THE CRAMÉR-RAO BOUND FOR DAMPED AND UNDAMPED SINUSOIDS IN GAUSSIAN NOISE

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Abstract: A new expression for a Fisher information matrix for the problem of estimating the parameters of damped or undamped sinusoidal signals in Gaussian noise is derived for both complex and real valued time series data cases. The expression in each data case is in such a form that some relations between the Cramér-Rao bound and the signal parameters are easily seen.

Key Words: Cramér-Rao bound, Sinusoidal signals, Fisher information matrix.

Gauss Gürültü İçindeki Sönümlü ve Sönümsüz Sinüslere İlişkin Cramér-Rao Sınırı

Özet: Gauss gürültü içindeki sönümlü ya da sönümsüz sinüzoidal sinyallerin parametrelerinin kestirimi problemine ilişkin bir Fisher bilgi matrisinin yeni bir ifadesi hem kompleks hem de reel değerli zaman serisi veri durumları için çıkarılmıştır. Her bir veri durumundaki matris ifadesi Cramér-Rao sınırı ile sinyal parametreleri arasındaki bazı ilişkilerin kolayca görüleceği bir biçimdedir.

Anahtar Kelimeler: Cramér-Rao sınırı, Sinüzoidal sinyaller, Fisher bilgi matrisi.

1. INTRODUCTION

The problem of estimating the parameters (amplitudes, phases, damping factors and frequencies) of sinusoidal signals in Gaussian noise is considered for both complex and real valued time series data cases.

The Cramér-Rao (C-R) bound provides a lower bound on the variance of any unbiased estimator of a nonrandom parameter. It is often used to investigate the optimality of parametric estimators. The C-R bound is calculated by inverting a Fisher information matrix for the estimation problem under consideration (Kay, 1993).

For the complex data case and when the noise is white Hua and Sarkar (1990) provided a useful expression for a Fisher information matrix that reveals the dependence of the C-R parameter bounds on some signal parameters. However, their expression is not readily applicable to the real data case. The real data case is probably more common in practice.

In this paper, we extend the work of Hua and Sarkar to the real data and colored noise cases. Our approach differs from their approach in that we introduce a decomposition of the Fisher information matrix which is applicable to both complex and real data cases.

The complex data case is considered in Section 2. The real data case is discussed in Section 3.

2. THE COMPLEX DATA

The complex data sequence is described by

$$\begin{aligned} y_k &= x_k + n_k \\ &= \sum_{i=1}^{M} \alpha_i e^{\beta_i k} e^{j(\omega_i k + \varphi_i)} + n_k \end{aligned} \tag{1}$$

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$k = 0,1,\ldots, N-1$. n_k's are the noise. α_i's and φ_i's are the amplitudes and the phases, respectively. β_i's and ω_i's are the damping factors and the frequencies, respectively. M is the number of sinusoids. The signal parameter vector θ defined as

$$\theta = [\alpha_1, \varphi_1, \beta_1, \omega_1, \alpha_2, \ldots, \omega_M]^T \tag{2}$$

is to be estimated from the data vector $y = [y_0, y_1, \ldots, y_{N-1}]^T$. If the probability density function (pdf) of $n = [n_0, n_1, \ldots, n_{N-1}]^T$ is complex Gauss, i.e., $CN(0, C)$ (e.g., see Kay (1993), p. 507), then the pdf of y is $CN(x, C)$ where $x = [x_0, x_1, \ldots, x_{N-1}]^T$.

Let θ_i denote the ith element of θ. Then the (i, j)th element of the Fisher information matrix J for the estimation problem in (1) and (2) can be shown to be

$$(J)_{i,j} = 2\,\mathrm{Re}\left\{(dx/d\theta_i)^H C^{-1} (dx/d\theta_j)\right\} \tag{3}$$

where $d(\ )/d\theta_i$ is partial derivative. But J can be partitioned as

$$J = \{J_{i,j};\quad i, j = 1,2,\ldots, M\}$$

where $J_{i,j}$ is a 4×4 (i, j)th block matrix of J. It can be shown from (3) that the $J_{i,j}$ can be expressed as

$$J_{i,j} = 2 D_i Q_i X_{i,j} Q_j^T D_j \tag{4}$$

where

$$D_i = \mathrm{diag}\{1, \alpha_i, \alpha_i, \alpha_i\}$$

$$Q_i = \begin{bmatrix} Q_i' & 0 \\ 0 & Q_i' \end{bmatrix},\quad Q_i' = \begin{bmatrix} \cos\varphi_i & \sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{bmatrix}$$

$$X_{i,j} = \mathrm{Re}\{Z_{i,j}\}$$

$$Z_{i,j} = \begin{bmatrix} \zeta_{i,j,0} & j\zeta_{i,j,0} & \zeta_{i,j,1} & j\zeta_{i,j,1} \\ -j\zeta_{i,j,0} & \zeta_{i,j,0} & -j\zeta_{i,j,1} & \zeta_{i,j,1} \\ \zeta_{i,j,2} & j\zeta_{i,j,2} & \zeta_{i,j,3} & j\zeta_{i,j,3} \\ -j\zeta_{i,j,2} & \zeta_{i,j,2} & -j\zeta_{i,j,3} & \zeta_{i,j,3} \end{bmatrix}$$

$$\zeta_{i,j,0} = \psi(z_i)^H C^{-1} \psi(z_j)$$

$$\zeta_{i,j,1} = \psi(z_i)^H C^{-1} \psi'(z_j)$$

$$\zeta_{i,j,2} = \psi'(z_i)^H C^{-1} \psi(z_j)$$

$$\zeta_{i,j,3} = \psi'(z_i)^H C^{-1} \psi'(z_j)$$

$$\psi(z) = [1, z, \ldots, z^{N-1}]^T,\quad \psi'(z) = z\frac{d\psi(z)}{dz}$$

and

$$z_i = \exp(\beta_i + j\omega_i).$$

The decomposition of J in (4) slightly differs from the decomposition employed in (Hua and Sarkar, 1990). We see in the next section that our decomposition, unlike theirs, is also applicable to the real data case.

It can be shown from (4) that the 4×4 (i, j)th block matrix of J^{-1} is

$$J^{i,j} = \frac{1}{2} D_i^{-1} Q_i X^{i,j} Q_j^T D_j^{-1}$$

where $X^{i,j}$ is the 4×4 (i, j)th block matrix of $X^{-1} = \{X_{i,j}\}^{-1}$ (which is independent of α_i's and φ_i's). Here we have used the property that $Q_i^{-1} = Q_i^T$. Note that we also have

$$\begin{aligned} J^{i,j} &= \frac{1}{2} D_i^{-1} X^{i,j} Q_i Q_j^T D_j^{-1} \\ &= \frac{1}{2} D_i^{-1} Q_i Q_j^T X^{i,j} D_j^{-1}. \end{aligned}$$

Thus, the ith diagonal block matrix of J^{-1} is

$$J^{i,i} = \frac{1}{2} D_i^{-1} X^{i,i} D_i^{-1}.$$

Since the 4 diagonal elements of $J^{i,i}$ are the C-R bounds for α_i, φ_i, β_i and ω_i, respectively, the following results can be stated:

R1: The C-R bounds for φ_i, β_i and ω_i are independent of α_j for j not equal to i but proportional to $1/\alpha_i^2$, the bound for α_i is independent of α_j for all j.

R2: The bounds for all parameters are independent of phases φ_j for all j.

R3: If the noise is white, i.e., C is diagonal, the bounds are independent of the group shift of frequencies; they depend upon the frequencies only through their differences $\omega_i - \omega_j$.

If β_i's are known (e.g., $\beta_i = 0$ for all i—the undamped sinusoid case), the results R1 and R3 are still valid but the result R2 no longer holds. The C-R bounds now depend upon the phases (but not the group shift of the phases). This is because a symmetry in (4) is destroyed when β_i's are known.

3. THE REAL DATA

The real data sequence is described by

$$\begin{aligned} y_k &= x_k + n_k \\ &= \sum_{i=1}^{M} \alpha_i e^{\beta_i k} \cos(\omega_i k + \varphi_i) + n_k \end{aligned} \tag{5}$$

$k = 0,1,\ldots,N-1$. Here, the pdf of $n = [n_0, n_1, \ldots, n_{N-1}]^T$ is real Gauss, i.e., $N(0, C)$, and hence the pdf of $y = [y_0, y_1, \ldots, y_{N-1}]^T$ is $N(x, C)$ where $x = [x_0, x_1, \ldots, x_{N-1}]^T$. The (i, j)th element of the Fisher information matrix J for estimation of the signal parameters in (5) can be shown to be

$$(J)_{i,j} = (dx/d\theta_i)^T C^{-1}(dx/d\theta_j). \tag{6}$$

We can show from (6) that the 4×4 (i, j)th block matrix of J can be expressed as

$$J_{i,j} = \frac{1}{2} D_i Q_i X'_{i,j} Q_j^T D_j \tag{7}$$

where

$$X'_{i,j} = \mathrm{Re}\{Z_{i,j} + Z'_{i,j}\}$$

$Z_{i,j}$ is as given before, and

$$Z'_{i,j} = \begin{bmatrix} \zeta'_{i,j,0} & j\zeta'_{i,j,0} & \zeta'_{i,j,1} & j\zeta'_{i,j,1} \\ j\zeta'_{i,j,0} & -\zeta'_{i,j,0} & j\zeta'_{i,j,1} & -\zeta'_{i,j,1} \\ \zeta'_{i,j,2} & j\zeta'_{i,j,2} & \zeta'_{i,j,3} & j\zeta'_{i,j,3} \\ j\zeta'_{i,j,2} & -\zeta'_{i,j,2} & j\zeta'_{i,j,3} & -\zeta'_{i,j,3} \end{bmatrix}$$

$$\zeta'_{i,j,0} = \psi(z_i)^T C^{-1}\psi(z_j)$$
$$\zeta'_{i,j,1} = \psi(z_i)^T C^{-1}\psi'(z_j)$$
$$\zeta'_{i,j,2} = \psi'(z_i)^T C^{-1}\psi(z_j)$$
$$\zeta'_{i,j,3} = \psi'(z_i)^T C^{-1}\psi'(z_j).$$

Comparing (4) with (7) we see that the decomposition technique we have employed for the complex data case directly carries over to the real data case.

The 4×4 (i, j)th block matrix of J^{-1} can be shown from (7) to be

$$J^{i,j} = 2D_i^{-1} Q_i X'^{i,j} Q_j^T D_j^{-1}$$

where $X'^{i,j}$ is the 4×4 (i, j)th block matrix of $X'^{-1} = \{X'_{i,j}\}^{-1}$ (which is independent of α_i's and φ_i's). It can be shown that $X'^{i,j}$, unlike $X^{i,j}$, does not commute with Q_i and Q_j^T in general, and thus the ith diagonal block matrix of J^{-1} in the real data case is

$$J^{i,i} = 2D_i^{-1} Q_i X'^{i,i} Q_i^T D_i^{-1}.$$

We have the following results:

R4: The C-R bounds for φ_i, β_i and ω_i are independent of α_j for j not equal to i but proportional to $1/\alpha_i^2$, the bound for α_i is independent of α_j for all j.

R5: The bounds for α_i, φ_i, β_i and ω_i depend upon the phase φ_i but are independent of φ_j for j not equal to i.

R6: If C is diagonal, the bounds depend upon the frequencies only through $\omega_i \pm \omega_j$.

If β_i's are known, the results R4 and R6 still apply but the result R5 becomes no longer valid. In this case, the C-R bounds depend upon the phases φ_j for all j (but only through $\varphi_i \pm \varphi_j$).

4. CONCLUSIONS

We have introduced a new decomposition of a Fisher information matrix for the problem of estimating the parameters of sinusoidal signals in the presence of (white or colored) Gaussian noise. It differs from a previous one in that it is applicable to both complex and real valued data. The decomposition reveals clearly the dependence of the C-R bound on some signal parameters in each data case.

5. REFERENCES

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