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Research Article

ON PERFECT CODES IN THE LEE-ROSENBLOOM-TSFASMAN-JAIN METRIC

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ABSTRACT

In this paper, we study perfect codes in the Lee-Rosenbloom-Tsfasman-Jain (LRTJ) metric over the finite field $\mathbb{Z}_p$. We begin by deriving some new upper bounds focusing on the number of parity check digits for linear codes correcting all error vectors of LRTJ weight up to w, $1 \leq w \leq 4$. Furthermore, we establish sufficient conditions for the existence of perfect codes correcting all error vectors with certain weights. We also search for linear codes which attain these bounds to determine the possible parameters of perfect codes. Moreover, we derive parity check matrices corresponding linear codes correcting all error vectors of LRTJ weight 1 over $\mathbb{Z}_p$, and correcting all error vectors of LRTJ weight up to 2 over $\mathbb{Z}_3$ and $\mathbb{Z}_{11}$. We also construct perfect codes for these cases. Lastly, we obtain non-existence results on w-perfect linear codes over $\mathbb{Z}_p$ for $2 \leq w \leq 4$.

Keywords: Linear codes, perfect codes, LRTJ weight, LRTJ metric.

1. INTRODUCTION

The study of perfect codes is one of the major topics for optimum solutions in coding theory. The readers may refer to the articles [3, 4, 5, 8, 9, 10]. The investigation of non-trivial perfect codes plays an important role in the theory of error-correcting codes. These approaches include techniques from studies in group theory, cryptography, graph theory and geometry.

The existence of perfect codes is one of the most important problem in the field. Many researchers have studied the existence of perfect codes in different metrics [5, 7, 10, 11]. Hamming metric is one of the most used metric. The only known examples in the Hamming metric are the binary repetition codes for odd length, the Hamming codes and the Golay codes which are the code with parameters [23,12,7] over the binary field and the code with parameters [11,6,5] over the ternary field. Hamming codes are single error correcting codes, Golay codes over the binary field are three-error-correcting codes and Golay codes over the ternary field are two-error-correcting codes. In addition, it has been shown that there exist no perfect Hamming codes over prime alphabet other than the above known examples [7, 11].

In this paper, we consider perfect codes in the Lee-Rosenbloom-Tsfasman-Jain (LRTJ) metric over $\mathbb{Z}_p$. LRTJ metric [5] is obtained by a combination of Lee metric [6] and RT metric [13].

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Firstly, we st ate some basic terminology and cover some preliminary definitions and facts.

For every prime p, the ring of integers modulo p is a finite field of order p and is denoted by $\mathbb{Z}_p$. Let $\mathbb{Z}_p^n$ denote the set of all vectors of length n with entries in $\mathbb{Z}_p$. Then $\mathbb{Z}_p^n$ is a vector space over $\mathbb{Z}_p$. A code C is said to be an (n, M) −linear code if and only if C is a subspace of $\mathbb{Z}_p^n$ of size M. A linear code C of length n and dimension k over $\mathbb{Z}_p$ is called a p −ary $[n, k]$ −code.

Lee weight for a single-letter a is defined as

$$|a| = \begin{cases} a & \text{if} \quad 0 \le a \le \frac{p}{2} \\ p - a & \text{if} \quad \frac{p}{2} < a \le p - 1 \end{cases} \tag{1}$$

and RT weight of y is defined as

$$\max_{j=1,\dots,n} \{j : y_j \neq 0\}. \tag{2}$$

The LRTJ weight $\Gamma(Y)$ for $Y = (y_0, y_1, \dots, y_{n-1}) \in \mathbb{Z}_p^n$, is defined as

$$\Gamma(Y) = \begin{cases} \max\limits_{j=0,1,\dots,n-1} |y_j| + \max\limits_{j=0,1,\dots,n-1} \{j : y_j \neq 0\} & \text{if} \quad Y \neq 0 \\ 0 & \text{if} \quad Y = 0 \end{cases} \tag{3}$$

For any $u, v \in \mathbb{Z}_p^n$, the LRTJ distance d_{LRTJ} between two vectors u and v is given by

$$d(u, v) = \Gamma(u - v). \tag{4}$$

Note that Lee distance coincides with Hamming distance but LRTJ distance does not coincide with Hamming distance over $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Next, we give the definition of a w-perfect linear code.

Definition 1. A p −ary $[n, k]$ −code is called a w-perfect linear code if for a given positive integer w, the code corrects all error vectors of weight up to w but no error vectors of weight greater than w.

For a perfect linear code correcting errors of weight up to w, the number of error vectors of weight up to w having the vector of all zeros is equal to the number of available cosets.

The organization of this paper is as follows. In Section 2, we obtain formulas for the number of error vectors of LRTJ weight. In Section 3, we obtain an upper bound on the number of parity check digits for linear codes correcting all error vectors of LRTJ weight 1. Also, we characterize all 1-perfect linear codes and their parity-check matrices. In Section 4, upper bounds on the number of parity check digits for linear codes correcting all error vectors of LRTJ weight up to 2 is obtained over $\mathbb{Z}_3$ and $\mathbb{Z}_{11}$. Also, parity-check matrices are characterized for these cases. Moreover, we show that there does not exist a 2-perfect linear code over $\mathbb{Z}_p$ when $p \neq 3,7$. In Sections 5 and 6, we prove nonexistence results for 3-perfect linear codes over $\mathbb{Z}_p$ and 4-perfect linear codes over $\mathbb{Z}_p$. Finally, the paper concludes in Section 7.

2. THE NUMBER OF ERROR VECTORS OF LRTJ WEIGHT

In this section, we enumerate the number of all error vectors of a given LRTJ weight w. Then, we generalize this to the number of all error vectors of LRTJ weight up to w.

For a p-ary $[n, k]$-code, the number of all error vectors of LRTJ weight w will be denoted by s_w. Clearly, $s_0 = 1$ and $s_1 = 2$. For $w \ge 2$, the value of s_w is given in the following lemma.

Lemma 1. For a p-ary $[n, k]$-code, the number of all error vectors of LRTJ weight w, $w \ge 2$, is

$$s_w = 2 + \sum_{i=1}^{w-1} \left[(2i + 1)^{w-i}(2i) - (2i - 1)^{w-i}(2i - 2)\right] \tag{5}$$

Proof. Assume that $p \geq 5$ is prime, LRTJ we ight is $w \geq 2$ and $u = (u_1, \ldots, u_n)$ is a vector. Then we will consider the equation

$$\max |u_i| + wt_{RT}(u) = w. \tag{6}$$

If RT weight is 0, then $\max |u_i| + wt_{RT}(u) = w + 0 = w$. In this case, the number of all error vectors of LRTJ weight w are 2. These vectors are as follows:

$$w, 0,0, \ldots ,0,$$
$$-w, 0,0, \ldots ,0.$$

If RT weight is r, then we have $\max |u_i| + wt_{RT}(u) = (w - r) + r = w$. Assume that maximum Lee weight of a vector is $w - r = i$. For the first r digits, we have $2i + 1$ different choices for each digit. For the $(r + 1)$th digit, zero is not a possible choice so we have $2i$ choices. Thus, we obtained $(2i + 1)^r 2i$ vectors. But if the Lee weight of each of the first $r + 1$ digits are less than or equal $i - 1$, then this contradicts with having weight w. There are $(2i - 1)^r(2i - 2)$ such vectors. We subtract these and we obtain Equation (5).

For a p-ary $[n, k]$-code, the number of all error vectors of LRTJ weight up to w will be denoted by $S_w = \sum_{t=0}^{w} s_t$. For $w = 0,1$, we have $S_0 = 1$ and $S_1 = 3$, respectively. For $w \geq 2$, the value of S_w is given in the following lemma.

Lemma 2. The number of all error vectors of LRTJ weight up to w over $\mathbb{Z}_p$, $w \geq 2$ and $p \geq 5$ prime, is

$$S_w = 3 + \sum_{t=2}^{w} \left[2 + \sum_{i=1}^{t-1} \left[(2i + 1)^{t-i}(2i) - (2i - 1)^{t-i}(2i - 2)\right]\right] \tag{7}$$

Proof. By definition, $S_w = \sum_{t=0}^{w} s_t = s_0 + s_1 + \sum_{t=2}^{w} s_t$. Then, the result follows directly from Lemma 2.

In Table 1, we tabulate the values of s_w and S_w for $1 \leq w \leq 15$. In this table, $S_1 = 3$ and $S_2 = 11$ are prime. We will analyze these cases in details in later sections.

Table 1. The number of error vectors for LRTJ weights over $\mathbb{Z}_p$, $p \geq 5$.

w	s_w	S_w	primality of S_w
1	2	3	yes
2	8	11	yes
3	34	45	no
4	160	205	no
5	834	1039	yes
6	4776	5815	no
7	29762	35577	no
8	200192	235769	no
9	1444354	1680123	no
10	11120008	12800131	yes
11	90948450	103748581	no
12	787057440	890806021	no
13	7181085506	8071891527	no
14	68861316008	76933207535	no
15	692064556162	768997763697	no

3. PERFECT LINEAR CODES CORRECTING ALL ERROR VECTORS OF LRTJ WEIGHT 1

In this section, we will examine whether there exist 1-perfect linear codes; that is, perfect linear codes correcting all error vectors of LRTJ weight 1 but no error vectors of LRTJ weight greater than 1. For this purpose, we will first give an upper bound on the number of parity check digits for linear codes correcting all error vectors of LRTJ weight 1 over $\mathbb{Z}_p$. Then, we will examine this bound to determine the possible 1-perfect linear code parameters. Lastly, we will illustrate this with an example.

Theorem 1. Let C be a p-ary $[n,k]$-linear code. If all errors of weight 1 of C can be corrected, then the parity check digits satisfy $n-k \geq \log_p 3$.

Proof. Firstly we will count all error vectors of LRTJ weight 1. These error vectors are $1,0,0,\dots,0$ and $(p-1),0,0,\dots,0$. If we also include the zero vector, then we obtain three vectors; that is, $S_1 = 3$, as given in Table 1.

Error vectors must be in distinct cosets and the number of all cosets is p^{n-k}. Therefore, we have $p^{n-k} \geq 3$.

Hence the proof is completed.

Note that the number of all error vectors with LRTJ weight 1 does not depend on the value of n.

Lemma 3. Let C be a p-ary $[n,k]$-linear code. If C is a 1-perfect linear code, then $p^{n-k} = 3$.

$p^{n-k} = 3$ holds only when $n-k = 1$ and $p = 3$.

Next, we will find the possible perfect code parameters $[n,k]$ satisfying this equality. The set of $[n,k]$ values is as follows:

$$M_{p^{n-k}} = \{\ [n,k]:\ k = n-1,\ n \geq 2,\ n \in \mathbb{Z}\}. \quad (9)$$

Theorem 2. Let C be a $1-$perfect linear code. The parity check matrix is in $[x_1, x_2, \dots, x_n]$ form where $x_1 \neq 0$ and $x_i \in \mathbb{Z}_3$, $i = 2, \dots, n$.

Proof. The error vectors of LRTJ weight 1 and the zero vectors are as follows:

$$0,0,0,\dots,0,$$
$$1,0,0,\dots,0,$$
$$2,0,0,\dots,0.$$

Let $H = [x_1, x_2, \dots, x_n]_{1\times n}$ be the parity check matrix of C. The syndromes of error vectors can be calculated as follows:

Error vector	Syndrome
$0,0,\dots,0$	0
$1,0,\dots,0$	x_1
$2,0,\dots,0$	$2x_1$

All syndromes are disjoint in $\mathbb{Z}_3$.

Corollary 1. The $[n,k]$ elements of $M_{3^{n-k}}$ satisfy parity check matrices for $n \geq 2$ and $k \geq 1$.

Example 1. Consider a 3-ary [2,1]-code C with parity check matrix $H = [1\ \ 2]$.

The error patterns and their syndromes are tabulated in Table 2.

Table 2. Error vectors of LRTJ weight up to 1 and their syndromes

Error vector	Syndrome
0,0	0
1,0	1
2,0	2

The associated syndromes of error vectors with LRTJ weight of up to 1 are independent. Therefore, C is a 1-perfect linear code.

4. PERFECT LINEAR CODES CORRECTING ALL ERROR VECTORS OF LRTJ WEIGHT 2

In this section, we will first obtain an upper bound on the number of parity check digits for linear codes correcting all error vectors of LRTJ weight up to 2 over $\mathbb{Z}_3$ and $\mathbb{Z}_p$. Then, we will study the existence of 2-perfect linear codes over $\mathbb{Z}_3$ and $\mathbb{Z}_p$.

Theorem 3. Let C be a 3-ary $[n,k]$-linear code. If all errors of weight up to 2 of C code can be corrected, then the parity check digits satisfy $n-k \geq \log_p 9$.

Proof. Firstly, we will determine all error vectors of LRTJ weight 2 over $\mathbb{Z}_3$. Each error vector has maximum two nonzero positions and the possible values are 0, 1 and $p-1=2$ for the first part, and 1 and 2 for the second part. So, all error vectors of LRTJ weight 2 over $\mathbb{Z}_3$ are listed as:

$$0,1,0,\ldots,0,$$
$$1,1,0,\ldots,0,$$
$$2,1,0,\ldots,0,$$
$$0,2,0,\ldots,0,$$
$$1,2,0,\ldots,0,$$
$$2,2,0,\ldots,0.$$

From Table 1, the number of vectors with LRTJ weight up to 1 is 3. Consequently, the number of parity check digits is at least 9. Thus,

$$p^{n-k} \geq 9. \tag{10}$$

Next, we will analyze the case for 2-perfect linear codes.

Lemma 4 Let C be a 3-ary $[n,k]$-linear code. If C is a 2-perfect linear code, then

$$p^{n-k} = 3^2. \tag{11}$$

Equation (11) holds only when $n-k=2$ and $p=3$. The set of possible $[n,k]$ values is as follows

$$M_{3^2} = \{\ [n,k]:\ k = n-2,\ n \geq 3, n \in \mathbb{Z}\}. \tag{12}$$

In the following example, we will illustrate Lemma 4.

Example 2. Consider a 3-ary [3,1]-code $C = \{000,101,202\}$ with parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}.$$

Linear code C corrects all error vectors of LRTJ weigh up to 2. We tabulate error patterns and their syndromes in Table 3.

Indeed, the error vectors of LRTJ weight up to 2 and their corresponding syndromes are distinct. Therefore, C is a 2-perfect linear code.

Table 3. Error vectors of LRTJ weight up to 2 and their syndromes

Error vector	Syndrome
0,0,0	00
1,0,0	11
2,0,0	22
0,1,0	10
1,1,0	21
2,1,0	02
0,2,0	20
1,2,0	01
2,2,0	12

Theorem 4. Let C be a p-ary $[n,k]$-linear code. If all errors of weight up to 2 of C can be corrected, then parity check digits satisfy $n-k \geq \log_p 11$.

Proof. Firstly, we will generate all error vectors of LRTJ weight 2 over $\mathbb{Z}_p$. Each error vector has maximum two nonzero positions and the possible values are 2, 1, 0, $(p-1)$ and $p-2$ for the first part, and 1 and $p-1$ for the second part. So, all error vectors of LRTJ weight 2 over $\mathbb{Z}_p$ are listed as:

$$\begin{array}{lllll} 2 & 0 & 0 & \ldots & 0, \\ (p-2) & 0 & 0 & \ldots & 0, \\ 1 & 1 & 0 & \ldots & 0, \\ 0 & 1 & 0 & \ldots & 0, \\ (p-1) & 1 & 0 & \ldots & 0, \\ 1 & (p-1) & 0 & \ldots & 0, \\ 0 & (p-1) & 0 & \ldots & 0, \\ (p-1) & (p-1) & 0 & \ldots & 0. \end{array}$$

The value S_1 is 3. Therefore, we have 11 error vectors of LRTJ weight up to 2 over $\mathbb{Z}_p$. See also Table 3. Each error vector must be in different cosets. So, we have the following inequality:

$$p^{n-k} \geq 11.$$

Next, we will analyze the case for 2-perfect linear codes over $\mathbb{Z}_p$.

Lemma 5. Let C be a p-ary $[n,k]$-linear code. If C is a 2-perfect linear code, then

$$p^{n-k} = 11. \quad (13)$$

Equation 8 holds only when $n-k=1$ and $p=11$. The set of possible $[n,k]$ values is as follows:

$$M_{11^1} = \{\ [n,k]:\ k = n-1,\ n \geq 2, n \in \mathbb{Z}\}. \quad (14)$$

The length of error vectors must at least 2. Otherwise, it is not possible to calculate the number of all error vectors of LRTJ weight 2.

Theorem 5. Let $0 \neq a \in \mathbb{Z}_{11}$ and $n \geq 2$. The parity check matrix for a 2 −perfect linear code is a $1 \times n$ matrix and has two possible forms:

i. $[a \quad 4a \quad 0 \quad \ldots \quad 0]$,
ii. $[a \quad p-4a \quad 0 \quad \ldots \quad 0]$.

Proof. The parity check matrices in Forms i. and ii. cover all possibilities when we consider the error vectors with LRTJ weight up to 2.

Example 3. Consider a 11-ary [3,2]-code C with parity check matrix

$$H = [6\ \ 2\ \ 5].$$

The syndrome values of corresponding error vectors with LRTJ weight of up to 2 are tabulated in Table 4. These values cover all possibilities in $\mathbb{Z}_{11}$. So C is a 2-perfect code over $\mathbb{Z}_{11}$.

Table 4. Error vector and their corresponding syndromes

Error vector	Syndrome
0,0,0	0
1,0,0	6
10,0,0	5
2,0,0	1
9,0,0	10
1,1,0	8
0,1,0	2
10,1,0	7
1,10,0	4
0,10,0	9
10,10,0	3

Theorem 6. There does not exist a 2-perfect linear code over $\mathbb{Z}_p$ expect $\mathbb{Z}_3$ and $\mathbb{Z}_{11}$.

Proof. The proof follows from Lemma 4 and Lemma 5.

5. PERFECT LINEAR CODES CORRECTING ALL ERROR VECTORS OF LRTJ WEIGHT 3

In this section, we will first obtain an upper bound on the number of parity check digits for linear codes correcting all error vectors of LRTJ weight up to 3 over $\mathbb{Z}_p$. Then we will show that there does not exist a 3-perfect linear code over $\mathbb{Z}_p$.

Theorem 7. Let C be a p-ary [n, k]-linear code. If all errors of weight up to 3 of C can be corrected, then parity check digits satisfy $n - k \geq \log_p 45$.

Proof. The value of S_3 is 45 as given in Table 1. These vectors are tabulated in Table 5. Each of these vectors must belong to a distinct coset and we have p^{n-k} cosets in total. Therefore, we have an upper for the number of parity check digits for LRTJ weight up to 3 as $p^{n-k} \geq 45$.

Theorem 8. There does not exist a 3 −perfect linear code over $\mathbb{Z}_p$.

Proof. Suppose that there exists a 3 −perfect linear code C over $\mathbb{Z}_p$ If C is a 3 −perfect linear code, then equation $p^{n-k} = 45$ is satisfied for a prime p. This is not possible as 45 is not a prime power.

Table 5. All error vectors of LRTJ weight up to 3

$(p-2),(p-2),0,\ldots,0$	$(p-1),2,0,\ldots,0$
$(p-1),(p-2),0\ldots,0$	$0,2,0,\ldots,0$
$0,(p-2),0,\ldots,0$	$1,2,0,\ldots,0$
$1,(p-2),0,\ldots,0$	$2,2,0,\ldots,0$
$2,(p-2),0,\ldots,0$	$(p-1),(p-1),(p-1),0,\ldots,0$
$(p-2),(p-1)0\ldots0$	$0,(p-1),(p-1),0,\ldots,0$
$(p-1),(p-1)0\ldots0$	$1,(p-1),(p-1),0,\ldots,0$
$0,(p-1)0\ldots0$	$(p-1),0,(p-1),0,\ldots,0$
$1,(p-1)0\ldots0$	$0,0,(p-1),0,\ldots,0$
$2,(p-1)0\ldots0$	$1,0,(p-1),0,\ldots,0$
$(p-3),0,0,\ldots,0$	$(p-1),1,(p-1),0,\ldots,0$
$(p-2),0,0,\ldots,0$	$0,1,(p-1),0,\ldots,0$
$(p-1),0,0,\ldots,0$	$1,1,(p-1),0,\ldots,0$
$0,0,0,\ldots,0$	$(p-1),(p-1),1,0,\ldots,0$
$1,0,0,\ldots,0$	$0,(p-1),1,\ldots,0$
$2,0,0,\ldots,0$	$1,(p-1),1,0,\ldots,0$
$3,0,0,\ldots,0$	$(p-1),0,1,0,\ldots,0$
$(p-2),1,0,\ldots,0$	$0,0,1,0\ldots,0$
$(p-1),1,0,\ldots,0$	$1,0,1,0\ldots,0$
$0,1,0,\ldots,0$	$(p-1),1,1,0,\ldots,0$
$1,1,0,\ldots,0$	$0,1,1,0,\ldots,0$
$2,1,0,\ldots,0$	$(p-1),1,1,0,\ldots,0$
$(p-2),2,0,\ldots,0$	

6. PERFECT LINEAR CODES CORRECTING ALL ERROR VECTORS OF LRTJ WEIGHT 4

In this section, we will first obtain an upper bound on the number of parity check digits for linear codes correcting all error vectors of LRTJ weight up to 4 over $\mathbb{Z}_p$. Then, we will show that there does not exist a 4-perfect linear code over $\mathbb{Z}_p$.

Theorem 9. Let C be a p-ary $[n,k]$-linear code. If all errors of weight up to 4 of C can be corrected, then parity check digits satisfy $n-k \geq \log_p 205$.

Proof. The value of S_4 is 205 as given in Table 1. The proof is similar to the proof of Theorem 5.1.

Theorem 10. There does not exist a 4-perfect linear code over $\mathbb{Z}_p$.

Proof. As 205 is not a prime power, it is not possible to have for a prime p.

7. CONCLUSION

In this work, we have studied the existence of perfect linear codes over $\mathbb{Z}_p$ with respect to the LRTJ metric. We characterized all 1-perfect linear codes over $\mathbb{Z}_p$ and all 2-perfect linear codes over $\mathbb{Z}_3$ and over $\mathbb{Z}_{11}$. Moreover, we showed that there does not exist a 2-perfect linear code over $\mathbb{Z}_p$ when $p \neq 3,7$. Then, we showed that there does not exist a 3-perfect linear code over $\mathbb{Z}_p$ and a

4-perfect linear code over $\mathbb{Z}_p$. These results can be generalized for larger w values when S_w is not a prime power. The smallest unresolved case is when $w = 5$.

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