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ON THE PERFORMANCE OF MAXIMUM LIKELIHOOD ESTIMATORS OF SNR FOR NONCOHERENT BFSK SIGNALS IN RAYLEIGH FADING CHANNELS

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Abstract: In a recent paper, we have derived Cramér-Rao bounds for data and non-data-aided SNR estimation of noncoherent BFSK signals in slowly Rayleigh fading channels, and provided the corresponding true and approximate maximum-likelihood estimators for the data-aided and non-data-aided estimation, respectively. In this paper, the performances of the estimators are examined analytically in terms of means and variances. The results illustrate the efficiency of their performance.

Key Words: Signal-to-Noise Ratio Estimation, Maximum Likelihood, Rayleigh Channels, Noncoherent BFSK.

Rayleigh Sönümlü Kanallarda Evre Uyumsuz BFSK İşaretlerin İşaret Gürültü Oranının En Büyük Olabilirlik Kestiricilerinin Başarımı Hakkında

Özet: Yakın geçmişte yayınlanan bir makalemizde yavaş Rayleigh sönümlü kanallarda evre uyumsuz BFSK işaretlerinin veri destekli ve veri desteksiz işaret gürültü oranı kestirimine ilişkin Cramér-Rao sınırları çıkarılmış, ve bunlara karşılık gelen tam ve yaklaşık en büyük olabilirlik kestiricileri, sırasıyla, veri destekli ve veri desteksiz kestirim için elde edilmiştir. Bu makalede, kestiricilerin başarımı beklenti ve değişinti büyüklükleri yoluyla analitik olarak incelenmiştir. Sonuçlar kestiricilerin başarımlarının verimliliğini göstermektedir.

Anahtar Kelimeler: İşaret Gürültü Oranı Kestirimi, En Büyük Olabilirlik, Rayleigh Kanalları, Evre Uyumsuz BFSK.

1. INTRODUCTION

Diversity combining has long been recognised as a powerful technique for mitigating the destructive effects of channel fading, and can be implemented either coherently or noncoherently (Proakis, 1983). Noncoherent detection is normally desired when the transmission channel is such that reliable carrier recovery is difficult or impractical to obtain. It is well-known (Simon and Alouini, 2000) that the optimum receiver for the noncoherent communication employs square-law detection in each diversity channel, and applies weights to the output of each diversity channel determined from the average fading signal-to-noise ratio (SNR) estimated in practice from measurement on each channel before combining them. Since the combined system performance depends on the SNR estimates in each branch (Simon and Alouini, 2003), it is therefore of great interest to find its actual maximum likelihood estimators and assess their performance against the Cramér-Rao bound (CRB) (Kay, 1993), a well-known lower bound for the variance of any unbiased estimator for a given observation data.

In Dilaveroğlu and Ertaş (2005), we obtained three possible CRB expressions for the data-aided (DA) and the non-data-aided (NDA) SNR estimation of noncoherent binary frequency-shift-keying (NCBFSK) modulated signals with square-law detection, and provided the corresponding maximum likelihood estimators (MLEs) for the DA and the NDA estimation. Due to the restrictions on the length of the manuscript, derivation and therefore the proof of the mean and variance of the MLEs have not been unfortunately included. In this paper, we derive the mean and variance of the MLEs for the NDA and the DA estimation given in Dilaveroğlu and Ertaş (2005), and compare their performance against the corresponding CRBs derived therein.

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2. SYSTEM DEFINITIONS

Assuming a flat slowly Rayleigh fading channel with a receiver using square-law detection for NCBFSK signalling (Simon and Alouini, 2000), three data sets (models) out of many are adopted in Dilaveroğlu and Ertaş (2005), from which the average SNR may be estimated. Data models are referred to as Data Model 1 (DM1), Data Model 2 (DM2), and Data Model 3 (DM3), which are obtained from appropriately selected stages of the square-law detector and can be written as:

$$\text{DM1: } \underline{v}_k = \mathrm{Re}\{\underline{w}_k\},\ k = 1, 2, ..., K, \tag{1}$$

$$\text{DM2: } \underline{v}_k = \left[|u_k|^2, |z_k|^2\right]^T,\ k = 1, 2, ..., K, \tag{2}$$

$$\text{DM3: } \underline{v}_k = |u_k|^2 - |z_k|^2,\ k = 1, 2, ..., K, \tag{3}$$

respectively, where $\underline{w}_k = [u_k, z_k]^T = \sqrt{\gamma}\,\underline{s}_k \alpha_k e^{j\theta_k} + \underline{n}_k$, $\gamma = E_b/N_0$ is the average SNR in which E_b is the transmitted bit energy and N_0 is the spectral density of the additive white Gaussian noise (AWGN), $\underline{s}_k = [1, 0]^T$ or $[0, 1]^T$ with equal probability, assuming sufficient channel interleaving, α_k is a Rayleigh distributed fading amplitude with $E\{(\alpha_k)^2\} = 1$, θ_k is the random phase uniformly distributed in $[0, 2\pi]$, $\underline{n}_k = \left[n_{c1,k} + j\, n_{s1,k}, n_{c2,k} + j\, n_{s2,k}\right]^T$ is the complex noise vector in which $n_{c1,k}$, $n_{s1,k}$, $n_{c2,k}$, and $n_{s2,k}$ are i.i.d. zero-mean Gaussian random variables with variance $1/2$. $\underline{s}_k$, α_k, θ_k, and $\underline{n}_k$ are all independent of each other. Also, the $\underline{v}_k$'s for $k = 1, 2, ..., K$ are assumed to be i.i.d. random vectors (of size 2×1 or 1×1 depending on the data model). The average received SNR is $\frac{E_b}{N_0} E\{(\alpha_k)^2\} = \frac{E_b}{N_0} = \gamma$, and our interest is to find an unbiased estimator of γ, by using the observed data $\{\underline{v}_k\}_{k=1}^K$ for each data model above. We denote the CRB on the variance of any unbiased estimator of γ by CRB_γ.

3. CRAMÉR-RAO BOUNDS AND MAXIMUM LIKELIHOOD ESTIMATORS

Given the kth data sample $\underline{v}_k = [x_k, y_k]^T$ in (1) and (2), and $\underline{v}_k = [v_k]$ in (3), the likelihood functions of γ for the $\underline{v}_k$ respectively for the data sets (1)-(3) in the case of NDA estimation are given as (Dilaveroğlu and Ertaş, 2005)

$$p_{NDA,DM1}(\underline{v}_k; \gamma) = \frac{1}{2\pi\sqrt{1+\gamma}}\left[\exp\left(-\frac{x_k^2}{1+\gamma} - y_k^2\right) + \exp\left(-x_k^2 - \frac{y_k^2}{1+\gamma}\right)\right], \tag{4}$$

$$p_{NDA,DM2}(\underline{v}_k; \gamma) = \frac{1}{2(1+\gamma)}\left[\exp\left(-\frac{x_k}{1+\gamma} - y_k\right) + \exp\left(-x_k - \frac{y_k}{1+\gamma}\right)\right], \tag{5}$$

$$p_{NDA,DM3}(\underline{v}_k; \gamma) = \frac{1}{2(2+\gamma)}\left[\exp(-|v_k|) + \exp\left(-\frac{|v_k|}{1+\gamma}\right)\right], \tag{6}$$

with the corresponding CRB_γ's as

$$\mathrm{CRB}_{\gamma,NDA,DM1} = \frac{1}{K}\left[\frac{1}{2(1+\gamma)^2} - \frac{f(\gamma)}{4\pi(1+\gamma)^{9/2}}\right]^{-1}, \tag{7}$$

$$\mathrm{CRB}_{\gamma,NDA,DM2} = \frac{1}{K}\left[\frac{1}{(1+\gamma)^2} - \frac{g(\gamma)}{4(1+\gamma)^5}\right]^{-1}, \tag{8}$$

$$\mathrm{CRB}_{\gamma,NDA,DM3} = \frac{1}{K}\left[\frac{\zeta\left(3,\frac{1}{2\gamma}\right)-\zeta\left(3,\frac{1+\gamma}{2\gamma}\right)}{4\gamma^3(1+\gamma)(2+\gamma)} - \frac{1}{(2+\gamma)^2}\right]^{-1}, \tag{9}$$

respectively, where

$$f(\gamma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left(x^2-y^2\right)^2 \exp\left[-\frac{2+\gamma}{2(1+\gamma)}\left(x^2+y^2\right)\right]}{\cosh\left[\frac{\gamma}{2(1+\gamma)}\left(x^2-y^2\right)\right]} dx\, dy,$$

$$g(\gamma) = \int_{0}^{\infty}\int_{0}^{\infty} \frac{(x-y)^2 \exp\left[-\frac{2+\gamma}{2(1+\gamma)}(x+y)\right]}{\cosh\left[\frac{\gamma}{2(1+\gamma)}(x-y)\right]} dx\, dy,$$

and $\zeta(s,a) = \sum_{k=0}^{\infty}(k+a)^{-s}$ is the generalised Riemann's zeta function. For the DA estimation, eliminating the dependency of the $\underline{v}_k$ on the $\underline{s}_k$ gives rise to a scalar $\underline{v}_k = [v_k]$ for all the data models DM1-DM3, and the likelihood functions are obtained as (Dilaveroğlu and Ertaş, 2005)

$$p_{DA,DM1}(\underline{v}_k;\gamma) = \frac{1}{\sqrt{\pi}\sqrt{1+\gamma}}\exp\left(-\frac{v_k^2}{1+\gamma}\right), \tag{10}$$

$$p_{DA,DM2}(\underline{v}_k;\gamma) = \frac{1}{(1+\gamma)}\exp\left(-\frac{v_k}{1+\gamma}\right), \tag{11}$$

$$p_{DA,DM3}(\underline{v}_k;\gamma) = \begin{cases} \frac{1}{(2+\gamma)}\exp(v_k), & v_k < 0, \\ \frac{1}{(2+\gamma)}\exp\left(-\frac{v_k}{1+\gamma}\right), & v_k \geq 0, \end{cases} \tag{12}$$

with the corresponding CRB_{γ}'s given as

$$\mathrm{CRB}_{\gamma,DA,DM1} = \frac{2(1+\gamma)^2}{K}, \tag{13}$$

$$\mathrm{CRB}_{\gamma,DA,DM2} = \frac{(1+\gamma)^2}{K}, \tag{14}$$

$$\mathrm{CRB}_{\gamma,DA,DM3} = \frac{(1+\gamma)(2+\gamma)^2}{K(3+\gamma)}. \tag{15}$$

The MLE $\hat{\gamma}$ of γ is obtained from the maximisation of the applicable (NDA or DA) likelihood function $\prod_{k=1}^{K} p(\underline{v}_k;\gamma)$ with respect to γ. It is unfortunately prohibited to find a closed-form expression for the $\hat{\gamma}$ for the NDA estimation. However, an approximate MLE can be obtained in closed-form for a sufficiently large γ. For the DM1-DM3, the approximate MLEs are obtained as (Dilaveroğlu and Ertaş, 2005)

$$\hat{\gamma}_{NDA,DM1} \cong \frac{2}{K}\sum_{k=1}^{K}\frac{x_k^2}{1+e^{-x_k^2+y_k^2}} + \frac{y_k^2}{1+e^{x_k^2-y_k^2}} - 1, \tag{16}$$

$$\hat{\gamma}_{NDA,DM2} \cong \frac{1}{K}\sum_{k=1}^{K}\frac{x_k}{1+e^{-x_k+y_k}}+\frac{y_k}{1+e^{x_k-y_k}}-1, \tag{17}$$

$$\hat{\gamma}_{NDA,DM3} \cong \frac{T+\sqrt{T(T+4K)}}{2K}-1,\; T=\sum_{k=1}^{K}\frac{|v_k|}{1+e^{-|v_k|}}, \tag{18}$$

for the NDA estimation. Fortunately, for the DA estimation, the true MLEs can be obtained as (Dilaver-oğlu and Ertaş, 2005)

$$\hat{\gamma}_{DA,DM1} = \frac{2}{K}\sum_{k=1}^{K}v_k^2-1, \tag{19}$$

$$\hat{\gamma}_{DA,DM2} = \frac{1}{K}\sum_{k=1}^{K}v_k-1, \tag{20}$$

$$\hat{\gamma}_{DA,DM3} = \frac{T+\sqrt{T(T+4K)}}{2K}-1,\; T=\sum_{k:v_k>0}v_k\,. \tag{21}$$

4. MEAN AND VARIANCE OF THE APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATORS FOR THE NDA ESTIMATION

In this section we derive the mean and variance of the approximate MLEs (16)-(18) for the NDA estimation of γ for data models DM1-DM3, respectively. However, due to the complexity of the likelihood functions and the MLEs for the data models, it is unfortunately prohibited to obtain the results in closed forms. We therefore present the results most conveniently in the form of plots.

We begin with the data model DM1. Let us define an auxiliary random variable as

$$r_k(x_k,y_k)=\frac{x_k^2}{1+e^{-x_k^2+y_k^2}}+\frac{y_k^2}{1+e^{x_k^2-y_k^2}}\,.$$

Since the $\underline{v}_k=[x_k,y_k]^T$'s are i.i.d., the r_k 's are also i.i.d. Thus, we get the mean and variance of the MLE $\hat{\gamma}_{NDA,DM1}$ as

$$E\{\hat{\gamma}_{NDA,DM1}\}=\frac{2}{K}\sum_{k=1}^{K}E\{r_k\}-1=2E\{r_k\}-1, \tag{22}$$

$$\mathrm{var}\{\hat{\gamma}_{NDA,DM1}\}=\frac{4}{K^2}\sum_{k=1}^{K}\mathrm{var}\{r_k\}=\frac{4}{K}\mathrm{var}\{r_k\}, \tag{23}$$

where the mean and variance of r_k are, of course, given by

$$E\{r_k\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}r_k(x,y)p_{NDA,DM1}(x,y;\gamma)\,dx\,dy,$$

$$\mathrm{var}\{r_k\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[r_k(x,y)-E\{r_k\}]^2\,p_{NDA,DM1}(x,y;\gamma)\,dx\,dy.$$

However, the above integrals need to be evaluated numerically.

For the data model DM2, if we similarly define

$$r_k(x_k,y_k)=\frac{x_k}{1+e^{-x_k+y_k}}+\frac{y_k}{1+e^{x_k-y_k}},$$

the mean and variance of the MLE $\hat{\gamma}_{NDA,DM2}$ become

$$E\{\hat{\gamma}_{NDA,DM2}\}=\frac{1}{K}\sum_{k=1}^{K}E\{r_k\}-1=E\{r_k\}-1, \tag{24}$$

$$\text{var}\{\hat{\gamma}_{NDA,DM2}\} = \frac{1}{K^2}\sum_{k=1}^{K}\text{var}\{r_k\} = \frac{1}{K}\text{var}\{r_k\}, \tag{25}$$

where, again, the integrals

$$E\{r_k\} = \int_0^\infty \int_0^\infty r_k(x,y) p_{NDA,DM2}(x,y;\gamma)\,dx\,dy,$$

$$\text{var}\{r_k\} = \int_0^\infty \int_0^\infty [r_k(x,y) - E\{r_k\}]^2 p_{NDA,DM2}(x,y;\gamma)\,dx\,dy,$$

are evaluated numerically.

For the data model DM3, we first compute the mean and variance of

$$r_k(v_k) = \frac{|v_k|}{1+e^{-|v_k|}}$$

by numerically evaluating the integrals

$$E\{r_k\} = \int_{-\infty}^{\infty} r_k(x) p_{NDA,DM3}(x;\gamma)\,dx,$$

$$\text{var}\{r_k\} = \int_{-\infty}^{\infty} [r_k(x) - E\{r_k\}]^2 p_{NDA,DM3}(x;\gamma)\,dx.$$

Then, the mean and variance of the random variable T in (18) are calculated by

$$E\{T\} = KE\{r_k\},$$

$$\text{var}\{T\} = K\,\text{var}\{r_k\}.$$

Now, if we assume that the probability density function (PDF) of T is negligible outside a small interval about its mean, the mean and variance of the MLE $\hat{\gamma}_{NDA,DM3}$ can be approximated as (Papoulis, 1991, p. 112)

$$E\{\hat{\gamma}_{NDA,DM3}\} \cong \left.\frac{T+\sqrt{T(T+4K)}}{2K} - 1\right|_{T=E\{T\}}, \tag{26}$$

$$\text{var}\{\hat{\gamma}_{NDA,DM3}\} \cong \left\{\frac{d}{dT}\left[\frac{T+\sqrt{T(T+4K)}}{2K} - 1\right]\Bigg|_{T=E\{T\}}\right\}^2 \text{var}\{T\}. \tag{27}$$

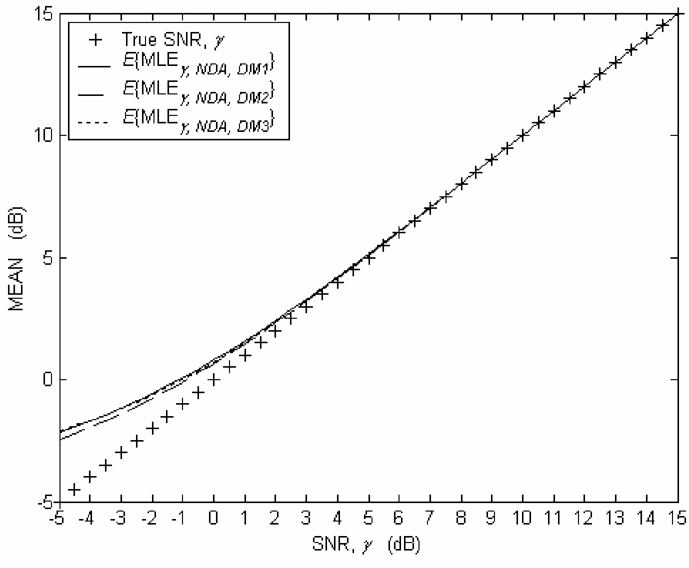

Figure 1:
Mean of the approximate MLEs for the NDA estimation for the three data models.

We computed the mean and variance of the MLEs for SNR values taken between $-5\,\mathrm{dB}$ and $15\,\mathrm{dB}$ with increments of $0.5\,\mathrm{dB}$. The means, (22), (24) and (26), are shown in Figure 1 and the variances, (23), (25) and (27), together with the corresponding CRBs from Dilaveroğlu and Ertaş (2005), repeated here in (7)-(9), are shown in Figure 2. We observe from the figures that the estimators are practically unbiased and efficient, i.e., achieve the corresponding CRBs, for all SNR values greater than or equal to, say, $3\,\mathrm{dB}$. Note that a similar conclusion has also been drawn in Dilaveroğlu and Ertaş (2005) based on the simulations performed therein. Also, note that the abovementioned assumption on the PDF of the random variable T for the data model DM3 becomes valid for $\gamma \geq 3\,\mathrm{dB}$. This follows from the related analytical results presented in the figures here and the corresponding simulation results given in (Dilaveroğlu and Ertaş, 2005, Table 1).

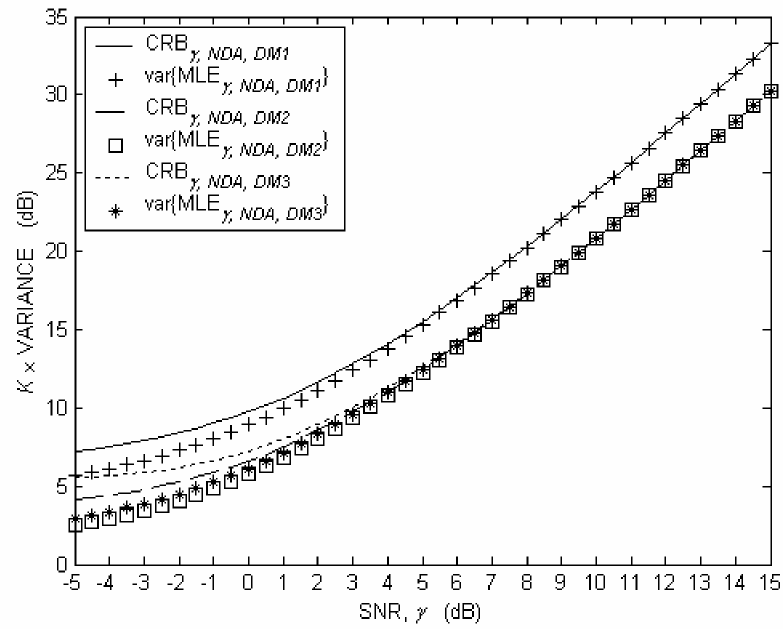

Figure 2:
Variance of the approximate MLEs for the NDA estimation for the three data models.

5. MEAN AND VARIANCE OF THE EXACT MAXIMUM LIKELIHOOD ESTIMATORS FOR THE DA ESTIMATION

We next consider the DA estimation case. Fortunately, for this case we can get closed-form expressions for the mean and variance of the exact MLEs given in (19)-(21) for the data models DM1-DM3, respectively. We shall show that the MLEs are unbiased and efficient.

For the data model DM1, since the v_k 's are i.i.d. random variables, we have

$$\begin{aligned} E\{\hat{\gamma}_{DA,DM1}\} &= \frac{2}{K}\sum_{k=1}^{K} E\{v_k^2\} - 1 = 2E\{v_k^2\} - 1 = 2\left(\frac{1+\gamma}{2}\right) - 1 \\ &= \gamma, \end{aligned} \tag{28}$$

$$\begin{aligned} \mathrm{var}\{\hat{\gamma}_{DA,DM1}\} &= \frac{4}{K^2}\sum_{k=1}^{K} \mathrm{var}\{v_k^2\} = \frac{4}{K}\mathrm{var}\{v_k^2\} \\ &= \frac{4}{K}\left[\frac{(1+\gamma)^2}{2}\right] = \frac{2(1+\gamma)^2}{K} \\ &= \mathrm{CRB}_{\gamma,DA,DM1}, \end{aligned} \tag{29}$$

c.f. (13). Thus, the MLE $\hat{\gamma}_{DA,DM1}$ is unbiased and attains the corresponding CRB regardless of the value of γ.

Similarly, for DM2

$$E\{\hat{\gamma}_{DA,DM2}\} = \frac{1}{K}\sum_{k=1}^{K} E\{v_k\} - 1 = E\{v_k\} - 1 = 1 + \gamma - 1 \\ = \gamma, \tag{30}$$

$$\operatorname{var}\{\hat{\gamma}_{DA,DM2}\} = \frac{1}{K^2}\sum_{k=1}^{K} \operatorname{var}\{v_k\} = \frac{1}{K}\operatorname{var}\{v_k\} \\ = \frac{1}{K}\left[(1+\gamma)^2\right] = \frac{(1+\gamma)^2}{K} \\ = \mathrm{CRB}_{\gamma,DA,DM2}, \tag{31}$$

c.f. (14). Hence the MLE $\hat{\gamma}_{DA,DM2}$ is also an unbiased and efficient estimator of γ for all γ.

For the data model DM3, we first calculate the mean and variance of the random variable T in (21). To this end, let A_l for $l = 0,1,\ldots,K$ denote the event that l out of the K random variables $v_1, v_2, \ldots, v_K$ are positive. Also, define

$$m_0 = \int_{-\infty}^{0} p_{DA,DM3}(x;\gamma)dx, \quad M_0 = \int_{0}^{\infty} p_{DA,DM3}(x;\gamma)dx,$$
$$M_1 = \int_{0}^{\infty} x p_{DA,DM3}(x;\gamma)dx, \quad M_2 = \int_{0}^{\infty} x^2 p_{DA,DM3}(x;\gamma)dx.$$

Then

$$\begin{aligned}
E\{T\} &= \sum_{l=0}^{K} E\{T/A_l\}\Pr\{A_l\} \\
&= \binom{K}{1} M_1 m_0^{K-1} + \binom{K}{2} 2M_1 M_0 m_0^{K-2} + \ldots + \\
&\quad + \binom{K}{K-1}(K-1) M_1 M_0^{K-2} m_0 + \binom{K}{K} K M_1 M_0^{K-1} \\
&= M_1 \sum_{l=1}^{K} \binom{K}{l} l M_0^{l-1} m_0^{K-l} = M_1 K \sum_{l=0}^{K-1} \binom{K-1}{l} M_0^{l} m_0^{K-1-l} \\
&= M_1 K (M_0 + m_0)^{K-1} = M_1 K \\
&= \frac{K(1+\gamma)^2}{2+\gamma},
\end{aligned}$$

$$\begin{aligned}
E\{T^2\} &= \sum_{l=0}^{K} E\{T^2/A_l\}\Pr\{A_l\} \\
&= \binom{K}{1} M_2 m_0^{K-1} + \binom{K}{2}\left(2M_2 M_0 + 2M_1^2\right) m_0^{K-2} \\
&\quad + \binom{K}{3}\left(3M_2 M_0^2 + 6M_1^2 M_0\right) m_0^{K-3} + \ldots + \\
&\quad + \binom{K}{K-1}\left[(K-1) M_2 M_0^{K-2} + (K-1)(K-2) M_1^2 M_0^{K-3}\right] m_0 \\
&\quad + \binom{K}{K}\left[K M_2 M_0^{K-1} + K(K-1) M_1^2 M_0^{K-2}\right] \\
&= M_2 \sum_{l=1}^{K} \binom{K}{l} l M_0^{l-1} m_0^{K-l} + M_1^2 \sum_{l=2}^{K} \binom{K}{l} l(l-1) M_0^{l-2} m_0^{K-l} \\
&= M_2 K \sum_{l=0}^{K-1} \binom{K-1}{l} M_0^{l} m_0^{K-1-l} + M_1^2 K(K-1) \sum_{l=0}^{K-2} \binom{K-2}{l} M_0^{l} m_0^{K-2-l} \\
&= M_2 K (M_0 + m_0)^{K-1} + M_1^2 K(K-1)(M_0 + m_0)^{K-2} \\
&= M_2 K + M_1^2 K(K-1),
\end{aligned}$$

and, thus

$$\begin{aligned}\mathrm{var}\{T\} &= E\{T^2\} - [E\{T\}]^2 \\ &= K\left(M_2 - M_1^2\right) \\ &= \frac{K(1+\gamma)^3(3+\gamma)}{(2+\gamma)^2}.\end{aligned}$$

Now, under the assumption that the PDF of T is concentrated near its mean, the mean and variance of the MLE $\hat{\gamma}_{DA,DM3}$ are related to those of the T as

$$\begin{aligned}E\{\hat{\gamma}_{DA,DM3}\} &\cong \left.\frac{T+\sqrt{T(T+4K)}}{2K} - 1\right|_{T=E\{T\}} \\ &= \gamma,\end{aligned} \tag{32}$$

$$\begin{aligned}\mathrm{var}\{\hat{\gamma}_{DA,DM3}\} &\cong \left\{ \frac{d}{dT}\left[\frac{T+\sqrt{T(T+4K)}}{2K} - 1\right]\Bigg|_{T=E\{T\}} \right\}^2 \mathrm{var}\{T\} \\ &= \frac{(1+\gamma)(2+\gamma)^2}{K(3+\gamma)} \\ &= \mathrm{CRB}_{\gamma,DA,DM3},\end{aligned} \tag{33}$$

c.f. (15). So, $\hat{\gamma}_{DA,DM3}$ is unbiased and attains the CRB under the abovementioned assumption. The fact that this assumption is valid for all practical values of the SNR follows from the simulation results given in (Dilaveroğlu and Ertaş, 2005, Table 2), where the SNR values were considered in the range from -3 dB to 15 dB.

6. CONCLUSIONS

The performances of the MLEs proposed in Dilaveroğlu and Ertaş (2005) for the data-aided and the non-data-aided estimation of the SNR for noncoherent binary frequency-shift-keying signals in Rayleigh fading channels have been investigated. Analytical results show that the proposed MLEs for the DA estimation are unbiased and efficient regardless of the value of SNR. However, the MLEs proposed for the NDA estimation are unbiased and efficient only for some moderate to large values of SNR, such as 3 dB onwards.

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