

SIMILARITY SOLUTION OF THE OSCILLATORY PRESSURE DRIVEN FULLY DEVELOPED FLOW IN A CHANNEL

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Abstract: Two dimensional channel flow is considered under the action of periodic oscillatory pressure field. The flow is assumed to be fully developed laminar and incompressible, so that dimensionless form of the momentum equation has been solved analytically by using similarity transformation. Variations of the velocity profiles and skin friction coefficient over a cycle have been obtained together with behavior of the flow for various oscillation frequencies. Results were also compared to numerical values of the two dimensional laminar flow equations, based on the finite volume technique.

Key Words: Oscillatory flow, Channel flow, Unsteady flow, Analytical solution.

Salınımlı Basınca Maruz Tam Gelişmiş Kanal Akışının Benzerlik Transformasyonu İle Çözümü

Bu çalışmada, periyodik olarak değişen basınca maruz iki boyutlu bir akış ele alınmıştır. Tam gelişmiş, laminar ve sıkıştırılamaz akış için yazılan momentum denkleminin benzerlik transformasyonu ile analitik çözümü yapılmıştır. Bir çevrim boyunca hız profillerinin ve sürtünme katsayısının değişimleri elde edilmiş, akışın davranışı değişik salınımlı frekans için analiz edilmiştir. Ayrıca, elde edilen analitik çözüm sonuçları, iki boyutlu laminar akışın sonlu hacimler tekniğine dayanan sayısal çözüm sonuçları ile mukayese edilmiştir.

Anahtar Kelimeler: Salınımlı akış, Kanal akışı, Geçici rejim, Analitik çözüm.

1. INTRODUCTION

A specific type of unsteady flows includes oscillatory and pulsatile flows which occur in biological systems, such as human respiratory and vascular systems, as well as in many engineering areas, for example, the flow in hydraulic and pneumatic systems, ic engines, pumping systems or applications of heat transfer.

Oscillatory and pulsating flows in branching pipes have been extensively studied by investigators concerned especially in biology. Additionally, a number of work have been reported in literature concerning oscillatory or pulsatile flows in straight pipes (see for example, Womersley 1955, Uchida 1956, Gerrard and Hughes 1971, Kerczek and Davis, 1974, Schneck and Ostrach 1975, Hino et al. 1976, Muto and Nakane 1980, Shemer et al. 1985, Elad et al. 1998)

The present study is dealt with the oscillatory channel flows of which most of the studies in literature devoted to heat transfer characteristics. Recently, numerical analysis of pulsating flow between parallel plates was carried out by Faghri et al. (1980), showing that flow pulsations do not produce any significant effect on time averaged Nusselt number when no flow reversal exists, and by Kim et al. (1993) who focused on thermally developing region and reported that both heat transfer enhancement and reduction could be expected in various axial locations of the channel. Enhancement of longitudinal heat conduction (Kurzweg, 1985) and heat convection (Cooper et al. 1994, Liao et al. 1994, Karagoz 1996, Li and Yang 2000) have been also investigated numerically and experimentally for oscillatory channel flow, who showed significantly enhanced heat transfer depending on the flow conditions.

On the other hand, two-phase flow and heat transfer were also considered in channels and pipes by Gidaspow (1986) and Chamkha (2000), and results were presented in terms of volumetric flow rates and skin friction coefficients in order to show the effects of particle loading and viscosity ratio on the solution.

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The aim of the present study is to solve laminar flow subjected to pressure field varying sinusoidally with time in a 2 dimensional channel. Solution of the governing equations was carried out analytically for fully developed flow by using similarity transformation, and numerically for two dimensional flow employing finite volume solution technique. Behavior of the flow was also analyzed for different oscillation frequencies.

2. MATHEMATICAL FORMULATION

Consider unsteady, laminar, 2-D flow in a horizontal channel under the action of an arbitrary time-varying longitudinal pressure gradient (Figure 1). Based on the assumption of constant properties, conservation of mass and momentum equations can be written in Cartesian coordinates as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3)$$

where u and v represent the x and y velocity components, ρ and μ are the density and viscosity of the fluid, respectively.

2.1 Analytical solution

If this flow is fully developed, governing equations (Eqs. 1-3) can be reduced to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} \quad (4)$$

Assuming that this flow is subjected to a longitudinal harmonic pressure field such as

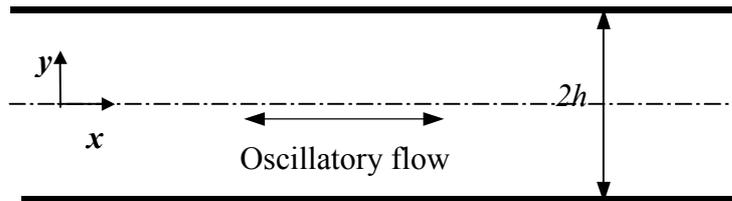


Figure 1:
Sketch of the flow geometry

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = P_0 \cos \omega t \quad (5)$$

where P_0 is the amplitude of pressure gradient parameter and ω is the angular frequency, Eq.4 leads to

$$\frac{\partial u}{\partial t} = P_0 \cos \omega t + \nu \frac{\partial^2 u}{\partial y^2} \quad (6)$$

where ν is the kinematic viscosity. Defining the characteristic parameters h , $1/\omega$ and (P_0/ω) for length, time and velocity, respectively, Eq. (6) can be written in dimensionless form as

$$\frac{\partial u^*}{\partial t^*} = \text{Cost}^* + \frac{\nu}{h^2 \omega} \frac{\partial^2 u^*}{\partial y^{*2}} \quad (7)$$

where

$$t^* = \omega t \quad y^* = \frac{y}{h} \quad u^* = \frac{u}{P_0 / \omega} \quad (8)$$

are the dimensionless time, position and velocity respectively, and h is the half of the channel height. Another nondimensional parameter appears in Eq.7 and can be defined as

$$\alpha = h \sqrt{\frac{\omega}{\nu}} \quad (9)$$

which gives a measure of ratio of inertial to viscous forces.

For similarity solutions, it is assumed that the velocity function has the following form

$$u^* = f(y^*) e^{it^*} \quad (10)$$

Substituting the derivatives of Eq.10 into Eq.7, and knowing that Cost^* is equal to the real part of the right hand side of the following identity

$$e^{it^*} = \text{Cost}^* + i \text{Sint}^* \quad (11)$$

one can obtain the following equation

$$f'' - i\alpha^2 f = \alpha^2 \quad (12)$$

The total solution of this ordinary differential equation can be obtained as

$$f = C_1 e^{-\alpha\sqrt{i}y^*} + C_2 e^{\alpha\sqrt{i}y^*} + i \quad (13)$$

and Eq.10 leads to the velocity function

$$u^* = (C_1 e^{-\alpha\sqrt{i}y^*} + C_2 e^{\alpha\sqrt{i}y^*} + i) e^{it^*} \quad (14)$$

Using the boundary conditions

$$u^* = 0 \text{ at } y^* = -1 \quad (15a)$$

$$u^* = 0 \text{ at } y^* = 1 \quad (15b)$$

for this flow, the integration constants of Eq.13, and then the velocity profile can be obtained as

$$C_1 = C_2 = \frac{e^{\alpha\sqrt{i}} - e^{-\alpha\sqrt{i}}}{e^{-2\alpha\sqrt{i}} - e^{2\alpha\sqrt{i}}} i \quad (17)$$

and

$$u^* = \left(-i \frac{e^{-\alpha\sqrt{i}y^*} + e^{\alpha\sqrt{i}y^*}}{e^{\alpha\sqrt{i}} + e^{-\alpha\sqrt{i}}} + i \right) e^{it^*} \quad (18)$$

respectively. In order to put this equation in appropriate form for easy elaboration, the identity $i + I = \sqrt{2}i$ can be used, so that Eq.18 gives

$$u^* = -i \frac{e^{\frac{\alpha y^*}{\sqrt{2}}} e^{(t^* + \frac{\alpha y^*}{\sqrt{2}})i} + e^{-\frac{\alpha y^*}{\sqrt{2}}} e^{(t^* - \frac{\alpha y^*}{\sqrt{2}})i}}{e^{\frac{\alpha}{\sqrt{2}}} e^{\frac{\alpha}{\sqrt{2}}i} + e^{-\frac{\alpha}{\sqrt{2}}} e^{-\frac{\alpha}{\sqrt{2}}i}} + i e^{it^*} \quad (19)$$

or with the identity of Eq.11, one can calculate the real velocity profile as

$$u^* = \frac{e^{\frac{\alpha y^*}{\sqrt{2}}} \left(c \sin(t^* + \frac{\alpha y^*}{\sqrt{2}}) - d \cos(t^* + \frac{\alpha y^*}{\sqrt{2}}) \right)}{c^2 + d^2} + \frac{e^{-\frac{\alpha y^*}{\sqrt{2}}} \left(c \sin(t^* - \frac{\alpha y^*}{\sqrt{2}}) - d \cos(t^* - \frac{\alpha y^*}{\sqrt{2}}) \right)}{c^2 + d^2} - \sin t^* \quad (20)$$

where

$$c = \left(e^{\frac{\alpha}{\sqrt{2}}} + e^{-\frac{\alpha}{\sqrt{2}}} \right) \cos \frac{\alpha}{\sqrt{2}} \quad (21a)$$

$$d = \left(e^{\frac{\alpha}{\sqrt{2}}} - e^{-\frac{\alpha}{\sqrt{2}}} \right) \sin \frac{\alpha}{\sqrt{2}} \quad (21b)$$

Skin friction coefficient is defined as

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho \left(\frac{P_0}{\omega} \right)^2} \quad (22)$$

where τ_w is the shear stress at the wall and can be calculated for fully developed flow as

$$\tau_w = \rho \nu \frac{\partial u}{\partial y} \quad (23a)$$

Using the dimensionless parameters (Eq. 8), the wall shear stress and the skin friction coefficient can be rewritten in dimensionless form as

$$\tau_w = \frac{\rho P_0 h}{\alpha^2} \frac{\partial u^*}{\partial y^*} \quad (23b)$$

$$C_f = \frac{2\omega^2 h}{P_0 \alpha^2} \frac{\partial u^*}{\partial y^*} \Big|_w \quad (24a)$$

respectively. Deriving the velocity profile at the wall and then inserting it in Eq.24a give the skin friction coefficient in the final form

$$C_f = \frac{2\omega^2 h}{\sqrt{2}P_0\alpha (c^2 + d^2)} q \quad (24b)$$

where

$$q = e^{-\frac{\alpha}{\sqrt{2}}} \left((c+d)\text{Sin}(t^* - \frac{\alpha}{\sqrt{2}}) + (c-d)\text{Cos}(t^* - \frac{\alpha}{\sqrt{2}}) \right) - e^{\frac{\alpha}{\sqrt{2}}} \left((c+d)\text{Sin}(t^* + \frac{\alpha}{\sqrt{2}}) + (c-d)\text{Cos}(t^* + \frac{\alpha}{\sqrt{2}}) \right) \quad (24c)$$

and c and d are given in Eqs. 21a and 21b respectively.

2.2. Numerical solution

Oscillatory two dimensional laminar flow described above was also solved numerically in developing and fully developed regions of the channel by using finite volume technique. The governing equations (Eqs.1-3) were transformed into the finite difference equations by integrating them over control volumes formed by grid lines. A nonuniform staggered grid of 70x30 was used for the predictions, pressure values are stored the nodal points, whereas u and v velocity components were considered at the cell faces. In this transformation, central difference and hybrid scheme are used for the diffusive and convective terms, respectively. Fully implicit resultant linear equations can be written in the following general form

$$A_p \varphi_p = A_N \varphi_N + A_S \varphi_S + A_E \varphi_E + A_W \varphi_W + S \quad (25)$$

where φ_i 's stand for any dependent variable, A_i 's are coefficients combining the effects of convection and diffusion, and they are calculated from the grid sizes, physical properties and dependent variable values known from previous iteration. The quantity S includes any source term such as pressure gradients and the term concerning the time derivatives. This equation relates the value of φ at the grid point P to its four neighboring points, denoted by N , S , E and W . The numerical procedure for the solution of these coupled linear equations is based on the SIMPLE pressure correction iterative procedure. Application of this procedure can be found in literature (see for example Patankar, 1980 and van Doormaal, 1984). The trapezoidal rule is used for the numerical integration necessary for the parameters described above. Details of this numerical solution were given in previous work by Karagoz (1996).

3. RESULTS and DISCUSSION

This work covers unsteady fully developed flow subjected to sinusoidal oscillatory pressure field in a 2-D channel. Analytical solution of the momentum equation was carried out for Newtonian fluids to give time-space variations of velocity profiles. The solutions were obtained for two different frequency parameters, $\alpha = 4$ and 13, by taking air properties at the standard atmospheric conditions. The channel height and the period of the oscillating pressure field were chosen as 7.5 cm, and 3 and 33 s. respectively, in the actual calculations.

The time history of the normalized velocity profiles over a cycle with a time step of $t/T = 1/12$ (T is the period) corresponding to 30 degrees are shown in Figure 2, for $\alpha = 4$. As can be seen from Figure 4, time dependency destroys the parabolic shape of the profiles and the boundary layer thickness varies with time. It should be noted that sign change of velocity is earlier in boundary layer than the core flow.

In order to demonstrate the relationship between pressure and velocity variations, time variation of the pressure is plotted together with the velocities at various stations, $y^* = 0, 0.75, 0.85$ and 0.95 (Figure 4). The phase lag between the driven pressure and velocity varies with y coordinate as can be seen from the figure. The flow near the walls reverses its direction before the core flow as expected, since inertial forces are bigger in the core region.

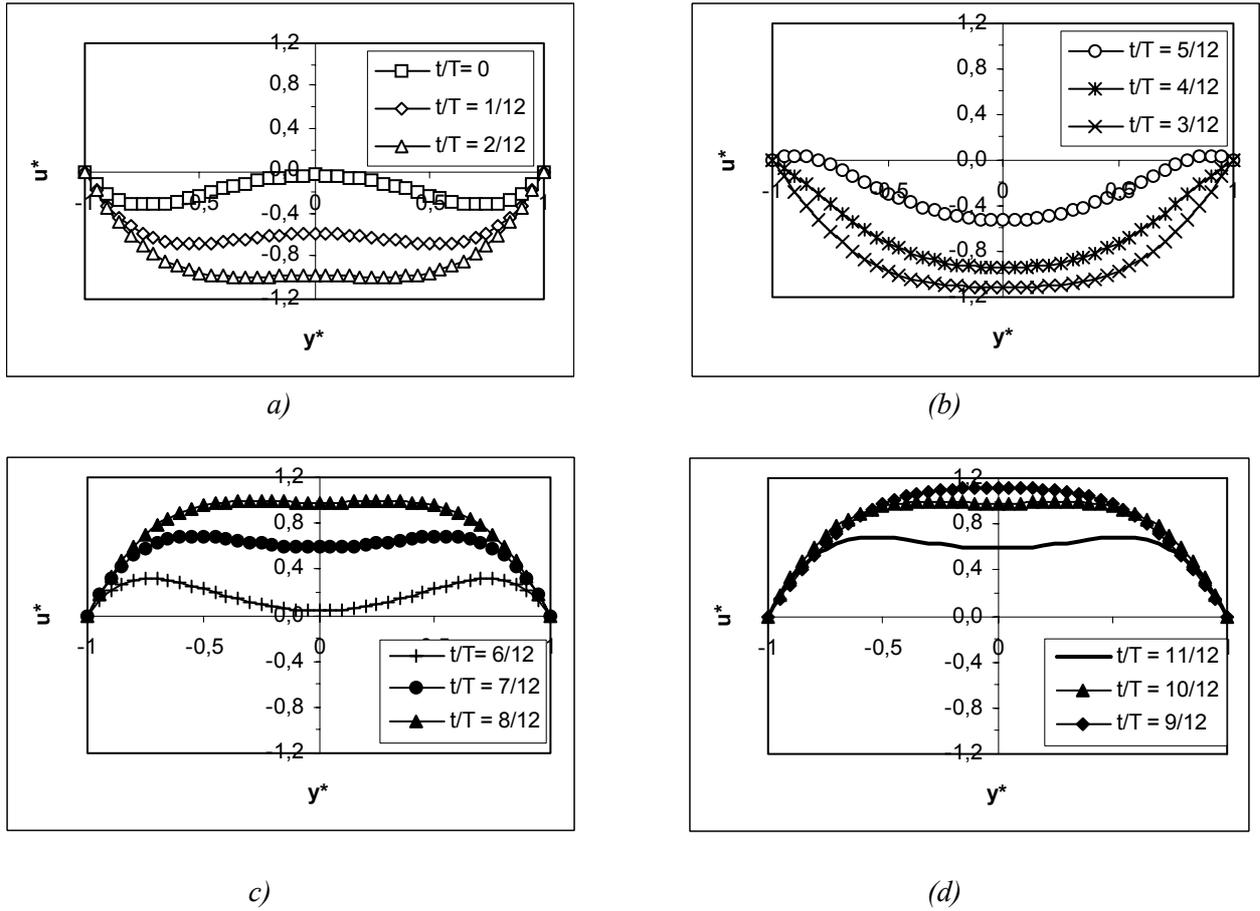
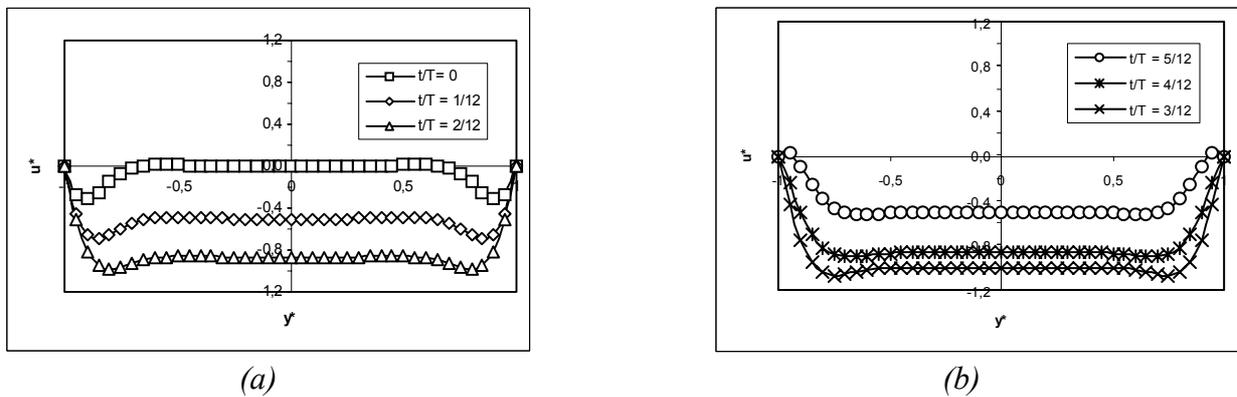
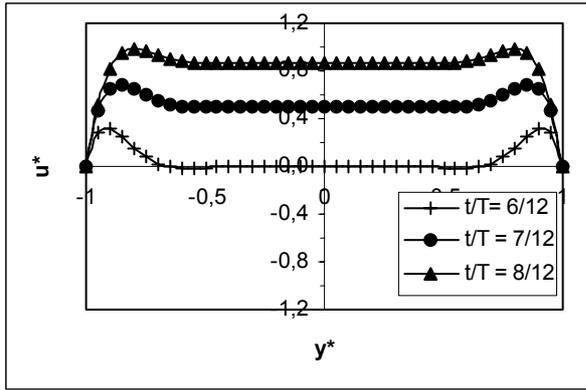


Figure 2:
Time history of the velocity profiles within a cycle for $\alpha = 4$

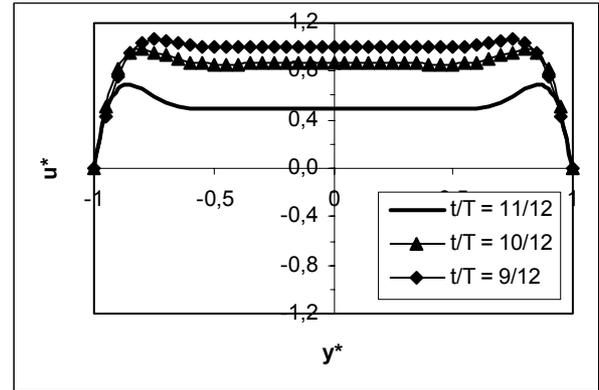
The temporal behavior of the skin friction coefficient (Eq. 24) is shown in Figure 5. The phase lead of the skin friction coefficient over the pressure approaches 45 degrees. When α is high, pulsation effects are noticeable and amplitude of the skin friction coefficient is high.

Numerical results were also obtained for different flow conditions with a converge criterion of 10^{-5} . One example from the numerical results is given here, more results could be found in literature (Karagoz 1996, Karagoz 2001). Figure 6 shows a comparison of the numerical velocity profiles obtained in fully developed flow region (Fig. 6a) with the analytical ones (Fig. 6b) over a flow cycle at time intervals of $t^* = 30$ degrees, for $\alpha = 13$. As can be seen from Figure 6, the numerical solutions are in satisfactory overall agreement with the analytical results, especially from the point of view of flow reversals.





(c)



(d)

Figure 3:
Time history of the velocity profiles within a cycle for $\alpha = 13$

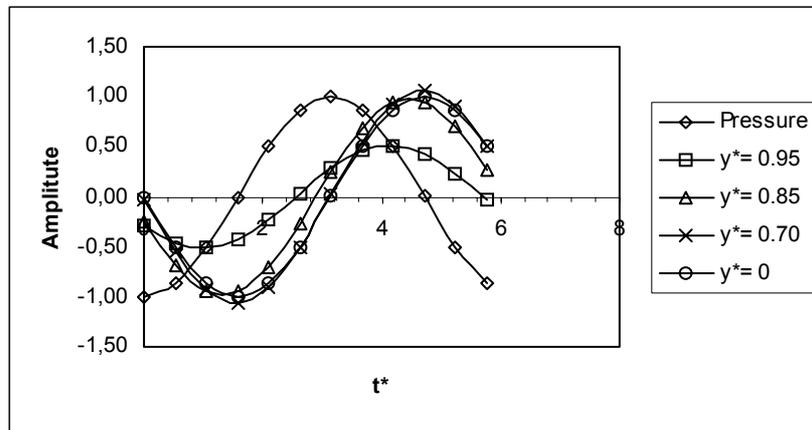


Figure 4:
Phase lag between pressure and velocities at different locations for $\alpha = 13$

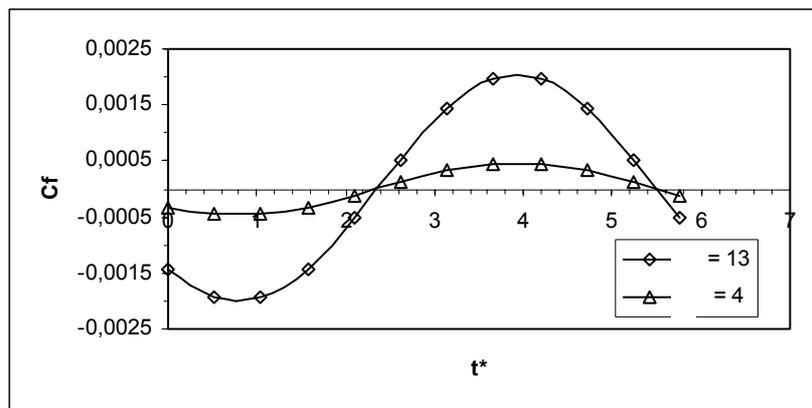


Figure 5:
Time variation of skin friction coefficient

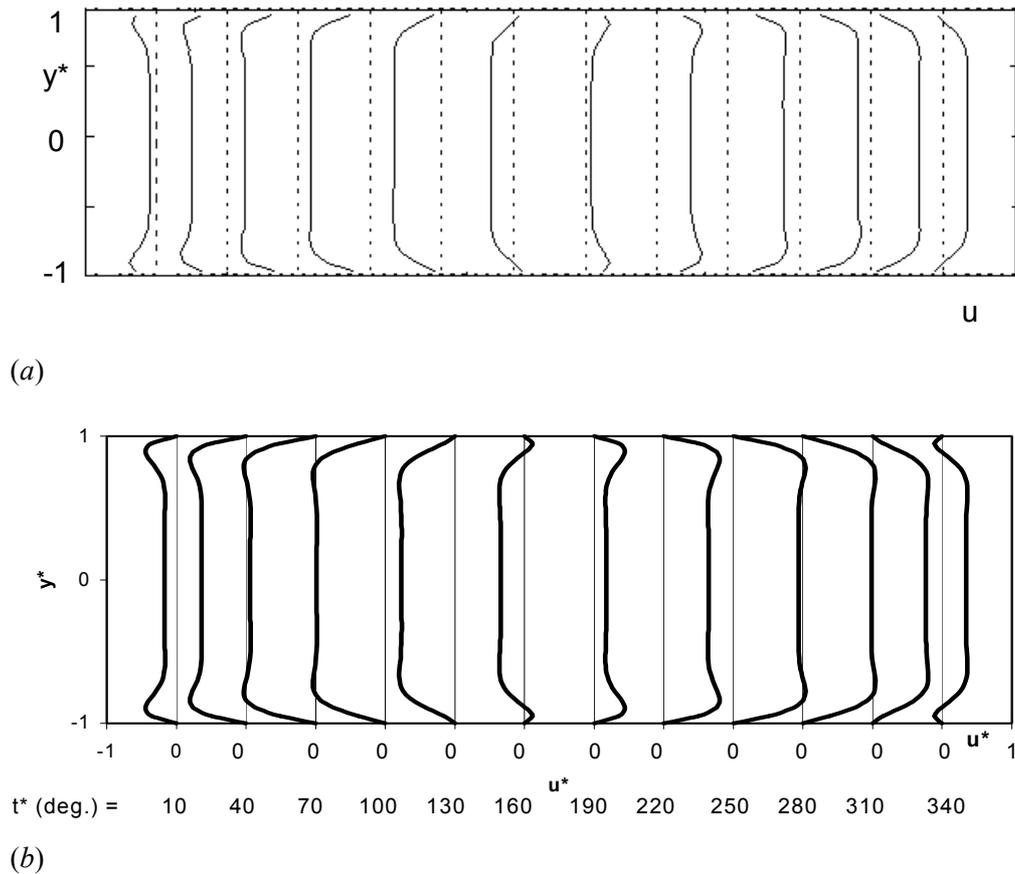


Figure 6
Time history of the velocity profiles obtained from the numerical solution (a)
and analytical solution (b), for $\alpha = 13$.

4. CONCLUSION

Solution of the unsteady momentum equation has been carried out analytically, for a fully developed laminar flow driven by oscillatory pressure field in a two dimensional channel. Time history velocity profiles, skin friction coefficient, and phase relationship between pressure and velocity variations have been plotted for two different values of α , 4 and 13. The following conclusion can be drawn from the results given above.

When the frequency parameter is low, velocity profiles resemble much of the quasi steady solution. An increase of α leads to a wider core region where the fluid moves as if it were frictionless slug flow. The effects of oscillation become dominant in a narrow zone adjacent to the walls in the case of high α .

Phase lag occurs between pressure and velocity variations so that it is higher in the core region than in the boundary layers, due to variation of the inertial forces across the flow section.

Skin friction coefficient was also affected by oscillation frequency of which high values give rise to the amplitude of the skin friction coefficient.

Extension of this study would include analytical and numerical solutions of heat transfer problems for oscillating laminar flow in straight and branching channels.

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