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A Numerical Method for Solving Singularly Perturbed Quasilinear Boundary Value Problems on Shishkin Mesh

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Abstract. In this paper, singularly perturbed quasilinear boundary value problems are taken into account. With this purpose, a finite difference scheme is proposed on Shishkin-type mesh (S-mesh). Quasilinearization technique and interpolating quadrature rules are used to establish the numerical scheme. Then, an error estimate is derived. A numerical experiment is demonstrated to verify the theory.

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1. Introduction

It is well known that singularly perturbed problems present the important mathematical structures. Many different phonemena in science can be modeled by them. They emerge in electrochemistry [22], control theory [7], nuclear engineering [1], fluid dynamics [12] and plasma physics [6] (see, also the references therein).

In singularly perturbed problems, standard numerical methods don't give reliable results when the perturbation parameter is small. Therefore, some important techniques have been introduced in the literature. These involve: finite difference methods [1, 4, 5, 8, 11, 12, 15], fitted mesh method [23], reproducing kernel method [7], factorization method [25], Galerkin method [18, 26], differential transform method [9], variational iteration methods [2, 10] and so on [6, 16, 19, 20, 27].

In this paper, we consider the following singularly perturbed boundary value problem:

$$\varepsilon u'' - b(x)u + f(x, u') = 0, \ x \in (0, l), \tag{1.1}$$

$$u'(0) = \frac{u_0}{\varepsilon}, \quad u(l) = u_1, \tag{1.2}$$

where $0 < \varepsilon < 1$ is perturbation parameter and $0 < b(x) \le b^*$ is sufficiently smooth function. Moreover, the function $f(x, u')$ holds

$$\left|\frac{\partial f(x,u')}{\partial x}\right| \le M_1 \in \mathbb{R}, \ \left|\frac{\partial f(x,u')}{\partial u'}\right| \le M_2 \in \mathbb{R},$$

and $\left|f(x, u_1') - f(x, u_2')\right| \le L\left|u_1' - u_2'\right|$. Here M_1 and M_2 are positive constants.

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The solution of the problem (1.1)-(1.2) has an initial layer within a neighborhood of $x = 0$. In recent years, some discretizations have been suggested for singularly perturbed quasilinear problems. In [11,12], nonlinear singularly perturbed reaction-diffusion type problems were discretized on layer-adapted meshes. The finite difference scheme with exponential coefficient on piecewise equidistant mesh is established for delay form of singularly perturbed quasilinear problems [13, 14, 17]. High order difference scheme was proposed for singularly perturbed quasilinear convection-diffusion problems in [24].

The novelty of this study is that the quasilinear term contains the derivative of the function $u(x)$. Motivated by the works in [11,12], this research aims to present an efficient and new difference scheme for solving singularly perturbed quasilinear boundary value problems on S-mesh.

The present paper is arranged as follows: In Section 2, asymptotic estimates for the continuous problem are given. The finite difference scheme is constructed on S-mesh in Section 3. The convergence analysis for the approximate solution is discussed in Section 4. Numerical results are presented in Section 5. The paper ends with a brief conclusion.

2. Asymptotic Behavior of the Exact Solution

In this section, some preliminary results are presented to estimate convergence analysis of the proposed method in later sections. For this, we write the equation (1.1) in the form

$$f(x,u') = f(x,0) + \frac{\partial f(x,\tilde{u}')}{\partial u'}u',\ \tilde{u}' = \gamma u', 0 < \gamma < 1.$$

Then, the equation (1.1) is expressed with

$$Lu \equiv \varepsilon u'' + a(x)\, u' - b(x)u = F(x),$$

where

$$a(x) = \frac{\partial f(x,\tilde{u}')}{\partial u'}, b(x) = b(x),\ F(x) = -f(x,0).$$

Moreover,

$$a(x) = \frac{\partial f(x,\tilde{u}')}{\partial u'} \geq \alpha > 0$$

and

$$F(x) \geq 0.$$

Lemma 2.1. *Let $v(x) \in C^2[0,l]$ be the function that satisfies the following conditions:*

$$Lv(x) \leq 0\ (0 \leq x \leq l),\ v'(0) \leq 0, v(l) \geq 0.$$

Then, $v(x) \geq 0\ (0 \leq x \leq l)$.

Proof. For the proof of the lemma, it can be referred to [12]. □

Lemma 2.2. *For the solution of the problem (1.1)-(1.2), following relations are held:*

$$|u(x)| \leq |u_0|\,\alpha^{-1}e^{-\alpha x/\varepsilon} + |u_1| + \alpha^{-1}(l-x)\,\|F(x)\|_{C[0,l]},\ 0 \leq x \leq l, \tag{2.1}$$

$$\begin{aligned} \left|u'(x)\right| &\leq \frac{|u_0|}{\varepsilon}e^{-\alpha x/\varepsilon} + \alpha\left(\|b(x)\|_{C[0,l]} + 1\right)\|F(x)\|_{C[0,l]} \\ &+ \alpha^{-1}\|b(x)\|_{C[0,l]}\left(\alpha^{-1}|u_0| + |u_1|\right),\ 0 \leq x \leq l. \end{aligned} \tag{2.2}$$

Proof. Firstly, we prove the estimation (2.1). Thus, we use the following barrier function:

$$\Psi(x) = |u_0|\,\alpha^{-1}e^{-\alpha x/\varepsilon} + |u_1| + \alpha^{-1}(l-x)\,\|F(x)\|_{C[0,l]} \pm u(x).$$

Taking into account Lemma 2.1, it is found that

$$L\Psi(x) \leq 0, \Psi'(0) \leq 0, \Psi(l) \geq 0.$$

According to the maximum principle, we get $\Psi(x) \geq 0$. From here, the relation (2.1) is true. Now, we show the proof of the inequality (2.2). By using the (1.1), it is obtained that

$$\begin{aligned} u'(x) &= u'(0)\,e^{-\frac{1}{\varepsilon}\int_0^x a(\eta)d\eta} \\ &+\frac{1}{\varepsilon}\int_0^x [b(s)\,u(s) + F(s)]\,e^{-\frac{1}{\varepsilon}\int_0^x a(\eta)d\eta}ds. \end{aligned}$$

From (2.1), we have

$$\begin{aligned} \left|u'(x)\right| &\leq \frac{|u_0|}{\varepsilon}e^{-\alpha x/\varepsilon} + \alpha^{-1}\max|b(s)\,u(s) + F(s)|\,(1 - e^{-\alpha x/\varepsilon}) \\ &\leq \frac{|u_0|}{\varepsilon}e^{-\alpha x/\varepsilon} + \alpha^{-1}\left\{\|b\|_{C[0,l]}\left(|u_0| + |u_1| + \alpha^{-1}\|F\|_{C[0,l]}\right) + \|F\|_{C[0,l]}\right\} \\ &= \frac{|u_0|}{\varepsilon}e^{-\alpha x/\varepsilon} + \alpha^{-1}\left\{\left(\alpha^{-1}\|b\|_{C[0,l]} + 1\right)\|F\|_{C[0,l]} + \alpha^{-1}\|b\|_{C[0,l]}\left(|u_0| + |u_1|\right)\right\}. \end{aligned}$$

Therefore, the proof of the lemma is done. □

3. The Difference Scheme

In this section, the finite difference scheme is constructed to discretize the problem (1.1)-(1.2) on Shishkin mesh. When the establishing the difference scheme, we use the interpolating quadrature rules and linear basis functions. Let ω_N be any non-uniform mesh on $[0,l]$:

$$\varpi_N = \{0 = x_0 < x_1 < ... < x_i < ... < x_N = l\}.$$

Here, mesh stepsizes are defined by

$$h_i = x_i - x_{i-1},\ \hbar_i = \frac{1}{2}(h_i + h_{i+1}).$$

Moreover, for any mesh function $v(x)$ defined on $\bar{\omega}_N$, we use

$$v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}},\ v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}$$

and

$$v_{\mathring{x},i} = \frac{1}{2}(v_{x,i} + v_{\bar{x},i}),\ v_{\bar{x}x,i} = \frac{1}{\hbar_i}(v_{x,i} - v_{\bar{x},i}).$$

Now, we give the definition of Shishkin type mesh. For an even number N, we divide each of the subintervals $[0,\sigma]$ and $[\sigma,l]$ into $\frac{N}{2}$ equidistant subintervals. The transition point σ is determined as $\sigma = \min\{\frac{l}{2}, \alpha^{-1}\varepsilon\ln N\}$. x_i node points are denoted by

$$\bar{\omega}_N = \begin{cases} x_i = ih^{(1)}, & i = 0,1,...,\frac{N}{2},\ x_i \in [0,\sigma] \\ x_i = \sigma + \left(i - \frac{N}{2}\right)h^{(2)}, & i = \frac{N}{2}+1,...,N,\ x_i \in [\sigma,l], \end{cases}$$

where $h^{(1)} = \frac{2(l-\sigma)}{N}$ and $h^{(2)} = \frac{2(l-\sigma)}{N}$. To generate the difference approach, we use the following integral identity:

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu\varphi_i dx := \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\left[\varepsilon u'' - b(x)\right]u\varphi_i dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,u')\varphi_i dx,$$

where the basis function $\varphi_i(x)$ is as follows:

$$\varphi_i = \begin{cases} \varphi_i^{(1)} = \frac{(x-x_{i-1})}{h_i} &, x \in (x_{i-1},x_i) \\ \varphi_i^{(2)} = \frac{(x_{i+1}-x)}{h_{i+1}} &, x \in (x_i,x_{i+1}) \\ 0 &, x \notin (x_{i-1},x_{i+1}). \end{cases}$$

Then, we use the interpolating quadrature rules in [3] to each term of (1.1). For the first term $\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} -\varepsilon u''\varphi_i dx$, we can write

$$
\begin{aligned}
-\hbar_i^{-1}\varepsilon\int\limits_{x_{i-1}}^{x_{i+1}}\varepsilon u'\varphi_i' dx &= -\hbar_i^{-1}\varepsilon\int\limits_{x_{i-1}}^{x_i} u'\varphi_i^{(1)'}dx-\hbar_i^{-1}\varepsilon\int\limits_{x_i}^{x_{i+1}} u'\varphi_i^{(2)'}dx\\
&= -\hbar_i^{-1}\varepsilon u_{\bar{x}}\int\limits_{x_{i-1}}^{x_i}\varphi_i^{(1)'}dx-\hbar_i^{-1}\varepsilon u_x\int\limits_{x_i}^{x_{i+1}}\varphi_i^{(2)'}dx\\
&= \hbar_i^{-1}\varepsilon\left(u_x-u_{\bar{x}}\right)=\varepsilon u_{\hat{x}x,i}. \qquad (3.1)
\end{aligned}
$$

For the second term $-\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} b(x)u(x)\varphi_i dx$, we obtain

$$
\begin{aligned}
-\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} b(x)u(x)\varphi_i dx &= -\hbar_i^{-1}\left[\int\limits_{x_{i-1}}^{x_{i+1}}\left[b(x)-b(x_i)\right]u(x)\varphi_i dx-b(x_i)\int\limits_{x_{i-1}}^{x_{i+1}} u(x)\varphi_i dx\right]\\
&= -b_i u_i+R_i^{(1)}+R_{b,i}, \qquad (3.2)
\end{aligned}
$$

where

$$
R_{b,i}=-\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}}\left[b(x)-b(x_i)\right]u\left(x\right)\varphi_i dx
$$

and

$$
R_i^{(1)}=-b_i\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} dx\varphi_i\int\limits_{x_{i-1}}^{x_{i+1}}\frac{du\left(\xi\right)}{dx}T_0\left(x-\xi\right)d\xi.
$$

Finally, the right-side of the equation (1.1), we find

$$
\begin{aligned}
\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} f(x,u')\varphi_i dx &= \hbar_i^{-1}\left[\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x,u')-f(x_i,u')\right]\varphi_i dx+\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x_i,u')-f(x_i,u_i')\right]\varphi_i dx\right.\\
&\left.+\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x_i,u_i')-f(x_i,u_{\mathring{x},i})\right]\varphi_i dx\right]+\int\limits_{x_{i-1}}^{x_{i+1}} f(x_i,u_{\mathring{x},i})\varphi_i dx.
\end{aligned}
$$

Accordingly, it is written

$$
\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} f(x,u')\varphi_i dx=f(x_i,u_{\mathring{x},i})+R_2, \qquad (3.3)
$$

where

$$
\begin{aligned}
R_2 &= \hbar_i^{-1}\left[\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x,u')-f(x_i,u')\right]\varphi_i dx+\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x_i,u')-f(x_i,u_i')\right]\varphi_i dx\right.\\
&\left.+\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x_i,u_i')-f(x_i,u_{\mathring{x},i})\right]\varphi_i dx\right].
\end{aligned}
$$

For the boundary condition $u'(0)=\frac{u_0}{\varepsilon}$, we get

$$
\int_{x_0}^{x_1}\left[\varepsilon u''\left(x\right)-b\left(x\right)+f\left(x,u'\right)\right]\varphi_0\left(x\right)dx=0.
$$

Here the function $\varphi_0(x)$ is specified as

$$\varphi_0 = \begin{cases} \varphi_0^{(2)} = \frac{(x_1-x)}{h_1} \quad , & x \in (x_0, x_1) \\ 0 \,, & x \notin (x_{i-1}, x_{i+1}) \,. \end{cases}$$

Using the interpolating quadrature rules and partial integration, is is found that

$$\begin{aligned} \int_{x_0}^{x_1} \varepsilon u''(x)\, \varphi_0 dx &= \varepsilon u'(x)\, \varphi_0\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u'(x)\, \varphi_0' dx \\ &= -\varepsilon u'(0) + \varepsilon u_{x,0}. \end{aligned}$$

From here, we have

$$-\varepsilon u'(0) + \varepsilon u_{x,0} = r_0, \tag{3.4}$$

where

$$r_0 = \int_{x_0}^{x_1} [b(x)\, u(x) - f(x, u')]\, \varphi_0(x)\, dx.$$

Combining the relations (3.1), (3.2), (3.3) and (3.4), we obtain the following difference problem:

$$\varepsilon u_{\hat{x}x,i} + b_i u_i + f(x_i, u_{\mathring{x},i}) = R_i,\ i = 1, 2, ..., N-1, \tag{3.5}$$

$$u_{x,0} = u_0/\varepsilon + r_0,\ \ u_N = u_1. \tag{3.6}$$

Here

$$R_i = R_{b,i} + R^{(1)} - R_2.$$

By omitting the remainder term in (3.5), the following finite difference problem is found:

$$\ell y_i \equiv \varepsilon y_{\hat{x}x,i} - b_i y_i + f(x_i, y_{\mathring{x},i}) = 0,\ i = 1, 2, ..., N-1, \tag{3.7}$$

$$y_{x,0} = \frac{u_0}{\varepsilon},\ \ y_N = u_1. \tag{3.8}$$

4. The Convergence Analysis

Let u_i be the solution of the problem (1.1)-(1.2) and let y_i be the solution of the problem (3.5)-(3.6).Then, for the function $z = y_i - u_i$, the following discrete problem can be written:

$$\varepsilon z_{\hat{x}x,i} - b_i z_i + f(x_i, y_{\mathring{x},i}) - f(x_i, u_{\mathring{x},i}) = R_i, \quad i = 1, ..., N-1,$$

$$z_{x,0} = \frac{r_0}{\varepsilon}, \quad z_N = 0.$$

Using the mean value theorem, it is found that

$$\begin{aligned} f(x_i, y_{\mathring{x},i}) - f(x_i, u_{\mathring{x},i}) &= \frac{\partial f(x_i, \tilde{u}_{\bar{x},i})}{\partial u'} (y_{\mathring{x},i} - u_{\mathring{x},i}) \\ &= \frac{\partial f(x_i, \tilde{u}_{\bar{x},i})}{\partial u'} z_{\mathring{x},i} = a(x_i)\, z_{\mathring{x},i} \\ &= a_i z_{\mathring{x},i}. \end{aligned}$$

In this instance, we obtain

$$\ell z \equiv \varepsilon z_{\hat{x}x,i} + a_i z_{\mathring{x},i} - b_i z_i = R_i, \quad i = 1, ..., N-1, \tag{4.1}$$

$$z_{x,0} = \frac{r_0}{\varepsilon},\ \ z_N = 0. \tag{4.2}$$

Lemma 4.1. *For the solution of the problem (4.1)-(4.2), the following relation is valid:*

$$\left|z_p\right| \leq 4\alpha^{-1} \left(|r_0| + \sum_{k=1}^{N-1} \hbar_k\, |R_k| \right).$$

Proof. The discrete maximum principle is true for the difference operator ℓz_i: If $\ell z_i \le 0$ $(i = 1, 2, ..., N-1)$, $z_{x,0} \le 0$, $z_N \ge 0$, $z_i \ge 0$ $(i = 1, 2, ..., N-1)$ is obtained. Thus, we obtain $|z_i| \le \eta_i$.
Here, η_i is the solution of the following problem:

$$\varepsilon \eta_{\hat{x}x,i} + a_i \eta_{\mathring{x},i} = -R_i,\ 1 \le i \le N-1, \tag{4.3}$$

$$\eta_{x,0} = -r_0/\varepsilon, \quad \eta_N = 0. \tag{4.4}$$

Substituting $\eta_{x,i} = v_i$ in (4.3)-(4.4), we get

$$\varepsilon \hbar_i^{-1} (v_i - v_{i-1}) + \frac{a_i}{2} (v_i + v_{i-1}) = -R_i,$$
$$v_0 = -r_0/\varepsilon.$$

From here, it is found that

$$v_i = v_0 \prod_{k=1}^{i} \left(\frac{1 - a_k \rho_k/2}{1 + a_k \rho_k/2} \right) - \sum_{k=1}^{i} \hbar_k \frac{R_i}{\varepsilon + a_k \hbar_k/2} \prod_{j=k+1}^{i} \left(\frac{1 - a_j \rho_j/2}{1 + a_j \rho_j/2} \right). \tag{4.5}$$

Taking into account $a_k \rho_k/2 \le 1$, $1 - x \le e^{-x/2}$ $(0 \le x \le 1)$, and the equation (4.5), we have

$$\begin{aligned} |v_i| \quad &\le \quad \varepsilon^{-1} |r_0| \, e^{-\sum_{k=1}^{i} \hbar_k \frac{a_k/\varepsilon}{1+a_k\rho_k/2}} \\ &\quad + \sum_{k=1}^{i} \hbar_k \frac{|R_k|}{\varepsilon} e^{-\left(\frac{a_k\rho_k/2}{1+a_k\rho_k/2} + \sum_{j=k+1}^{i} \hbar_j \frac{a_j/\varepsilon}{1+a_j\rho_j/2} \right)}. \end{aligned}$$

Then, by using the relation

$$\eta_p = \eta_N - \sum_{k=p}^{N-1} \hbar_k v_k = \eta_N - \sum_{k=1}^{N-p} \hbar_{k+p-1} v_{k+p-1},$$

we find

$$\left| \eta_p \right| \le \sum_{k=1}^{N-p} \hbar_{k+p-1} v_{k+p-1}$$

and

$$\begin{aligned} \left| \eta_p \right| \quad &\le \quad \varepsilon^{-1} |r_0| \sum_{i=1}^{N-p} \hbar_k e^{-\sum_{k=1}^{i+p-1} \hbar_k \frac{a_k/\varepsilon}{1+a_k\rho_k/2}} \\ &\quad + \sum_{i=1}^{N-p} \hbar_i \left\{ \sum_{k=1}^{i+p-1} \hbar_k \frac{|R_k|}{\varepsilon} e^{-\left(\frac{a_k\rho_k/2}{1+a_k\rho_k/2} + \sum_{j=k+1}^{i} \hbar_j \frac{a_j/\varepsilon}{1+a_j\rho_j/2} \right)} \right\}. \end{aligned} \tag{4.6}$$

For the first term of right side of (4.6), it is written that

$$\begin{aligned} |r_0| \sum_{i=1}^{N-p} \rho_i e^{-\sum_{k=1}^{i+p-1} \hbar_k \frac{a_k/\varepsilon}{1+a_k\rho_k/2}} \quad &\le \quad |r_0| \sum_{i=1}^{N-p} \rho_i e^{-\frac{\alpha x_{i-1}}{\varepsilon + \alpha h_k/2}} \\ &\le \quad |r_0| \, \rho_i \frac{1}{1 - e^{-\frac{\alpha \hbar_i}{\varepsilon + \alpha h_i/2}}}. \end{aligned}$$

From here, we obtain

$$\begin{aligned} |r_0| \sum_{i=1}^{N-p} \rho_i e^{-\sum_{k=1}^{i+p-1} \hbar_k \frac{a_k/\varepsilon}{1+a_k\rho_k/2}} \quad &\le \quad |r_0| \, \rho_i \left(1 + \frac{\alpha h_i/2 + \varepsilon}{\alpha h_i} \right) \\ &= \quad |r_0| \left(\frac{3\rho_i}{2} + \alpha^{-1} \right) \le 4\alpha^{-1} |r_0| \, . \end{aligned} \tag{4.7}$$

Similarly, for the right side of the inequality (4.6), it is found that

$$\sum_{i=1}^{N-p} \hbar_i \left\{ \sum_{k=1}^{i+p-1} \hbar_k \frac{|R_k|}{\varepsilon} e^{-\left(\frac{a_k\rho_k/2}{1+a_k\rho_k/2} + \sum_{j=k+1}^{i} \hbar_j \frac{a_j/\varepsilon}{1+a_j\rho_j/2} \right)} \right\} \le 4\alpha^{-1} e^{-\frac{\alpha\rho_i/2}{1+\alpha\rho_i/2}} \left(\sum_{j=1}^{N-1} \hbar_j \left| R_j \right| \right) \le 4\alpha^{-1} \sum_{k=1}^{N-1} \hbar_k \, |R_k| \, . \tag{4.8}$$

Rewriting (4.7) and (4.8) in (4.6), the lemma is proven. □

Lemma 4.2. *Under the conditions $a(x)$, $b(x)$ and $F(x) \in C^1[0,l]$, the following estimates are satisfied:*

$$\sum_{i=1}^{N-1} h_i |R_i| \leq Ch_i, \; i = 0, ..., N, \tag{4.9}$$

$$|r_0| \leq Ch_i. \tag{4.10}$$

Proof. Firstly, we estimate each remainder term seperately. The error term R_i is as follows:

$$R_i = R_{b,i} + R^{(1)} - R_2.$$

Here

$$R_{b,i} = -\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [b(x) - b(x_i)]\, u(x)\, \varphi_i dx \leq Ch_i,$$

$$\begin{aligned} R_2 &= \hbar_i^{-1} \left[\int_{x_{i-1}}^{x_{i+1}} [f(x,u') - f(x_i,u')]\, \varphi_i dx + \int_{x_{i-1}}^{x_{i+1}} [f(x_i,u') - f(x_i,u_i')]\, \varphi_i dx \right. \\ &\left. + \int_{x_{i-1}}^{x_{i+1}} [f(x_i,u_i') - f(x_i,u_{\bar{x},i})]\, \varphi_i dx \right] \\ &\leq \hbar_i^{-1} \left[\int_{x_{i-1}}^{x_{i+1}} \frac{\partial f(\xi,u')}{\partial x} \varphi_i dx + \int_{x_{i-1}}^{x_{i+1}} \frac{\partial f(x_i,\tilde{u}')}{\partial u'} \varphi_i dx + \int_{x_{i-1}}^{x_{i+1}} K \left| u_i' - u_{\mathring{x},i} \right| \varphi_i dx \right], \end{aligned}$$

and

$$R_i^{(1)} = -b_i \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i \int_{x_{i-1}}^{x_{i+1}} \frac{du(\xi)}{dx} T_0(x-\xi)\, d\xi \leq Ch_i.$$

Similarly, the error term r_0 is written as the form

$$\begin{aligned} r_0 &= \int_{x_0}^{x_1} f(x,u')\, \varphi_0(x) dx - \int_{x_0}^{x_1} b(x) u(x) \varphi_0(x) dx \\ &= \int_{x_0}^{x_1} \left[f(x,0) + \frac{\partial f(x,\tilde{u}')}{\partial u'} u'(x) \right] \varphi_0(x) dx - \int_{x_0}^{x_1} b(x) u(x) \varphi_0(x) dx \\ &= \int_{x_0}^{x_1} \frac{\partial f(x,\tilde{u}')}{\partial u'} u'(x)\, \varphi_0(x) dx + \int_{x_0}^{x_1} f(x,0)\, \varphi_0(x) dx - \int_{x_0}^{x_1} b(x) u(x) \varphi_0(x) dx. \end{aligned}$$

Therefore, we obtain

$$|r_0| \leq Ch_1 \left(1 + \int_{x_0}^{x_1} \left| u'(x) \right| dx \right).$$

Now, we estimate the remainder terms for node points of the piecewise equidistant mesh. Firstly, we consider the condition $\sigma = \frac{l}{2}$. Taking $\frac{l}{2} \leq \alpha^{-1}\varepsilon \ln N$, we find $h^{(1)} = h^{(2)} = h = lN^{-1}$. Thus, it can be written that

$$|R_i| \leq C \left\{ N^{-1} + \varepsilon^{-1} l N^{-1} \right\} \leq C \left\{ N^{-1} + \alpha^{-1} N^{-1} \ln N \right\}$$

and

$$|R_i| \leq CN^{-1} \ln N, \quad i = 1, 2, ..., N.$$

Then, we take into account the condition $\sigma = \alpha^{-1}\varepsilon \ln N$ and $\alpha^{-1}\varepsilon \ln N < \frac{l}{2}$. For the interval $[0,\sigma]$, we get

$$|R_i| \leq C \left(1 + \varepsilon^{-1} \right) h^{(1)} \leq C \left(1 + \varepsilon^{-1} \right) \frac{2\alpha^{-1}\varepsilon \ln N}{N}.$$

From here, it is obtained that $|R_i| \leq CN^{-1}\ln N,\ \ i = 1, 2, ..., \frac{N}{2}$. For the interval $[\sigma, l]$, we get

$$|R_i| \leq C\left\{h^{(2)} + \alpha^{-1}\left(e^{-\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_i}{\varepsilon}}\right)\right\},\quad i = \frac{N}{2}, ..., N.$$

Since $x_i = \alpha^{-1}\varepsilon \ln N + \left(i - \frac{N}{2}\right)h^{(2)}$, we can write

$$\begin{aligned} e^{-\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_i}{\varepsilon}} &= e^{\frac{-\alpha\left(\alpha^{-1}\varepsilon\ln N + \left(i-1-\frac{N}{2}\right)\right)h^{(2)}}{\varepsilon}} - e^{\frac{-\alpha\left(\alpha^{-1}\varepsilon\ln N + \left(i-\frac{N}{2}\right)\right)h^{(2)}}{\varepsilon}} \\ &= \frac{1}{N}\left(e^{\frac{-\alpha\left(i-1-\frac{N}{2}\right)h^{(2)}}{\varepsilon}} - e^{\frac{-\alpha\left(i-\frac{N}{2}\right)h^{(2)}}{\varepsilon}}\right) \\ &= \frac{1}{N}e^{\frac{-\alpha\left(i-1-\frac{N}{2}\right)h^{(2)}}{\varepsilon}}\left(1 - e^{\frac{-\alpha h^{(2)}}{\varepsilon}}\right) \leq N^{-1}. \end{aligned}$$

Finally, we have $|R_i| \leq CN^{-1}$. □

Theorem 4.3. *Under the conditions of Lemma 4.2, the solution of the difference problem (3.7)-(3.8) converges to the solution of the problem (1.1)-(1.2) with the order of $O(h_i)$ in the norm $C(\omega_N)$. Thus, the following relation is true:*

$$\|y - u\|_{C(\omega_N)} \leq Ch_i.$$

Proof. From Lemma 4.1, it is clear that

$$\|z\|_{C(\omega_N)} \leq C\left(|r_0| + \|R\|_{L_1(\omega_N)}\right).$$

Considering (4.9) and (4.10), the proof of the theorem is shown. □

5. Numerical Results

The difference problem (3.7)-(3.8) can be written as the form:

$$\begin{aligned} &-\varepsilon y^{(n)}_{\hat{x}x,i} + b_i y^{(n)}_i = f(x_i, y^{(n-1)}_{\bar{x},i}) + \frac{\partial f}{\partial u'}\left(x_i, y^{(n-1)}_{\bar{x},i}\right)\left(y^{(n)}_{\bar{x},i} - y^{(n-1)}_{\bar{x},i}\right),\quad i = 1, 2, ..., N-1;\quad n = 1, 2, 3, ..., \\ &y_{x,0} = u_0/\varepsilon,\quad y_N = u_1. \end{aligned}$$

Now, we edit this expression according to the following format:

$$\begin{aligned} A_i y^{(n)}_{i-1} - C_i y^{(n)}_i + B_i y^{(n)}_{i+1} &= -F_i,\ i = 1, ..., N-1, \\ y_0 &= y_1 - h_1 u_0/\varepsilon,\ y_N = u_1, \end{aligned}$$

where

$$\begin{aligned} &A_i = \varepsilon\hbar_i^{-1}h_{i-1}^{-1} - f_{u'}\left(x_i, y^{(n-1)}_{\mathring{x},i}\right),\ B_i = \varepsilon\hbar_i^{-1}h_{i+1}^{-1} - f_{u'}\left(x_i, y^{(n-1)}_{\mathring{x},i}\right), \\ &C_i = \varepsilon\hbar_i^{-1}\left(\frac{1}{h_{i+1}} + \frac{1}{h_i}\right) + b_i,\ F_i = f(x_i, y^{(n-1)}_i) - y^{(n-1)}_i\frac{\partial f(x_i, y^{(n-1)}_{\mathring{x},i})}{\partial u'}. \end{aligned}$$

Then, we consider the eliminaton method in [21]. The coefficients of elimination method are as follows:

$$\begin{aligned} \overset{(\rightarrow)}{\alpha}_{i+1} &= \frac{B_i}{C_i - \alpha_i A_i},\ \alpha_1 = 0,\ i = 1, ..., N-1, \\ \overset{(\rightarrow)}{\beta}_{i+1} &= \frac{F_i + A_i\beta_i}{C_i - \alpha_i A_i},\ \beta_1 = 1,\ i = 1, ..., N-1, \end{aligned}$$

and

$$y_i = y_{i+1}\alpha_{i+1} + \beta_{i+1},\ i = N-1, ..., 1.$$

For the problem (1.1)-(1.2), we take

$$b(x) = x(1-x),\quad f(x, u') = x + u' + (u')^2$$

with

$$u'(0) = 0.5,\quad u(1) = 1.$$

Also, $\alpha_1 = 1$ and $\beta_1 = -u_0 h_1/\varepsilon$. Maximum point-wise errors are computed as

$$e^N = \max_{0<i<N}\left|y_h^i - y_{\frac{h}{2}}^i\right|, \quad e^{2N} = \max_{0<i<N}\left|y_{\frac{h}{2}}^i - y_{\frac{h}{4}}^i\right|$$

and the order of convergence is denoted by $p = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}$. The computed results are displayed in Table 1.

Table 1. Exact errors and the order of convergence for $\varepsilon = 10^{-w}$ $(w = 1, 2, ..., 7)$

ε		N			
		32	64	128	256
10^{-1}	e^N	0.0163519564	0.0079127727	0.0039092726	0.0019455932
	e^{2N}	0.0058047143	0.0034066209	0.0018237924	0.0009409825
	p	1.4941662806	1.2158419013	1.0999586604	1.0479702698
10^{-2}	e^N	0.0162910015	0.0068288495	0.0028604105	0.0012399372
	e^{2N}	0.0039093324	0.0017790949	0.0009776340	0.0005132421
	p	0.0039093324	1.9404990722	1.5488558570	1.2725555217
10^{-3}	e^N	0.0176982963	0.0072888653	0.0030716365	0.0013293514
	e^{2N}	0.0072888653	0.0030716365	0.0013269710	0.0005506774
	p	1.2798443477	1.2466867323	1.2108706880	1.2714430460
10^{-4}	e^N	0.0184691055	0.0076487847	0.0032564528	0.0014663558
	e^{2N}	0.0076487847	0.0032564528	0.0014663558	0.0006891447
	p	1.2718115558	1.2319292249	1.1510661157	1.0893562822
10^{-5}	e^N	0.0202671764	0.0076930051	0.0032748267	0.0014751728
	e^{2N}	0.0076930051	0.0032748267	0.0014751728	0.0006947771
	p	1.3975259400	1.2321286886	1.1505346131	1.0862617977
10^{-6}	e^N	0.0232378294	0.0076985043	0.0032770770	0.0014761317
	e^{2N}	0.0076985043	0.0032770770	0.0014761317	0.0006952306
	p	1.5938252298	1.2321686303	1.1505881391	1.0862579666
10^{-7}	e^N	0.0262939090	0.0076991507	0.0032773440	0.0014762467
	e^{2N}	0.0076991507	0.0032773440	0.0014762467	0.0006952825
	p	1.7719574133	1.2321721997	1.1505932622	1.0862625869

According to the obtained results in Table 1, for large values of N, the exact errors decrease and the convergence rate is close to 1. This indicates that the proposed scheme is stable for solving these problems.

6. Conclusions

We described a robust finite difference scheme for singularly perturbed quasilinear boundary value problems. The stability and convergence of the developed scheme were analyzed in the discrete maximum norm. A numerical example was solved and the computed results were tabulated. According to these results, we observe that the suggested scheme is uniformly convergent and it is almost first-order accuracy on Shishkin mesh. Concisely, the method is stable and appropriate for solving such problems. To advance the numerical studies, higher order difference scheme can be prososed or different types of singularly perturbed quasilinear problems can be considered.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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