-t^2)\ {}_{0^+}d_qt\end{array}\right].\end{aligned}$$

Applying Lemma 7, we get

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\ \leq\ &\frac{b-a}{4}\left(\frac{q}{1+q+q^2}\right)\left[\begin{array}{c}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|+\left|{}_{b-}D_qf\left(\frac{a+b}{2}\right)\right|\\ +\frac{q^2}{1+q}\left(\left|{}_{a^+}D_qf(a)\right|+\left|{}_{b^-}D_qf(b)\right|\right)\end{array}\right].\end{aligned}$$

This complete the proof. ☐

Remark 7. *If f is differentiable on $[a,b]$ and $q\to 1$, then the inequality* (38) *reduce to*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\frac{b-a}{12}\left(2\left|f'(\frac{a+b}{2})\right|+\frac{|f'(a)|+|f'(b)|}{2}\right),\end{aligned}$$

See also [18, Theorem 2.1 for $s=m=1$].

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and*
${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$. If $\left|{}_{a^+}D_qf\right|^r$ and $\left|{}_{b^-}D_qf\right|^r$ are convex on $[a,b]$ where $r>0$, then the following inequality holds:

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\ \leq\ &\frac{q(b-a)}{4(1+q)}\left(\frac{1}{(1+q+q^2)}\right)^{\frac{1}{r}}\\ &\times\left[\begin{array}{c}\left(\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r(1+q)+\left|{}_{a^+}D_qf(a)\right|^r q^2\right)^{\frac{1}{r}}\\ +\left(\left|{}_{b-}D_qf\left(\frac{a+b}{2}\right)\right|^r(1+q)+\left|{}_{b^-}D_qf(b)\right|^r q^2\right)^{\frac{1}{r}}\end{array}\right].\end{aligned}\tag{39}$$

Proof. Since $\left|{}_{a^+}D_qf\right|^r$ and $\left|{}_{b^-}D_qf\right|^r$ are convex functions, so from Lemma 12 and using the power mean inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\ \leq\ &\left|\frac{b-a}{4}\int_0^1 qt\left(\begin{array}{c}{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\\ -{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{array}\right)\ {}_{0^+}d_qt\ \right|\end{aligned}$$

$$\leq \frac{q(b-a)}{4}\left[\begin{array}{c}\left(\int_0^1 t \ _{0^+}d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 t\left| \ _{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^r \ _{0^+}d_qt\right)^{\frac{1}{r}} \\ +\left(\int_0^1 t \ _{0^+}d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 t\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|^r \ _{0^+}d_qt\right)^{\frac{1}{r}}\end{array}\right]$$

$$\leq \frac{q(b-a)}{4}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left[\begin{array}{c}\left(\begin{array}{c}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t^2 \ _{0^+}d_qt \\ +\left|{}_{a^+}D_qf(a)\right|^r\int_0^1 t(1-t) \ _{0^+}d_qt\end{array}\right)^{\frac{1}{r}} \\ +\left(\begin{array}{c}\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t^2 \ _{0^+}d_qt \\ +\left|{}_{b^-}D_qf(b)\right|^r\int_0^1 t(1-t) \ _{0^+}d_qt\end{array}\right)^{\frac{1}{r}}\end{array}\right].$$

Applying Lemma 7, we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t) \ _{\frac{a+b}{2}}d_qt\right|$$

$$\leq \frac{q(b-a)}{4}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left[\begin{array}{c}\left(\begin{array}{c}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r\frac{1}{1+q+q^2} \\ +\left|{}_{a^+}D_qf(a)\right|^r\frac{q^2}{(1+q)(1+q+q^2)}\end{array}\right)^{\frac{1}{r}} \\ +\left(\begin{array}{c}\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r\frac{1}{1+q+q^2} \\ +\left|{}_{b^-}D_qf(b)\right|^r\frac{q^2}{(1+q)(1+q+q^2)}\end{array}\right)^{\frac{1}{r}}\end{array}\right]$$

$$= \frac{q(b-a)}{4(1+q)}\left(\frac{1}{(1+q+q^2)}\right)^{\frac{1}{r}}$$

$$\times\left[\begin{array}{c}\left(\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r(1+q)+\left|{}_{a^+}D_qf(a)\right|^r q^2\right)^{\frac{1}{r}} \\ +\left(\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r(1+q)+\left|{}_{b^-}D_qf(b)\right|^r q^2\right)^{\frac{1}{r}}\end{array}\right].$$

This complete the proof. □

Remark 8. *If f is differentiable on $[a,b]$ and $q\to 1$, then the inequality (39) reduce to*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t) \ dt\right|$$

$$\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{r}}\left[\begin{array}{c}\left(2\left|f'\left(\frac{a+b}{2}\right)\right|^r+\left|f'(a)\right|^r\right)^{\frac{1}{r}} \\ +\left(2\left|f'\left(\frac{a+b}{2}\right)\right|^r+\left|f'(b)\right|^r\right)^{\frac{1}{r}}\end{array}\right].$$

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and*

${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)$ *are left quantum integrable on* $[0,1]$. *If* $\left|{}_{a^+}D_qf\right|^r$ *and* $\left|{}_{b^-}D_qf\right|^r$ *are convex on* $[a,b]$ *where* $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, *then the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right| \leq \frac{q(b-a)}{4}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{a^+}D_qf(a)\right|^r}{1+q}\right)^{\frac{1}{r}}\\ +\left(\frac{\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{b^-}D_qf(b)\right|^r}{1+q}\right)^{\frac{1}{r}}\end{array}\right]. \tag{40}$$

is true.

Proof. From Lemma 12 and using Holder's inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\ =&\left|\frac{b-a}{4}\int_0^1 qt\left(\begin{array}{c}{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\\ -\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\end{array}\right)\ {}_{0^+}d_qt\ \right|\\ \leq&\frac{q(b-a)}{4}\left[\begin{array}{c}\left(\int_0^1 t^p\ {}_{0^+}d_qt\ \right)^{\frac{1}{p}}\\ \times\left(\int_0^1\left|\ {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\ \right|^r\ {}_{0^+}d_qt\ \right)^{\frac{1}{r}}\\ +\left(\int_0^1 t^p\ {}_{0^+}d_qt\ \right)^{\frac{1}{p}}\\ \times\left(\int_0^1\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\right|^r\ {}_{0^+}d_qt\ \right)^{\frac{1}{r}}\end{array}\right].\end{aligned}$$

Applying Lemma 8 and convexity of $\left|{}_{a^+}D_qf\right|^r$ and $\left|{}_{b^-}D_qf\right|^r$, we get

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\ \leq&\frac{q(b-a)}{4}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\begin{array}{c}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_qt\\ +\left|{}_{a^+}D_qf\left(a\right)\right|^r\int_0^1(1-t)\ {}_{0^+}d_qt\end{array}\right)^{\frac{1}{r}}\\ \left(\begin{array}{c}\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_qt\\ +\left|{}_{b^-}D_qf\left(b\right)\right|^r\int_0^1(1-t)\ {}_{0^+}d_qt\end{array}\right)^{\frac{1}{r}}\end{array}\right]\\ \leq&\frac{q(b-a)}{4}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{a^+}D_qf(a)\right|^r}{1+q}\right)^{\frac{1}{r}}\\ +\left(\frac{\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{b^-}D_qf(b)\right|^r}{1+q}\right)^{\frac{1}{r}}\end{array}\right].\end{aligned}$$

This complete the proof. $\square$

Remark 9. *If f is differentiable on $[a,b]$ and $q \to 1$, then the inequality* (40) *reduce to*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\begin{array}{c} \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^r + \left|f'(a)\right|^r}{2}\right)^{\frac{1}{r}} \\ + \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^r + \left|f'(b)\right|^r}{2}\right)^{\frac{1}{r}} \end{array} \right].$$

5. Conclusions

We have introduced a new quantum analogue of Hermite-Hadamard inequality and based on it we obtained two new quantum trapezoid and midpoint type identities. In [21] and [18], respectively by taking $x = \frac{a+b}{2}$ and $s = m = 1$, trapezoid and midpoint type inequalities for convex functions have been presented. We have established quantum analogs of some of these inequalities by using the new quantum trapezoid and midpoint type identities. For $q \to 1$ the obtained results give refinement of some trapezoid and midpoint type inequalities in [18, 21]. The idea and techniques of this paper may help the interested researcher in this field for further research.

Author Contribution Statements The authors contributed equally. All authors read and approved the final copy of the manuscript.

Declaration of Competing Interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement The authors thanks to the referees for their valuable comments and suggestions.

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