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Research Article

THE EXISTENCE OF GLOBAL ATTRACTORS FOR SUSPENSION BRIDGE EQUATIONS

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ABSTRACT

The aim of this paper is to establish a well-posedness result and the existence of finite- dimensional global attractors for a model of a coupled suspension bridge as well as the regularity of global attractor is achieved. This result generalizes the previous result in [6].
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1. INTRODUCTION

Taccoma Narrows bridge collapse is certainly the most impressive setback in the history. The crucial event in the collapse was a sudden change from vertical to torsional oscillations .

From the physical point of view, the suspension bridge equation describes the transverse deflection of road bed in the vertical plane. On the other hand, from the mathematical point the suspension bridge model describes the vibration of the vertical plane.

The mathematical model appears necessarily a precise description of the instability and the structural behavior of suspension bridge under the action of the load which reveals its lifelong, the nonlinear behavior of suspension bridge, which is by now well established, also plays a crucial role in causing oscillations. The reliable model for suspension should be nonlinear and it should have enough degrees of freedom to display torsional oscillations.

To motivate our work let us start with some works for example a single equation of suspension bridge, Messaoudi et al [8] suggested the following problem:

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$$\begin{cases} utt\ (x,y,t) + \delta ut(x,y,t) + \mu\Delta 2u(x,y,t) - \int_{-\infty}^{t} g(t-s)\Delta 2u(x,y,s)ds + h(u) = f & \in\ \Omega \times (0,T) \\ u(0,y,t) = uxx(0,y,t) = 0 \ \text{ for } (y,t) & \in\ (-L,L) \times (0,+\infty) \\ u(\pi,y,t) = uxx(\pi,y,t) = 0 \ \text{ for } (y,t) & \in\ (-L,L) \times (0,+\infty) \\ uyy(x,\pm L,t) +\ \delta uxx(x,\pm L,t) = 0 & \text{for } (y,t) \in\ (0,\pi) \times (0,+\infty) \\ uyyy(x,\pm L,t) + (2-\delta)uxxy(x,\pm L,t) = 0 & \text{for } (y,t) \in\ (0,\pi) \times (0,+\infty) \\ u(x,y,0) = u0(x,y), ut(x,y,0) = u1(x,y) & \text{in } \Omega \end{cases}$$

where $\delta,\mu >0$ are constants, $u(x,y,t)$ is the vertical displacement of the plate in the downward direction, $h(u)$ is a restoring force due to hangers of the suspension bridge, $f \in L^2(\Omega)$ is an external force which also includes the gravity. The memory kernel g : $IR_+ \rightarrow IR_+$ is an absolutely continuous function which may blow up at 0. They gave a rigorous well-posedness result and established the existence of a global attractor.

Recently, Lazer and McKenna [5] studied the nonlinear oscillation problems in suspension bridge and presented a (one-dimensional) mathematical model for a suspension as a new problem of nonlinear analysis where they modeled a suspension bridge as a rectangular plate since the plate is perfectly correct and corresponds mechanically to a vibrating suspension bridge. Gazzola [3] suggested an equation with linearized stretching term

$$\Delta^2 u - \delta\Delta u = f \quad in \quad \Omega.$$

Here $u = u(x;t)$ represents the vertical displacement of the plate, and f is an external force including the gravity.

The plate is assumed to be suspended by its vertical edges

$$u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0,$$

and the horizontal edges are free

$$u_{yy}(x,\pm l) + \sigma u_{xx}(x,\pm l) = u_{yyy}(x,\pm l) + (2-\sigma)u_{xxy}(x,\pm l) - u_y(x,\pm l) = 0$$

for all $y \in (-l,l)$ and all $x \in (0,\pi)$, where $0< \sigma < ½$ is the Poisson ratio.

It is well known that Ma and Wang [7] presented the following nonlinear problem which describes a vibrating beam equation coupled with a vibrating string equation

$$\begin{cases} utt + \alpha uxxxx + \delta 1ut + k(u-v) + fB(u) = hB(x\ t) & x \in [0\ L] \\ vtt - \beta vxx + \delta 2vt - K(u-v) + fs(v) = hS(x\ t) & x\ \in [0\ L] \end{cases}$$

with the simply supported boundary-value conditions

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad v(0,t) = v(L,t) = 0, t \geq \tau$$

such that

$$(u-v)^+ = \begin{cases} u - v\ , \text{if } u - v\ > 0 \\ 0, \text{if } u - v \leq 0 \end{cases}$$

where $k >0$ denotes the spring constant of the ties, $\alpha >0$ and $\beta >0$ are the flexural rigidity of the structure and coefficient of tensile strength of the cables, respectively

δ_1, $\delta_2 >0$ are constants, the forces term h_B, $h_S \in L^2_{loc}(IR,L^2(0,l))$ the nonlinear functions $f_B(u)$, $f_S(v) \in C^2(IR\ R)$ represent the source terms.

They proved the existence of pullback D-attractors for the non-autonomous coupled suspension bridge equations with suspended and clamped ends. Similar models have been studied by several authors, we refer the readers to ([6],[9],[10]) and the references therein. For example Jum-Ran Kang [4] investigated the long-time behavior of a solution to the following thermoelastic suspension bridge equation with linear memory

$$\begin{cases} utt + \alpha\Delta 2u - \Delta\, ut + K\, u \pm \int_0^{\infty} \mu(s)\;\; \Delta 2\; u(t-s)ds + \beta\Delta\, \theta \\ \qquad +f(u) = h(x) \qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (0,\infty) \\ \theta t - \Delta\, \theta\; - \beta\Delta\, ut - \int_0^{\infty} k(s)\;\; \Delta\, \theta\, (t-s)ds \qquad\qquad \text{in } \Omega \times (0,\infty) \end{cases}$$

such th at

$$(u\text{-}v)^{+} = \begin{cases} u - v\,, \text{if } u - v > 0 \\ 0, \text{if } u - v \le 0 \end{cases}$$

where **Ω** is a bounded domain in IR^2. Here α is the flexural rigidity of the structure, and $\beta > 0$ provides connection between deflection and temperature and depends on mechanical and thermal properties of the material. They showed the existence of a compact global attractor.

In [12] the authors suggested the following problem of suspension bridges:

$$\begin{cases} Mutt + EIuxxxx - Huxx + \frac{w}{H}\,\frac{EA}{L}\int_0^L u(z,t)dz = f(x,t) \qquad \text{in } (0,L)\times(0,+\infty) \\ I\, vtt + C1vxxxx - (C2 + Hwl2)vxx + \frac{l}{H}\;\frac{EA}{L}\;\int_0^L v(z,t)dz = g(x,t) \;\text{ on } (0,L)\times(0,+\infty) \end{cases} \tag{1}$$

such that M,E,A,w,H,C_1,C_2 l and `I are well determined in [12], by using a continuous model of the suspension bridge and by a quasi stationary approach, a simple formula of the combined vertical/torsional flutter wind speed is given. A good agreement is obtained comparing the predictions from the proposed formula with the flutter speeds of three modern suspension or cable stayed bridges. A more slightly sophisticated and complicated string-beam model was suggested by Lazer-McKenna [5]. They treated the cable as a vibrating string coupled with the vibrating beam of the roadway by piecewise linear springs having a spring constant k if expanded, but no restoring force if compressed. The sustaining cable is subject to some forcing term such as the wind or the motions in the towers. This leads to the following system:

$$\begin{cases} utt\; - c1uxx\; + \;\delta 1ut\; - K1(u\; - v) + = \;f(x,t) \qquad\qquad \text{in } (0,L) \times IR + \\ vtt\; + \;c2uxxxx\; + \;\delta 2ut\; + \;K2(u\; - v) + = \;W0 \qquad\qquad \text{on } (0,L) \times IR +, \end{cases}$$

such that

$$(u\text{-}v)^{+} = \begin{cases} u - v\,, \text{if } u - v > 0 \\ 0, \qquad \text{if } u - v \le 0 \end{cases}$$

where v is the displacement from equilibrium of the cable and u is the displacement of the beam, both measured in the downward direction. δ_1, δ_2 are respectively positive constants and the constants c_1 and c_2 represent the relative strengths of the cables and roadway respectively, whereas K_1 and K_2 are the spring constants and satisfy $K_2 \le K_1$. The two damping terms can possibly be set to 0, while f and W_0 are the forcing terms. They proved the existence and multiplicity of periodic solutions of mathematical model of nonlinearly supported bending beams, and they showed also some nonlinear behaviors as observed in large-amplitude flexings in suspension bridges.

In the present paper, we consider a plate model that better describes torsional oscillations in suspension bridges, we consider a variant of (1), we add to the equation the term $h_1(u)$, $h_2(v)$ which represent the restoring force due to the hangers of the suspension bridge, and a convolution term which means that the stress at any instant t depends on the whole history of strains, here f_1 and f_2 are a nonlinear source terms .

We omit the space variables x,y of $u(x,y,t)$, $\upsilon(x,y,t)$, $u_t(x,y,t)$ and $\upsilon_t(x,y,t)$ and for simplicity denote $u(x,y,t) = u$, $\upsilon(x,y,t) = \upsilon$, $u_t(x,y,t) = u_t$ and $\upsilon_t(x,y,t) = \upsilon_t$, when no confusion arises also the functions considered are all real valued, here $u_t = \partial u(t)/\partial t$, $u_{tt} = \partial^2 u(t)/\partial t^2$, $\upsilon_t = \partial\upsilon(t)/\partial t$ and $\upsilon_{tt} = \partial^2\upsilon(t)/\partial t^2$.

We consider the modified suspension bridge problem

$$\begin{cases} \mathrm{Mutt} + \mathrm{EI}\,\Delta 2\,\mathrm{u} - \mathrm{H}\,\Delta\mathrm{u} + \mathrm{ut} - \int_0^\infty \mu 1(s)\Delta 2\ \mathrm{u}(x,y,t-s)ds + h1(u) = f(x,t) \ \text{in}\ (0,L)\times(0,+\infty) \\ \mathrm{I0\ vtt} + \mathrm{C1}\,\Delta 2\,\mathrm{v} - (\mathrm{C2} + \mathrm{Hw}\ell)\Delta 2\mathrm{v} + \int_0^\infty \mu 2(s)\Delta 2\ \mathrm{v}(x,y,t-s)ds + h2(v) = g(x,t) \ \text{in}\ (0,L)\times(0,+\infty) \\ U(0,y,t) = uxx(0,y,t) = 0 \quad \text{for } (y,t) \text{ in } (-L,L)\times(0,+\infty) \\ U(0,y,t) = vxx(0,y,t) = 0 \quad \text{for } (y,t) \text{ in } (-L,L)\times(0,+\infty) \\ U(\pi,y,t) = uxx(\pi,y,t) = 0 \quad \text{for } (y,t) \text{ in } (-L,L)\times(0,+\infty) \\ v(\pi,y,t) = vxx(\pi,y,t) = 0 \ \text{for } (y,t) \text{ in } (-L,L)\times(0,+\infty) \\ Uyy(x,\pm L,t) + \sigma\, uxx(x,\pm L,t) = 0 \quad \text{for } (x,t) \text{ in } (0,\pi)\times(0,+\infty) \\ Vyy(x,\pm L,t) + \sigma\, uxx(x,\pm L,t) = 0 \quad \text{for } (x,t) \text{ in } (0,\pi)\times(0,+\infty) \\ uyyy(x,\pm L,t) + (2-\sigma)uxxy(x,\pm L,t) - uy\,(x,\pm L,t) = 0 \quad \text{for } (x,t) \text{ in } (0,\pi)\times(0,+\infty) \\ vyyy(x,\pm L,t) + (2-\sigma)vxxy(x,\pm L,t) - vy\,(x,\pm L,t) = 0 \quad \text{for } (x,t) \text{ in } (0,\pi)\times(0,+\infty) \\ u(x,y,t) = u0(x,y), ut(x,y,0) = u1(x,y) \quad \text{in } \Omega \\ v(x,y,t) = v0(x,y), vt(x,y,0) = v1(x,y) \quad \text{in } \Omega \end{cases} \tag{2}$$

where $\Omega = (0,\pi) \times (-l,l) \in \mathrm{IR}^2$, $f_1,f_2 \in \mathrm{IL}^2(\Omega)$. The memory kernel $\mu_i : \mathrm{IR}^+ \rightarrow \mathrm{IR}^+, i = 1,2$ is an absolutely continuous function which may blow up at 0. Δ

We identify some parameters that arise in the equation (2) , for example x and y are the space variables along the beam in the bounded domain Ω.

- t denotes the time variable.
- u and v denote respectively the vertical and torsional components of the oscillation of the bridge.
- ut,vt represent the damping terms, the damping are produced by processes that dissipate the energy stored in the oscillation.
- $f1(x,y)$ and $f2(x,y)$ are the lift and the moment for unit girder length of the self-excited forces
- $h1(u)$ and $h2(v)$ represent restoring force due to the hangers of the suspension bridge
- $\mu 1(.)$, $\mu 2(.)$ represent the viscoelastic materials are a kind of materials that have the properties of keeping past information (memories) and which will be used in the future.
- E and I are, respectively, the elastic modulus and the moment of inertia of the stiffening girder so that EI is the stiffness of the girder
- m denotes the mass per unit length
- $I0$ The polar moment of inertia of the girder section
- 2ℓ the roadway
- $w = mg$ is the weight which produces a cable stress whose horizontal component is H_w,
- $C1$ and $C2$ are, respectively the warping and torsion.

Motivated by the previous works, in the present paper, it is interesting to analyze the influence of the viscoelastic, source on the behavior of (2). Under suitable assumptions on the functions $\mu_i(.)$, $f_i(.,.)(i = 1.2)$, the initial data and the parameters in the equations, to the best of our knowledge, there are no results concerning coupled suspension bridge in the presence of memories terms and more general form of source terms, we establish several results concerning existence and regularity of global attractor.

The scope of this paper is as follows: In Section 2, we give some preliminaries and main result. In Section 3, we prove the existence of global attractor, firstly, we prove the existence of an absorbing set, then, establish the smoothness . In Section 4, we verify the regularity of global attractors.

2. MAIN RESULT

For simplicity, one denote $m = E = I = w = H_w = 1$, $C_1 = 1$ and $C2= \ell =\frac{1}{2}$

Now, we present the following conditions about memory kernel

Assumption (H)

i) $\mu 1,\mu 2 \in C1(0,+\infty)\cap L1(0,+\infty)$

$\dot{\mu 1}(s) \leq 0 \leq \mu 1(s), \quad \forall s \in (0,+\infty)$

$\dot{\mu 2}(s) \leq 0 \leq \mu 2(s), \quad \forall s \in (0,+\infty)$

(ii) $l1= 1 -\int_0^\infty \mu 1(s)ds = 1 - \mu_0 >0, \quad \forall s \in (0,+\infty)$;

$l2=1 -\int_0^\infty \mu 2(s)ds = 1 - \mu_0 >0, \quad \forall s \in (0,+\infty)$;

(iii) $\dot{\mu 1}(s) + \delta\mu 1(s) \leq 0, \quad \forall s \in (0,+\infty), \quad \delta >0,$

$\dot{\mu 2}(s) + \delta\mu 2(s) \leq 0, \quad \forall s \in (0,+\infty), \quad \delta >0,$

Assumption (G)

Concerning the forcing term h_i : IR → IR, $i = 1,2$, we assume that

(i) : $h_i(0) = 0, \quad |h_i(u) - h_i(\bar{u})| \leq K_0 (1 + |u|^p + |\bar{u}|^p)|u -\bar{u}| \quad \forall u, \bar{u} \in (0,+\infty)$

where $\delta >0$ and $p >0$. The condition $p >0$ implies that such that

$$\begin{cases} \lim\limits_{|s|\to\infty} inf \frac{hi(s)}{s} \geq \delta, \quad i = 1,2 \\ \lim\limits_{|s|\to\infty} sup \frac{|hi\prime(s)|}{s} = 0, \quad i = 1,2 \end{cases} \tag{4}$$

Where $0 \leq P <\infty$

As in Dafermos [2], we introduce the relative displacement past history functions as

$$\begin{cases} \phi 1t(x,y,s) = u(x,y,t) - u(x,y,t-s), \ \forall s \geq 0 \\ \phi 2t(x,y,s) = u(x,y,t) - u(x,y,t-s), \ \forall s \geq 0 \end{cases} \tag{5}$$

Then

$$\begin{cases} \phi 1t(x,y,s) + \phi 1s(x,y,s) - ut = 0, \phi 1(x,y,0) = 0 \\ \phi 10(x,y,s) = u0(x;y) - u(x,y,-s) = w1(s) \\ \phi 2(x,y,s) + \phi 1s(x,y,s) - Vt = 0, \phi 2(x,y,0) = 0 \\ \phi 20(x,y,s) = V0(x,y) - V(x,y,-s) = w2(s) \end{cases} \tag{6}$$

where $w1$,$w2$ represents the history of u, v. Consequently, the problem equivalent to

$$\begin{cases} utt + \int_0^s \mu 1(s)ds)\Delta 2\, u - \Delta\, u + ut + \int_0^\infty \mu 1(s)\, \Delta 2\, \phi 1t(x,y,s)ds + h1(s) = f1(x,y) \text{ in } \Omega \times (0,\infty) \\ Vtt + \int_0^s \mu 2(s)ds)\Delta 2\, V - \Delta\, V + Vt + \int_0^\infty \mu 2(s)\, \Delta 2\, \phi 2t(x,y,s)ds + h2(s) = f1(x,y) \text{ in } \Omega \times (0,\infty) \\ \phi 1t(x,y,s) + \phi 1s(x,y,s) - Vt = 0 \\ \phi 1t(x,y,s) + \phi 1s(x,y,s) - Vt = 0 \end{cases} \tag{7}$$

with the boundary conditions

$$\begin{cases} u(0,y,t) = uxx(0,y,t) = 0, \text{for } (y,t) \in (-L;L) \times (0,\infty) \\ u(0,y,t) = Vxx(0,y,t) = 0, \text{for } (y,t) \in (-L,L) \times (0,\infty) \\ u(\pi,y,t) = uxx(\pi,y,t) = 0, \text{for } (y,t) \in (-L,L) \times (0,\infty) \\ V(\pi,y,t) = Vxx(\pi,y,t) = 0, \text{for } (y,t) \in (-L,L) \times (0,\infty) \\ uyy(x,\pm L,t) + \delta uxx(x,\pm L,t) = 0 \text{ for } (x,t) \in (0,\pi) \times (0,\infty) \\ Vyy(x,\pm L,t) + \delta Vxx(x,\pm L,t) = 0 \text{ for } (x,t) \in (0,\pi) \times (0,\infty) \\ uyyyy(x,\pm L,t) + (2-\delta)uxxy(x,\pm L,t) - uy(x,\pm L,t) = 0 \text{for } (x,t) \in (0,\pi) \times (0,\infty) \\ Vyyyy(x,\pm L,t) + (2-\delta)Vxxy(x,\pm L,t) - Vy(x,\pm L,t) = 0 \text{for } (x,t) \in (0,\pi) \times (0,\infty) \\ \phi 1(0,y,s) = \phi 1xx(0,y,s) = 0,\ for\ (y,s) \in (-L,L) \times (0,\infty) \\ \phi 2(0,y,s) = \phi 2xx(0,y,s) = 0,\ for\ (y,s) \in (-L,L) \times (0,\infty) \\ \phi 1(\pi,y,s) = \phi 1xx(\pi,y,s) = 0,\ for\ (y,s) \in (-L,L) \times (0,\infty) \\ \phi 2(\pi,y,s) = \phi 2xx(\pi,y,s) = 0,\ for\ (y,s) \in (-L,L) \times (0,\infty) \\ \phi 1yy(0,\pm L,s) + \delta\phi 1xx(x,\pm L,s) = 0,\ for\ (y,s) \in (0,\pi) \times (0,\infty) \\ \phi 2yy(0,\pm L,s) + \delta\phi 2xx(x,\pm L,s) = 0,\ for\ (x,s) \in (0,\pi) \times (0,\infty) \\ \phi 1yyy(0,\pm L,s) + (2-\delta)\phi 1xxy(x,\pm L,s) = 0 - \phi 1y(x,\pm L,t) = 0\ for\ (y,s) \in (0,\pi) \times (0,\infty) \\ \phi 2yyy(0,\pm L,s) + (2-\delta)\phi 2xxy(x,\pm L,s) = 0 - \phi 2y(x,\pm L,t) = 0\ for\ (y,s) \in (0,\pi) \end{cases} \tag{8}$$

and initial conditions given by

$$\begin{cases} u(x,y,0) = u0(x,y), ut(x,y,0) = u1(x,y) \quad \text{in } \Omega \\ V(x,y,0) = V0(x,y), Vt(x,y,0) = V1(x,y), \text{in } \Omega \\ \phi 10(x,y,s) = u0(x,y) - u(x,y,-s), \text{in } \Omega \times (0,\infty) \\ \phi 20(x,y,s) = V0(x,y) - V(x,y,-s), \text{in } \Omega \times (0,\infty) \end{cases} \tag{9}$$

We will use the standard functional space and denote (.,.) be a $L2(\Omega)$- inner product and $\|.\|p$ be $Lp(\Omega)$ norm. Especially, we take

$$H = V0 = L2(\Omega), \qquad V = V1 = V2 = H2*(\Omega)$$

with

$$H2*(\Omega) = \{\xi \in H2(\Omega),\ \xi = 0 \quad on \quad \{0,\pi\} \times \{-L,L\}$$

equipped with the inner product and norm respectively

$$(u,v) = (\Delta u, \Delta v), \qquad \|u\|V = \|\Delta u\|2$$

Define

$$D(A) = \{u,v \in H4(\Omega) \cap H2*(\Omega),\ (8)\ hold\}$$

such that, $Au = \Delta 2u$, and equip this space with the inner product (Au, Av), and the norm $\|Au\|2=(Au, Au)$. We have the following continuous dense injections

$$D(A) \subset V \subset H = H* \subset V*$$

Where $H*$ and $V*$ are the dual spaces of H, V respectively.

We consider the relative displacement $\phi 1$, $\phi 2$ as new variables and we introduce the weighted IL^2-space as follows

$$\mathbf{L2(R^+, Vi) = \{\phi i: R^+ \to Vi \,/ \int_0^\infty \mu i(s)\|\phi i(s)\|2ds < \infty \ i=1,2\}}$$

which is a Hilbert space endowed with the inner product and norm

$$(u,v)_{Vi} = \int_0^\infty \mu i(r)(u(r)V(r))_{Vi}\ dr$$

$$\|u\|_{\mu i,Vi} = \int_0^\infty \mu i(r)\|u(r)\|^2_{Vi}\, dr,\ i=1,2$$

espectively, where $V_3 = D(A^{\frac{3}{4}})$. Finally, we introduce the following Hilbert spaces

$$H0 = V \times V \times H \times H \times L_{\mu 1}(R^+; V) \times L_{\mu 2}(R^+; V)$$

$$H1 = D(A) \times D(A) \times V \times V \times L_{\mu}{}^2{}_1(IR^+; D(A)) \times L^2_{\mu 2}(IR^+; D(A)),$$

equipped with the norms

$$\|U,V,Ut,\phi1,\phi2\|_{H0}=\|\Delta U\|_2^2+\|\Delta V\|_2^2+\|Ut\|_2^2+\|Vt\|_2^2+\|\phi1\|_2^2+\|\phi2\|_2^2$$

and

$$\|U,V,Ut,\phi1,\phi2\|_{H0}= \|AU\|_2^2+\|AV\|_2^2+\|\Delta Ut\|_2^2+\|\Delta Vt\|_2^2+\|\phi1\|_{2.D(A)}^2+\|\phi2\|_{2\ D(A)}^2$$

We will assume a Poincare′ inequality

$$\tau\| v\|_2^2\leq +\|\Delta v\|_2^2 \ \ \forall \ \ v\in V$$

where τ denotes the first eigenvalue of $\Delta^2 v = \tau\, v$ in Ω.

Using the semigroup theory (see [11]) we can conclude the following theorem

Theorem 2.1 Assume that assumption (H) hold and $f1,f2 \in L^2(\Omega)$. Then the problem (7) − (9) has a weak solution $(u,v,ut,vt, \phi1 , \phi2) \in C([0,T],H0)$ with the initial data $(u_0,v_0,u_1,v_1, \phi1, \phi2) \in H_0$, satisfying

$$(u,v)\in L^\infty(0,T; V);\ (ut,vt)\in L^\infty(0,T; V);\ \ \phi^i \in L^\infty(0,T; L_{\mu i}{}^2(R^+;V)),\ i=1,2$$

and the mapping $\{u0, v0, u1, v1, \phi10, \phi20\} \to \{u(t),v(t),u_t(t),v_t(t),\varphi^{1t},\varphi^{2t}\}$ is continuous in H_0. In addition, if $z^n(t) = (u^n(t),v^n(t),u^nt (t),vt^n(t), \phi^{in})$ is a weak solution of the problem (7) − (9) corresponding to the initial data

$z^n(0)=(u_0{}^n, v_0{}^n, u_1{}^n, v_1{}^n, \phi_0{}^{in})$, then one has

$$\|z_1(t) - z_2(t)\|_{H0} \leq e^{ct}\ \|z_1(0) - z_2(0)\|_{H0}, \qquad t \in [0,T],$$

for some constant $c \geq 0$.

The well-posedness of the problem (7) − (9) implies that the family of operators $S(t) : H_0 \to H_0$ defined by

$$S(t)(u_0,v_0,u_1,v_1, \phi10, \phi20) = (u ,v ,ut ,vt, \phi1t, \phi2t), \quad t \geq 0,$$

where $(u ,v ,ut ,vt, \phi1t, \phi2t)$ is the unique weak solution of the problem (7) − (9), satisfies the semigroup properties and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on H_0.

Now, we recall some basic definitions and theorems concerning a global attractor.

Definition 2.1 *A dynamical system* $(H,S(t))$ *is dissipative if it possesses a bounded absorbing set, that is, a bounded set* $B \subset H$ *such that for any bounded set* $B \subset H$ *there exists* $t_B \geq 0$ *satisfying*

$$S(t)B \subset B,\ \forall\ \ t \geq t_B.$$

Definition 2.2 *[10] Let X be Banach space and B a bounded subset of X. We call a function* $\Phi(.,.)$ *which is defined on* $X \times X$ *a contractive function on* $B \times B$ *if for any sequence* $\{x_n\}_{n\subset N} \subset B$, *there is a subsequence* $\{x_{nk}\}_{k\subset N} \subset \{x_n\}_{n\subset N}$, *such that*

$$\lim_{k\to\infty\ l\to\infty} \Phi(xnk, xnl) = 0. \tag{10}$$

Denote all such contractive functions on $B \times B$ *by* C.

Definition 2.3 [10] *Let* $\{S(t)\}_{t\geq 0}$ *be semigroup on a Banach space* $(X,\|.\|)$ *that has a bounded absorbing set* B_0. *Moreover, assume that for* $\epsilon > 0$ *there exist* $T=T(B_0 , \epsilon)$ *and* $\Phi T(.,.) \subset C(B_0)$ *such that*

$$\|S(T)x-S(T)y\|\leq \epsilon + \Phi_T(x,y),\ \forall\ (x,y) \in B_0$$

where ΦT depends on T. Then $\{S(t)\}_{t\geq 0}$ is asymptotically compact in X, i.e , for any bounded sequence $\{y_n\}_{n\subset N} \subset X$ *and* $\{t_n\}$ *with* $t_n \to \infty$ $\{S(t_n)y_n\}_{n\subset N}$ *is precompact in X.*

Theorem 2.2 [6] *A dissipative dynamical system* $(H, S(t))$ *has a compact global attractor if and only if it is asymptotically smooth.*

Our main result is the following:

Theorem 2.3 *Assume that assumptions* (*H*) , (*G*) *hold and* $h1, h2 \in C^1(\mathrm{IR},\mathrm{IR})$, $f1, f2 \in L^2(\Omega)$, *then the dynamical system* (H_0, $S(t)$) *corresponding to the system* (7) − (9) *has a compact global attractor* $A \subset H_0$, *which attracts any bounded set in* H_0 *with* $\|.\|_{H0}$.

3. GLOBAL ATTRACTOR IN H_0

In this section, we would like to prove Theorem 2.3 by showing that the dynamical system (H_0, $S(t)$) is dissipative, and verify the asymptotic compactness. Therefore, we get the existence of compact global attractor by Theorem 2.2.

3.1. Existing of absorbing set

We formally take the scalar product in H of the first equation of (7) with $\varphi = ut + \theta u$ and the second $w = vt+\theta v$ after a computation we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\varphi\|_2^2+\|w\|_2^2+l_1\|\Delta u\|_2^2+l_2\|\Delta v\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2\\
&+2\int H1(V)dx+2\int H2(V)dx-2\int f1(x,y)dx\\
&-2\int f1(x,y)dx)+\Theta l_1\|\Delta u\|_2^2+\Theta l_2\|\Delta v\|_2^2+\Theta\|\nabla u\|_2^2\\
&+\Theta\|\nabla v\|_2+(1-\Theta)(ut,\varphi)++(1-\Theta)(vt,\varphi)+(\phi 1,ut)_{\mu 1,V}\\
&+(\phi 2,vt)_{\mu,V}+\Theta(\phi 2,vt)_{\mu 2,V}+\Theta\int h1(u)u\,dx+\Theta\int h2(v)v\,dx\\
&-2\int f1(x,y)udx-2\int f2(x,y)vdx
\end{aligned}\tag{11}$$

Exploiting (*H*) and Hӧlder inequality, we have

$$(1-\theta)(ut,\varphi)=(1-\theta)\|\varphi\|_2^2-\theta(1-\theta)(u,\varphi),$$

$$(1-\theta)(vt,w)=(1-\theta)\|w\|_2^2-\theta(1-\theta)(v,w),$$

And

$$\begin{aligned}
&(\phi 1,ut)=(\phi 1,\phi 1t+\phi 1s)=\frac{1}{2}\frac{d}{dt}\|\phi 1\|_2^2+\int_0^\infty \mu_i(s)(\phi 1(s),\phi 1s(s))_V\,ds\\
&=\frac{1}{2}\frac{d}{dt}\|\phi 1\|_2^2+\frac{1}{2}\int_0^\infty \mu_i(s)\frac{d}{ds}\|\phi 1(s)\|_v^2ds\\
&=\frac{1}{2}\|\phi 1\|_2^2-\frac{1}{2}\int_0^\infty{}'\mu_i(s)\frac{d}{ds}\|\phi 1(s)\|_v^2ds\\
&\geq\frac{1}{2}\frac{d}{dt}\|\phi 1\|_2^2+\frac{\delta}{2}\int_0^\infty \mu_i(s)\|\phi 1(s)\|_v^2ds=\frac{1}{2}\frac{d}{dt}\|\phi 1\|_2^2+\frac{\delta}{2}\|\phi 1(s)\|_v^2
\end{aligned}\tag{12}$$

$$\begin{aligned}
&(\phi 2,vt)_V\geq\frac{1}{2}\frac{d}{dt}\|\phi 2\|_2^2+\frac{1}{2}\|\phi 2\|_2^2\\
&\theta(\phi 1,u)_V\geq\frac{-\delta}{4}\|\phi 1\|_2^2-\frac{(1-l1)\theta}{\delta}\|\Delta u\|_2^2\\
&\theta(\phi 2,v)_V\geq\frac{-\delta}{4}\|\phi 2\|_2^2-\frac{(1-l2)\theta}{\delta}\|\Delta v\|_2^2
\end{aligned}\tag{13}$$

We choose θ small enough, such that

$$1-\frac{(1-l1)\,\theta}{\delta}-\frac{\theta}{2\zeta}\geq 1-\theta,\ \left(\frac{1}{2}-\theta\right)\geq\frac{1}{4}\ ,\ 1-\frac{(1-l2)\,\theta}{\delta}-\frac{\theta}{2\zeta}\geq 1-\theta,\ \left(\frac{1}{2}-\theta\right)\geq\frac{1}{4}$$

hence, we conclude from Hӧlder, Young and Poincaré inequalities

$$\begin{aligned}
&\theta l1\left(1-\frac{(1-l1)\,\theta}{\delta}\right)\|\Delta u\|_2^2+(1-\theta)\|\varphi\|_2^2-\theta(1-\theta)(u,\varphi)\\
&\geq\theta\left(1-\frac{(1-l1)\,\theta}{\delta}\right)\|\Delta u\|_2^2+(1-\theta)\|\varphi\|_2^2-\frac{\varphi}{\sqrt{\zeta}}\|\Delta u\|_2^2\|\varphi\|_2^2\\
&\geq\theta\left(1-\frac{(1-l1)\,\theta}{\delta}\right)\|\Delta u\|_2^2+(1-\theta)\|\varphi\|_2^2-\left(\frac{\varphi}{\sqrt{\zeta}}\|\Delta u\|_2^2+\frac{1}{2}\|\varphi\|_2^2\right)\\
&+\left(\frac{1}{2}-\theta\right)\|\varphi\|_2^2\\
&\geq\theta l1\|\Delta u\|_2^2+\frac{1}{4}\|\varphi\|_2^2
\end{aligned}\tag{14}$$

Analogously

$$\Theta l2(1-\frac{(1-l2)\,\theta}{\delta})\ \|\Delta v\|_2^2+(1-\theta)\ \|w\|_2^2-\theta(1-\theta)(v,w) \geq \theta l2\ \|\Delta v\|_2^2+\frac{1}{4}\|w\|_2^2 \quad (15)$$

Combining (12) and (15), we have

$$\frac{1}{2}\frac{d}{dt}(\|\varphi\|_2^2+\|w\|_2^2+l1\|\Delta u\|_2^2+l2\|\Delta v\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\phi 1\|_2^2+\|\phi 2\|_2^2$$
$$2\int H1(V)dx\ +2\ \int H2(V)dx\ -2\int f1(x,y)dx\ -2\int f1(x,y)dx)$$
$$+\frac{1}{2}\|\varphi\|_2^2+\frac{1}{2}\|w\|_2^2+2\ \theta\ l1\ (1-\theta)\ \|\Delta u\|_2^2++2\ \theta\ l1\ (1-\theta)\ \|\Delta v\|_2^2$$
$$+2\ \theta\ \|\Delta u\|_2^2+2\ \theta\ \|\Delta v\|_2^2\ +\frac{\delta}{2}\|\phi 1\|_2^2+\frac{\delta}{2}\|\phi 2\|_2^2 \quad (16)$$
$$+2\ \Theta\int h1(u)u\ dx+\ 2\ \Theta\int h2(v)v\ dx$$
$$-2\int f1(x,y)udx\ -2\int f2(x,y)vdx\ \leq 0$$

Put

$$\Theta_0=min\{\ 2\ \theta\ l1((1-\ \theta)-\frac{1}{4});\ 2\ \theta\ l2((1-\ \theta)-\frac{1}{4});\ \frac{1}{2}\ ;\ \ \frac{\delta}{2}\}$$

Let

$$E(t)=\ \|\varphi\|_2^2+\ \|w\|_2^2+l_1\|\Delta u\|_2^2+l_2\|\Delta v\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\phi 1\|_2^2+\|\phi 2\|_2^2$$
$$2\int H1(u)dx\ +2\ \int H2(v)dx\ -2\int f1(x,y)udx\ -2\int f1(x,y)vdx \quad (17)$$

And

$$I(t)=\ \|\varphi\|_2^2+\ \|w\|_2^2+l_1\|\Delta u\|_2^2+l_2\|\Delta v\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\phi 1\|_2^2+\|\phi 2\|_2^2$$
$$2\int h1(u)dx\ +2\ \int h2(v)dx\ -2\int f1(x,y)udx\ -2\int f1(x,y)vdx \quad (18)$$

We have

$$\frac{d}{dt}E(t)+\Theta_0 I(t)\leq 0 \quad (19)$$

which implies that

$$E(t)\leq -\ \Theta_0\int_0^t I(\xi)d\xi+E(0) \quad (20)$$

Hence

$$E(0)=\ \|u1+\Theta\ u0\|_2^2+\ \|u1+\Theta v0\|_2^2+\ \|w\|_2^2+l2\|\Delta u0\|_2^2+l1\|\Delta v0\|_2^2+\|\nabla u0\|_2^2+\|\nabla v0\|_2+\|\phi 10\|_2^2+\|\phi 20\|_2^2$$
$$+\ 2\int H1(u0)dx\ +2\int H2(v0)dx\ -2\int f1(x,y)u0dx-2\int f1(x,y)v0dx$$

Noticing that (4) and (17) − (18), and using the compact Sobolev embedding theorem we get

$$E(t)\geq\ \|\varphi\|_2^2+\ \|w\|_2^2+(l1-\frac{\xi+2\Theta}{2\xi})\ \|\Delta u\|_2^2+(l1-\frac{\xi+2\Theta}{2\xi})\ \|\Delta v\|_2^2$$
$$+\|\nabla u\|_2^2++\|\nabla v\|_2^2+\|\phi 1\|_2^2+\|\phi 2\|_2^2-M \quad (21)$$

Similarly

$$I(t)\geq\ \|\varphi\|_2^2+\ \|w\|_2^2+(l1-\frac{\xi+2\Theta}{2\xi})\ \|\Delta u\|_2^2+(l1-\frac{\xi+2\Theta}{2\xi})\ \|\Delta v\|_2^2$$
$$+\|\nabla u\|_2^2++\|\nabla v\|_2^2+\|\phi 1\|_2^2+\|\phi 2\|_2^2-M \quad (22)$$

where

$M=-\frac{2}{\xi}(\|f1\|^2+(\|f2\|^2+2K1|\Omega|$, therefore $\frac{\xi+2\Theta}{2\xi}<l1$, and $0<\ \Theta_0<\ \xi(l2-\frac{1}{2})$
$\frac{\xi+2\Theta}{2\xi}<l1$, and $0<\ \Theta_0<\ \xi(l2-\frac{1}{2})$, we have

$$L1-\frac{\xi+2\Theta}{2\xi}>0 \quad (23)$$

$$L2-\frac{\xi+2\Theta}{2\xi}>0 \quad (24)$$

And

$$E(t) \geq \|\varphi\|_2^2 + \|w\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u\|_2^2 + + \|\nabla v\|_2^2 + \|\phi 1\|_2^2 + \|\phi 2\|_2^2 - M \quad (25)$$

$$I(t) \geq \|\varphi\|_2^2 + \|w\|_2^2 + (l1 - \frac{\xi+2\theta}{2\xi}) \|\Delta u\|_2^2 + (l1 - \frac{\xi+2\theta}{2\xi}) \|\Delta v\|_2^2 + \|\nabla u\|_2^2 + + \|\nabla v\|_2^2 + \|\phi 1\|_2^2 + \|\phi 2\|_2^2 - M \quad (26)$$

So, we deduce from (25)- (26) and (20) that

$$C_1(\|\varphi\|_2^2 + \|w\|_2^2 + l1\|\Delta u\|_2^2 + l2\|\Delta v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\phi 1\|_2^2 + \|\phi 2\|_2^2 - M_1) \leq \Theta \int_0^t (\|\varphi\|_2^2 + \|w\|_2^2 + l1\|\Delta u\|_2^2 + l2\|\Delta v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\phi 1\|_2^2 + \|\phi 2\|_2^2 - M)dt + E(0) \quad (27)$$

Thus, for any $\delta^2 > \frac{M}{C}$, there exists $t0 = t_0(B)$, such that

$$\|(t0)\|_2^2 + \|w(t0)\|_2^2 + l1\|\Delta u(t0)\|_2^2 + l2\|\Delta v(t0)\|_2^2 + \|\nabla u(t0)\|_2^2 + \|\nabla v(t0)\|_2^2 + \|\phi 1(t0)\|_2^2 + \|\phi 2(t0)\|_2^2 \leq \delta^2 \quad (28)$$

And we end up to

Lemma 3.1 *Assume that assumptions* (*H*) *and* (*G*) *hold and* $h1, h2 \in C^1(IR, IR)$, $f1, f2 \in L^2(\Omega)$, *then the ball of* H_0, $B_0 = B_{H0}(0, \rho_1)$, *centered at* 0 *of radius* ρ_1, *is an absorbing set in* H_0 *for the group* $S(t)$. *For any bounded subset B in* H_0, $S(t)B \subset B_0$ *for* $t \geq t_0$, *there exists a positive constant* $\rho_2 > \rho_1$ *such that* $\forall t \geq t_0$ *we have*

$$\|\varphi\|_2^2 + \|w\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\phi 1\|_2^2 + \|\phi 2\|_2^2 \leq \delta^2 \quad (29)$$

3.2. Attractor

First, we prove the following important Lemma:

Lemma 3.2 *Under the hypotheses of Theorem* 2.3, *there exists a constant* $\rho_3 > \delta$, *such that*

$$\|\varphi\|_2^2 + \|w\|_2^2 + \|\nabla\Delta u\|_2^2 + \|\nabla\Delta v\|_2^2 + \|\nabla ut\|_2^2 + \|\nabla vt\|_2^2 + \|\phi 1\|_2^2 + \|\phi 2\|_2^2 \leq \rho_3 \quad \forall t \geq t0 \quad (30)$$

Proof. Multiplying $(7)_1$ by $-\Delta\varsigma = -\Delta ut - \theta\Delta u$, and $(7)_2$ by $-\Delta\psi = -\Delta vt - \theta\Delta v$ and integrating over Ω, we get

Proof. Multiplying $(7)_1$ by $-\Delta\varsigma = -\Delta ut - \theta\Delta u$, and $(7)_2$ by $-\Delta\psi = -\Delta vt - \theta\Delta v$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + l1\|\nabla\Delta u\|_2^2 + l2\|\nabla\Delta v\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\nabla\psi\|_2^2) + \Theta\, l1\|\nabla\Delta u\|_2^2 + \Theta\, l2\|\nabla\Delta v\|_2^2 + \Theta\|\Delta u\|_2^2 + \Theta\|\Delta v\|_2^2 + (1-\Theta)(ut, -\Delta u) + (1-\Theta)(vt, -\Delta v) + (\phi 1, ut)_{D(A^{\frac{3}{4}})} + \Theta(\phi 1, u)_{D(A^{\frac{3}{4}})} + (\phi 2, vt)_{D(A^{\frac{3}{4}})} + (\phi 2, v)_{D(A^{\frac{3}{4}})} = (h1(u) - f1, \Delta\varsigma) = (h2(v) - f1, \Delta\psi) \quad (31)$$

Similar to previous estimates, we see that

$$(1-\theta)(ut, -\Delta\varsigma) = (1-\theta)\|\nabla\varsigma\|_2^2 - \theta(1-\theta)(\nabla u, \nabla\varsigma),$$
$$(1-\theta)(vt, -\Delta\psi) = (1-\theta)\|\nabla\psi\|_2^2 - \theta(1-\theta)(\nabla v, \nabla\psi),$$

And

$$(\phi 1, ut)_{D(A^{\frac{3}{4}})} \geq \frac{1}{2}\frac{d}{dt}\|\phi 1\|_{D(A^{\frac{3}{4}})} + \frac{\delta}{2}\|ut\|_{D(A^{\frac{3}{4}})}$$
$$(\phi 2, vt)_{D(A^{\frac{3}{4}})} \geq \frac{1}{2}\frac{d}{dt}\|\phi 2\|_{D(A^{\frac{3}{4}})} + \frac{\delta}{2}\|vt\|_{D(A^{\frac{3}{4}})}$$
$$\Theta(\phi 1, u)_{D(A^{\frac{3}{4}})} \geq \frac{-\delta}{4}\|\phi 1\|_{D(A^{\frac{3}{4}})} - \frac{(1-l1)\,\theta}{\delta}\|\nabla\Delta u\|_2^2$$
$$\Theta(\phi 2, v)_{D(A^{\frac{3}{4}})} \geq \frac{-\delta}{4}\|\phi 2\|_{D(A^{\frac{3}{4}})} - \frac{(1-l2)\,\theta}{\delta}\|\nabla\Delta v\|_2^2$$

We have

$$\frac{(1-l1)\,\Theta}{\delta}\ \|\nabla\Delta u\|_2^2 + (1-\Theta)\,\|\nabla\varsigma\|_2^2 - \Theta(1-\Theta)(\nabla u, \nabla\varsigma)$$
$$\geq \Theta(1-\Theta)\ \ \|\nabla\Delta u\|_2^2 + \frac{1}{4})\ \|\nabla\varsigma\|_2^2$$

and

$$\frac{(1-l2)\,\Theta}{\delta}\ \|\nabla\Delta v\|_2^2 + (1-\Theta)\,\|\nabla\psi\|_2^2 - \Theta(1-\Theta)(\nabla v, \nabla\psi)$$
$$\geq \Theta(1-\Theta)\ \ \|\nabla\Delta v\|_2^2 + \frac{1}{4})\ \|\nabla\psi\|_2^2$$

Then we get from (31)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta u\|_2^2+\|\Delta v\|_2^2+l1\|\nabla\Delta u\|_2^2+l2\|\nabla\Delta v\|_2^2+\|\nabla\varsigma\|_2^2\\
&+\|\nabla\psi\|_2^2+\ \|\phi 1\|_{D(A^{\frac{3}{4}})}+\ \|\phi 2\|_{D(A^{\frac{3}{4}})}\ )\\
&+\Theta\,(1-\Theta)l1\|\nabla\Delta u\|_2^2+\Theta\ (1-\Theta)l2\|\nabla\Delta v\|_2^2+\frac{1}{4}\|\nabla\varsigma\|_2^2+\frac{1}{4}\|\nabla\psi\|_2^2 \qquad (32)\\
&+\Theta\|\Delta u\|_2^2+\Theta\|\Delta v\|_2^2+\ \|\phi 1\|_{D(A^{\frac{3}{4}})}+\ \|\phi 2\|_{D(A^{\frac{3}{4}})}\\
&\leq(h1(u)-f1,\ \Delta\varsigma\ )+(h2(v)-f1,\ \Delta\ \psi)
\end{aligned}$$

Similarly, exploiting the bound $\|u\|_2^2 \leq c$, $\|v\|_2^2 \leq c$ which implies that

$\|h1(u)\|_2^2 \leq c$, $\|h2(u)\|_2^2 \leq c$

$$\begin{aligned}
&(h1(u)-f1,\ \Delta\,u_t+\Theta\Delta\,u)\\
&\leq\ \ (\ \|h1(u)\|_2^2+\ \|f1\|_2^2)(\ \|\Delta\,ut\,\|_2^2+\ \|\Delta u\|_2^2\ )\leq C \qquad (33)
\end{aligned}$$

$$\begin{aligned}
&(h2(u)-f2,\ \Delta\,vt+\Theta\Delta\,v)\\
&\leq\ \ (\ \|h2(v)\|_2^2+\|f2\|_2^2)(\ \|\Delta\,vt\,\|_2^2+\|\Delta v\|_2^2\ )\leq C \qquad (34)
\end{aligned}$$

So, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta u\|_2^2+\|\Delta v\|_2^2+l1\|\nabla\Delta u\|_2^2+l2\|\nabla\Delta v\|_2^2+\|\nabla\varsigma\|_2^2\\
&+\|\nabla\psi\|_2^2+\ \|\phi 1\|_{D(A^{\frac{3}{4}})}+\ \|\phi 2\|_{D(A^{\frac{3}{4}})}\ )\\
&+2\Theta\,(1-\Theta)l1\|\nabla\Delta u\|_2^2+2\,\Theta\,(1-\Theta)l2\|\nabla\Delta v\|_2^2+\frac{1}{2}\|\nabla\varsigma\|_2^2+\frac{1}{2}\|\nabla\psi\|_2^2 \qquad (35)\\
&+2\,\Theta\|\Delta u\|_2^2+2\Theta\|\Delta v\|_2^2+\frac{\delta}{2}\ \|\phi 1\|_{D(A^{\frac{3}{4}})}+\frac{\delta}{2}\ \|\phi 2\|_{D(A^{\frac{3}{4}})}\leq 4C
\end{aligned}$$

Thus, denote

$$F(t)=(\|\Delta u\|_2^2+\|\Delta v\|_2^2+l1\|\nabla\Delta u\|_2^2+l2\|\nabla\Delta v\|_2^2+\|\nabla\varsigma\|_2^2+\|\nabla\psi\|_2^2+\ \|\phi 1\|_{D(A^{\frac{3}{4}})}+\ \|\phi 2\|_{D(A^{\frac{3}{4}})}\ )$$

We deduce easily that

$$\frac{d}{dt}F(t)+\Theta_0\,F(t)\leq\ \breve{C}$$

Where $\Theta 0=\min\ \{2\ \Theta\ l1(1-\Theta),\ 2\ \Theta\ l2(1-\Theta),\ \ \frac{1}{2},\ \ \frac{\delta}{2}\}$ and $\breve{C} = 4c$. By Gronwall lemma , we get

$$F(t)\leq \exp\,(-\Theta_0\,t)F(0)+\frac{\breve{C}}{\Theta 0}$$

Using the fact that

$$F(t)\geq \|\Delta ut\|_2^2+\|\Delta vt\|_2^2+\|\nabla\Delta u\|_2^2+\|\nabla\Delta v\|_2^2+\ \|\phi 1\|_{D(A^{\frac{3}{4}})}+\ \|\phi 2\|_{D(A^{\frac{3}{4}})}\ )$$

we obtain (30).

We present important lemmas to prove Theorem 2.3.

Lemma 3.3 *(Stabilizability inequality) Under the hypotheses of Theorem* 2.3, *given a bounded set* $B \subset H_0$, *let* $z1=(u, v, ut, vt, \phi 1,\ \phi 2)$ *and* $z2 = (\bar{u}, \bar{v}, \bar{u}\,t\ , \bar{v}\,t, \xi 1, \xi 1)$ *be two weak solutions of problem* (7) − (9) *such that* $z_1(0) = (u0\ , v0, u1, v1, \phi 110, \phi 220)$ *and* $z_2(0) = (\bar{u}0\ \bar{v}0\ , \bar{u}\,1\ , \bar{v}\,1, \xi 110, \xi 220)$ *are in B. Then, for all* $t \geq 0$, *we have*

$\|z_1(t)-z_2(t)\|_{H0} \leq exp\ (-\nu t)\ \| z_1(0)-z_2(0)\|_{H0}+C_3 \int_0^t exp\,(-\nu(t-s))$
$(\ \|u(s)-\bar{u}(s)\|_{2(p-1)}+\|v(s)-\bar{v}(s)\|_{2(p+1)})ds$ (36)

where $\nu>0$ *is a small constant and* p, C_3 *are positive constants.*

Proof. Let us fix a bounded set $B \subset H_0$. We set $w = u-\bar{u}$, $V= v-\bar{v}$ and $\zeta = \phi 1-\xi 1$, $\rho = \phi 2-\xi 2$. Then (w, ζ) and (V, ρ) satisfy

$$\begin{cases} wtt + l1\Delta 2\, w - \Delta w + wt + \int_0^s \mu 1(s)\Delta 2\, \zeta t(s)ds + h1(u) - h1(\bar{u}) = 0 \\ Vtt + l1\Delta 2\, V - \Delta V + Vt + \int_0^s \mu 1(s)\Delta 2\, \rho t(s)ds + h2(u) - h2(\bar{u}) = 0 \\ \zeta t = -\zeta s + wt \\ \rho t = -\rho s + Vt \end{cases} \quad (37)$$

With initial condition

$$W(0)=u(0)-\bar{u}(0),\ wt(0)=u1-\bar{u}1,\ \zeta(0)= \phi 10-\xi 10$$
$$V(0)=v(0)-\bar{v}(0),\ Vt(0)=V1-\bar{V}1,\ \rho(0)=\phi 20-\xi 20$$

We take the scalar product in H of $(37)_1$ with $\zeta = w_t + \theta w$, and $(37)_2$ with $\psi = Vt+\theta V$, we get

$\frac{1}{2}\frac{d}{dt}(l1\|\Delta w\|_2^2+l2\|\Delta v\|_2^2+\|\zeta\|_2^2+\|\psi\|_2^2+\|\nabla w\|_2^2+\|\nabla V\|_2+\Theta l1\|\Delta u\|_2^2++\Theta l2\|\Delta V\|_2^2+\Theta\|\nabla V\|_2^2+(1-\Theta)(wt, \xi)$
$+(1-\Theta)(Vt, \psi)+(\zeta t, wt)_V+\Theta(\zeta t, w)_V+(\rho t, Vt)_V+\Theta(\rho t, wt)_V$ (38)
$+(h1(u)-h1(\bar{u}), \zeta)+(h2(V)-h2(\bar{V}), \psi)=0$

The same as the previous calculations

$$(1-\Theta)(wt, \zeta)=(1-\Theta)\|\zeta\|_2^2-\Theta(1-\Theta)(w, \zeta)$$
$$(1-\Theta)(Vt, \psi)=(1-\Theta)\|\psi\|_2^2-\Theta(1-\Theta)(V, \psi)$$
$$\Theta(\zeta t, wt)_V \geq \frac{1}{2}\frac{d}{dt}\|\zeta\|_2^2+\frac{\delta}{2}\|wt\|_2^2$$
$$\Theta(\rho t, Vt)_V \geq \frac{1}{2}\frac{d}{dt}\|\rho\|_2^2+\frac{\delta}{2}\|Vt\|_2^2$$

And

$$\Theta(\zeta t, w)_V \geq \frac{-\delta}{4}\|\zeta t\|_2^2+\frac{(1-l1)\theta}{\delta}\|\Delta w\|_2^2$$
$$\Theta(\rho t, V)_V \geq \frac{-\delta}{4}\|\rho t\|_2^2+\frac{(1-l1)\theta}{\delta}\|\Delta V\|_2^2$$

So, we have

$$\Theta l1\left(1-\frac{(1-l1)\theta}{\delta}\right)\|\Delta w\|_2^2+(1-\Theta)\|\zeta\|_2^2-\Theta(1-\Theta)(w, \zeta)$$
$$\geq \Theta l1\,(1-\Theta)\|\Delta w\|_2^2+\frac{\delta}{2}\|\zeta\|_2^2$$
$$\Theta l2\left(1-\frac{(1-l2)\theta}{\delta}\right)\|\Delta V\|_2^2+(1-\Theta)\|\psi\|_2^2-\Theta(1-\Theta)(V, \psi)$$
$$\geq \Theta l2\,(1-\Theta)\|\Delta V\|_2^2+\frac{\delta}{2}\|\psi\|_2^2$$

Then, we have

$\frac{1}{2}\frac{d}{dt}(l1\|\Delta w\|_2^2+l2\|\Delta v\|_2^2+\|\zeta\|_2^2+\|\psi\|_2^2+\|\nabla w\|_2^2+\|\nabla V\|_2^2+\|\zeta t\|_2^2+\|\rho\|_2^2)$
$+\Theta l1\,(1-\Theta)\|\Delta w\|_2^2+\Theta l1\,(1-\Theta)\|\nabla V\|_2^2+\frac{1}{4}\|\zeta\|_2^2+\frac{1}{4}\|\zeta t\|_2^2+\frac{\delta}{4}\|\rho\|_2^2$ (39)
$\leq h1(u)-h1(\bar{u}), \zeta)+(h2(V)-h2(\bar{V}), \psi)$

And

$$
\begin{aligned}
&|-\int_\Omega (h1(u) - h1(\bar{u}))(wt+\Theta w)dx| \\
&\le K\int_\Omega (1+|u|^p+|v|^p)\,|w||wt+\Theta w|dx \\
&\le K\int_\Omega (|\Omega|_{\frac{p}{2(p+1)}} + \|u\|_{2(p+1)}{}^p + \|v\|_{2(p+1)}{}^p)\ \|w\|_{2(p+1)}\ (\|wt\|_2{}^2 + \Theta\,\|w\|_2{}^2) \\
&\le (KC+\Theta)\,\|w\|_2{}^2 + \frac{\Theta}{4}\|\zeta\|_2{}^2
\end{aligned} \tag{40}
$$

By the same technique , we get

$$
|-\int_\Omega (h2(V) - h2(\bar{V}))(Vt+\Theta V)dx| \le (KC+\Theta)\,\|V\|_2{}^2 + \frac{\Theta}{4}\|\psi\|_2{}^2 \tag{41}
$$

We have used the fact that $\|wt\|_2{}^2 = \|\zeta - \Theta w\|_2{}^2$, that $\|Vt\|_2{}^2 = \|\psi - \Theta V\|_2{}^2$ and $c>0$ is an embedding constant for $L^{2(p+1)}(\Omega)\to L^2(\Omega)$. Integrating (37), we get from (39) − (41)

$$
\begin{aligned}
&\frac{d}{dt}(l1\|\Delta w\|_2{}^2 + l2\|\Delta v\|_2{}^2 + \|\zeta\|_2{}^2 + \|\psi\|_2{}^2 + \|\nabla w\|_2{}^2 + \|\nabla V\|_2{}^2 + \|\zeta t\|_2{}^2 + \|\rho t\|_2{}^2) \\
&+\Theta l1\,(1-\Theta)\,\|\Delta w\|_2{}^2 + 2\Theta l1\,(1-\Theta)\,\|\nabla V\|_2{}^2 + (\tfrac{1}{2} - \tfrac{\Theta}{2})\|\zeta\|_2{}^2 + (\tfrac{1}{2} - \tfrac{\Theta}{2})\|\psi\|_2{}^2 \\
&+2\Theta\|\nabla w\|_2{}^2 + 2\Theta\|\nabla V\|_2{}^2 + \frac{\delta}{2}\|\zeta t\|_2{}^2 + \frac{\delta}{2}\|\rho t\|_2{}^2 \\
&| \le (KC+\Theta)\,\|w\|_{2(p+1)}{}^2 + (KC+\Theta)\,\|V\|_{2(p+1}{}^2
\end{aligned} \tag{42}
$$

Choosing Θ small enough, such that

$$
2\Theta\,(1-\Theta) > 0, \qquad (\tfrac{1}{2} - \tfrac{\Theta}{2}) > 0
$$

Here

$$
E(t) = l1\|\Delta w\|_2{}^2 + l2\|\Delta v\|_2{}^2 + \|\zeta\|_2{}^2 + \|\psi\|_2{}^2 + \|\nabla w\|_2{}^2 + \|\nabla V\|_2{}^2 + \|\zeta t\|_2{}^2 + \|\rho t\|_2{}^2
$$

Hence

$$
\frac{d}{dt}E(t) + v1\,E(t) \le C(\|w\|_{2(p+1)}{}^2 + \|V\|_{2(p+1}{}^2)
$$

Where $v1 = \min\{+2\Theta\,(1-\Theta), \left(\frac{1}{2} - \frac{\Theta}{2}\right), \frac{\delta}{2}\}$ and $C = KC + \Theta$ which implies that

$$
E(t) \le \exp(-V1\,.\,t)E(0) + C(\int_0^t \exp(-V1(t-s))\|w\|_{2(p+1)}{}^2 + \|V\|_{2(p+1)}{}^2)ds
$$

Invoking the fact that $E(t) \ge \|z_1(t) - z_2(t)\|_{H0}$, we easly obtain (36).

Lemma 3.4 *(Asymptotic smoothness) Under assumptions of Theorem* 2.3*, the dynamical system* (H_0, $S(t)$) *corresponding to problem* (7) − (9) *is asymptotically smooth.*

Proof. Let B be a bounded subset of H_0 positively invariant with respect to $S(t)$. Denote by C several positive constants that are dependent on B but not on t. For (Z10, Z20) $\in B$, S(t) Z10=(U, Ut, ϕ,) and S(t) Z20=(V, Vt, ξ) are the solutions of (7) − (9). Then given $\epsilon > 0$ from inequality (41), we can choose $T>0$ such that

$$
\|S(t)Z10 - S(t)Z20\|_{H0} \le \epsilon + C\int_0^T (\|u(s) - \bar{u}(s)\|_{2(p+1)}{}^2 + \|V(s) - \bar{V}(s)\|_{2(p+1)}{}^2)^{1/2} ds \tag{43}
$$

where $C_B>0$ is a constant which depends only on the size of B. The condition $p>0$ implies that $2<2(p+1)<\infty$. Taking $= \frac{1}{2}(1 - \frac{1}{p+1})$ and applying Gagliardo-Nirenberg interpolation inequality, we have

$$
\begin{aligned}
\|u(t) - \bar{u}(t)\|_{2(p+1)}{}^2 &\le C\|\Delta u(t) - \bar{u}(t)\|_{2(p+1)}{}^{1/2}\,\|u(t) - \bar{u}(t)\|_{2(p+1)}{}^{1/2} \\
\|V(t) - \bar{V}(t)\|_{2(p+1)}{}^2 &\le C\|\Delta V(t) - \bar{V}(t)\|_{2(p+1)}{}^{1/2}\,\|V(t) - \bar{V}(t)\|_{2(p+1)}{}^{1/2}
\end{aligned}
$$

Since $\|\Delta w\|_2$ and $\|\Delta V\|_2{}^2$ are uniformly bounded, there exists a constant $C>0$ such that

$$
\|u(t) - \bar{u}(t\|_{2(p+1)}{}^2 \le C\|u(t) - \bar{u}(t)\|^{2(p+1)} \tag{44}
$$

$$
\|V(t) - \bar{V}(t)\|_{2(p+1)}{}^2 \le C\|V(t) - \bar{V}(t)\|^{2(p+1)} \tag{45}
$$

Then, from (43) and (44) − (45) we obtain

$$\|S(t)Z10-S(t)Z20\|_{H0} \leq \epsilon + \phi T(Z10, Z20)$$

With

$$\phi T(Z10, Z20) \leq \epsilon + C \int_0^T (\|u(s) - \bar{u}(s)\|_{2(p+1)}^{2(1-\ell)} + \|V(s) - \bar{V}(s)\|_{2(p+1)}^{2(1-\ell)} ds)^{1/2}$$

The following proof $\Phi T \in$ C namely ΦT satisfies (10). Indeed, give a sequence Zn0=(U0n, U1n , ϕ0n) $\in B$, let us write S(t)(Zn0)= =(Un, Unt , ϕnt) is uniformly bounded in H_0. On the other hand,(*Un,Unt*) is bounded in $C([0,T],V \times H), T > 0$. By the compact embedding $V \subset H$, the Aubin lemma implies that there exists a subsequence (u^{nk}) that converges strongly in $C([0,T],H)$. Therefore ,

$$\lim_{k \to \infty\, t \to \infty} \int_0^T \|Unk(s) - Un1(s)\|_{2(p+1)}^{2(1-\ell)} + \|Vnk(s) - Vn1(s)\|_{2(p+1)}^{2(1-\ell)} ds = 0$$

This completes the proof.

4. THE REGULARITY

Our main result is the following theorem

Theorem 4.1 *Under assumptions of Theorem* 2.3*, then the global attractor* A *is a bounded subset* H_1.

4.1. The semigroup decomposition

We fix a bounded set $B \subset H0$ and for $Z = (U0, U1, \phi 0) \in B$, we decompose the solution $S(t)Z = (U, Ut, \phi)$ of problem (7) − (9) into the sum

$$S(t)Z = D(t)Z + K(t)Z,$$

where

$$D(t)Z = z1(t), \quad K(t)Z = Z(t),$$

and

$$z = (U, Ut, \phi) = Z1 + Z2,$$

furthermore,

$$U = \bar{w} + w, \quad V = \bar{v} + v, \qquad \phi 1 = \zeta t + \xi 1t, \qquad \phi 2 = \rho t + \xi 2t$$
$$Z1 = (V, Vt, \zeta t), \qquad Z2 = (w, wt, \xi t),$$

where $Z(t)$ satisfies

$$\begin{cases} \bar{w}tt + l1\Delta 2\, \bar{w} - \Delta\bar{w} + \bar{w}t + \int_0^\infty \mu 1(s)\Delta 2\, \zeta t(s) ds = 0 \\ \qquad \bar{v}tt + l2\Delta 2\, \bar{v} - \Delta\bar{v} + \bar{v}t + \int_0^s \mu 1(s)\Delta 2\, \rho t(s) ds = 0 \\ \zeta t = -\zeta s + \bar{w}t \\ \rho t = -\rho s + \bar{v}t \end{cases} \tag{46}$$

$$\begin{cases} wtt + l1\Delta 2\, w - \Delta w + wt + \int_0^\infty \mu 1(s)\Delta 2\, \zeta t(s) ds = 0 \\ \qquad vtt + l2\Delta 2\, v - \Delta v + vt + \int_0^s \mu 1(s)\Delta 2\, \rho t(s) ds = 0 \\ \zeta t = -\zeta s + wt \\ \rho t = -\rho s + vt \end{cases} \tag{47}$$

The well-posedness of the problem (46) and (47) can be obtained by Faedo-Galerkin methods. Combining with the previous estimate about the solution $Z1(t)$ of equation (46) we obtain the exponential decay of $D(t)Z$

Lemma 4.1 *Under assumptions of Theorem* 2.3*, there exists a constant* $K > 0$*, such that the solution of* (46) *satisfies*

$$\|D(t)Z\|_{H0}^2 \leq Cexp(-k.t)$$

where C is a constant.

About the solution of equation (47), we have the next result that provides the boundedness of $K(t)z$ in a more regular space.

Lemma 4.2 *Under the assumptions of Theorem* 2.3*, there exists a constant N* >0 *such that the solution of* (47) *satisfies*

$$\|K(t)Z\|^2_{H1}{}^2 \leq Cexp(-k.t)$$

Proof. Taking the scalar product in H of $(47)_1$ with $A\zeta = Awt+\theta Aw$ and $(47)_2$ with $A\psi = AVt+\theta AV$, we obtain

$$\begin{aligned}&\frac{d}{dt}(l1\|Aw\|_2^2+l2\|Av\|_2^2+\|\Delta\zeta\|_2^2+\|\Delta\psi\|_2^2+\|\nabla\Delta w\|_2^2+\|\nabla\Delta V\|_2^2)\\&+\Theta l1\ \|Aw\|_2^2+\Theta l2\ \|\nabla V\|_2^2+\Theta\|\nabla\Delta w\|_2^2+\Theta\|\nabla\Delta V\|_2^2+(1-\Theta)(wt, Aw)\\&+(1-\Theta)(Vt, Aw)+(\xi 1t, wt)+(\xi 2t, Vt)+\Theta(\xi 1t, w)_{D(A)}\\&+\Theta(\xi 2t, V)_{D(A)}+(h1(u), A\zeta)+(h2(V), A\psi)\\&=(f1, A\zeta)+(f2, A\psi)\end{aligned} \tag{48}$$

It is the same as the previous estimate

$$(1-\Theta)(wt, A\zeta)=(1-\Theta)\ \|\Delta\zeta\|_2^2-\Theta(1-\Theta)\ (Aw, \zeta)$$
$$(1-\Theta)(Vt, A\psi)=(1-\Theta)\ \|\Delta\psi\|_2^2-\Theta(1-\Theta)\ (AV, \psi)$$
$$(\zeta 1t, wt)_{D(A)}\geq \frac{1}{2}\frac{d}{dt}\ \|\zeta 1t\|_{D(A)}^2+\frac{\delta}{2}\ \|wt\|_{D(A)}^2$$
$$(\zeta 2t, Vt)_{D(A)}\geq \frac{1}{2}\frac{d}{dt}\ \|\zeta 2t\|_{D(A)}^2+\frac{\delta}{2}\ \|Vt\|_{D(A)}^2$$

And

$$\Theta\ (\zeta 1t, w)_{D(A)}\geq \frac{-\delta}{4}\ \|\zeta 1t\|_{D(A)}^2-\frac{(1-l1)\ \Theta}{\delta}\ \|Aw\|_2^2$$
$$\Theta\ (\zeta 2t, V)_{D(A)}\geq \frac{-\delta}{4}\ \|\zeta 2t\|_{D(A)}^2-\frac{(1-l2)\ \Theta}{\delta}\ \|AV\|_2^2$$

We find from (48)

$$\begin{aligned}&\frac{d}{dt}(l1\|Aw\|_2^2+l2\|Av\|_2^2+\|\Delta\zeta\|_2^2+\|\Delta\psi\|_2^2+\|\nabla\Delta w\|_2^2+\|\nabla\Delta V\|_2^2+\|\nabla\Delta\psi\|_2^2)\\&+\ \|\zeta 1t\|_{D(A)}^2+\|\zeta 2t\|_{D(A)}^2\\&+(1-\frac{(1-l1)\ \Theta}{\delta\, l1})\ \|Aw\|_2^2+(1-\frac{(1-l2)\ \Theta}{\delta l2})\ \|AV\|_2^2+(1-\Theta)\ \|\Delta\zeta\|_2^2+(1-\Theta)\ \|\Delta\psi\|_2^2\\&\frac{\delta}{4}\|\zeta 1t\|_{D(A)}^2+\frac{\delta}{4}\|\zeta 2t\|_{D(A)}^2-\Theta(1-\Theta)(Aw,\zeta)-\Theta(1-\Theta)(AV, \psi)\\&(h1(u), A\zeta)+(h2(V), A\psi)\\&=(f1, A\zeta)+(f2, A\psi)\end{aligned} \tag{49}$$

Furthermore

$$\begin{aligned}&\Theta\ l1\ (1-\frac{(1-l1)\ \Theta}{\delta\, l1})\ \|Aw\|_2^2+(1-\Theta)\ )\ \|\Delta\zeta\|_2^2-\Theta(1-\Theta)(Aw, \zeta)\\&\geq\ \Theta l1\ (1-\Theta)\ \|Aw\|_2^2+\frac{1}{4}\ \|\Delta\zeta\|_2^2\end{aligned} \tag{50}$$

$$\begin{aligned}&\Theta\ l2\ (1-\frac{(1-l2)\ \Theta}{\delta\, l2})\ \|AV\|_2^2+(1-\Theta)\ )\ \|\Delta\psi\|_2^2-\Theta(1-\Theta)(AV, \psi)\\&\geq\ \Theta l1\ (1-\Theta)\ \|AV\|_2^2+\frac{1}{4}\ \|\Delta\psi\|_2^2\end{aligned} \tag{51}$$

By Lemma 3.1 and the Sobolev embedding theorem we know that $hi(u)$ and $h'i(u)$ are uniformly bounded in IL^∞ that there exists a constant M >0, such that

$$|h_i(u)| \leq M \quad \text{and}\ |h'_i(u)| \leq M.$$

Combining with the Hölder, Young, Cauchy and (29), (30), it follows that

$$\begin{aligned}
&(h1(u), A\zeta)\\
&= \frac{d}{dt}(h1(u), Aw)+\Theta\,(h1(u), Aw)\\
&\geq \frac{d}{dt}(h1(u), Aw)+\Theta\,(h1(u), Aw)\\
&\geq \frac{d}{dt}(h1(u), Aw)+\Theta\,(h1(u), Aw)-\Theta\,(h'1(u)ut, Aw)\\
&\geq \frac{d}{dt}(h1(u), Aw)+\Theta\,(h1(u), Aw)-\int_\Omega \Theta\, h'1(u)\,|ut|\,|Aw|dx\\
&\geq \frac{d}{dt}(h1(u), Aw)+\Theta\,(h1(u), Aw)-Ml1\,\|Aw\|_2^2\\
&\geq \frac{d}{dt}(h1(u), Aw)+\Theta\,(h1(u), Aw)-\frac{\Theta\, l1}{4}\|Aw\|_2^2-\frac{M\ell}{\Theta\, l1}
\end{aligned} \tag{52}$$

Analogously

$$(h2(V), A\psi) \geq \frac{d}{dt}(h2(V), AV)+\Theta\,(h2(V), AV)-\frac{\Theta\, l2}{4}\|AV\|_2^2-\frac{M\ell}{\Theta\, l2} \tag{53}$$

$$(f1, A\zeta)= (f1, Awt+\Theta Aw) =\frac{d}{dt}(f1, Aw)+\Theta\,(f1, Aw) \tag{54}$$

$$(f2, A\psi)= (f1, AVt+\Theta AV) =\frac{d}{dt}(f2, AV)+\Theta\,(f2, AV) \tag{55}$$

Thus, collecting (50) − (55) from (49) yields

$$\begin{aligned}
&\frac{d}{dt}(l1\|Aw\|_2^2+l2\|Av\|_2^2+\|\Delta\zeta\|_2^2+\|\Delta\psi\|_2^2+\|\nabla\Delta w\|_2^2+\|\nabla\Delta V\|_2^2+\|\nabla\Delta\psi\|_2^2)\\
&+\|\zeta 1t\|_{D(A)}^2+\|\zeta 2t\|_{D(A)}^2+2(h1(u), A\zeta)+2\,(h2(V), AV)\\
&-(f1, Aw)+(f2, AV)+2l1\,(\Theta(1-\Theta)-\frac{\Theta}{4})\,\|Aw\|_2^2\\
&+2l2\,(\Theta(1-\Theta)-\frac{\Theta}{4})\,\|AV\|_2^2+\frac{1}{2}\|\Delta\zeta\|_2^2+\frac{1}{2}\|\Delta\psi\|_2^2+2\Theta\|\nabla\Delta V\|_2^2+2\Theta\|\nabla\Delta w\|_2^2\\
&\frac{\delta}{2}\|\zeta 1t\|_{D(A)}^2+\frac{\delta}{2}\|\zeta 2t\|_{D(A)}^2-2\,\Theta\,(f1, Aw)-2\,\Theta\,(f2, AV)\\
&\leq \frac{M\ell}{\Theta\, l1}+\frac{M\ell}{\Theta\, l2}
\end{aligned} \tag{56}$$

Taking $\Theta_0=\min\{\, 2\Theta(1-\Theta)-\frac{\Theta}{2}, \Theta, \frac{\delta}{2}, \frac{1}{2}\}$ we can obtain from (56)

$$\begin{aligned}
&\frac{d}{dt}(l1\|Aw\|_2^2+l2\|Av\|_2^2+\|\Delta\zeta\|_2^2+\|\Delta\psi\|_2^2+\|\nabla\Delta w\|_2^2+\|\nabla\Delta V\|_2^2+\|\nabla\Delta\psi\|_2^2)\\
&+\|\zeta 1t\|_{D(A)}^2+\|\zeta 2t\|_{D(A)}^2+2(h1(u), Aw)+2\,(h2(V), AV)\\
&-2(f1, Aw)-2(f2, AV)+l1\Theta_0\,\|Aw\|_2^2+l2\,\|AV\|_2^2+\|A\zeta\|_2^2+\|\Delta\psi\|_2^2\\
&+\|\nabla\Delta w\|_2^2+\|\nabla\Delta V\|_2^2\\
&\leq \frac{M\ell}{\Theta\, l1}+\frac{M\ell}{\Theta\, l2}
\end{aligned} \tag{57}$$

On the other hand, by the Höder inequality, the Sobolev embedding theorem and (29), it follows that

$$\begin{aligned}
&\frac{d}{dt}\|\frac{l1}{2}Aw+\sqrt{\frac{l2}{2}}AV+\sqrt{\frac{2}{l1}}h1(u)+\sqrt{\frac{2}{l2}}h2(V)\|_2^2\\
&-\frac{4}{l1}\int_\Omega |h1(u)|\,|h'1(u)|\,|ut|\,dx\;-\frac{4}{l2}\int_\Omega |h2(V)|\,|h'1(V)|\,|Vt|\,dx\\
&\geq \frac{d}{dt}\|\sqrt{\frac{l1}{2}}Aw+\sqrt{\frac{l2}{2}}AV+\sqrt{\frac{2}{l1}}h1(u)+\sqrt{\frac{2}{l2}}h2(V)\|_2^2-\frac{4M\ell}{\Theta\, l1}-\frac{4M\ell}{\Theta\, l2}
\end{aligned} \tag{58}$$

$$\frac{d}{dt}(\frac{l1}{2}\|Aw\|_2^2-2(f1, Aw))=\frac{d}{dt}\|\sqrt{\frac{l1}{2}}Aw-\sqrt{\frac{2}{l1}}f1\|_2^2 \tag{59}$$

$$\frac{d}{dt}(\frac{l2}{2}\|AV\|_2^2-2(f2, AV))=\frac{d}{dt}\|\sqrt{\frac{l2}{2}}AV-\sqrt{\frac{2}{l2}}f2\|_2^2 \tag{60}$$

Therefore, integrating with (58), we get from (57)

$$\frac{d}{dt}\left(\frac{l1}{2}\|Aw\|_2^2-2(f1, Aw)\right)=\frac{d}{dt}\left\|\sqrt{\frac{l1}{2}}Aw-\sqrt{\frac{2}{l1}}f1-\sqrt{\frac{2}{l1}}f2\right\|_2^2$$

$$\|\Delta\zeta\|_2^2+\|\Delta\psi\|_2^2+\|\nabla\Delta w\|_2^2+\|\nabla\Delta V\|_2^2+\|\zeta 1t\|_{D(A)}^2+\|\zeta 2t\|_{D(A)}^2 \quad (61)$$

$$+\Theta_0\left\|\sqrt{\frac{l1}{2}}Aw+\sqrt{\frac{l1}{2}}AV-\sqrt{\frac{2}{l1}}f1-\sqrt{\frac{2}{l1}}f2+\sqrt{\frac{2}{l2}}h1(u)+\sqrt{\frac{2}{l2}}h2(V)\right\|_2^2\leq C$$

Where $C>0$

Applying the Gronwell lemma, there exists a constant *N* such that

$$\|Aw\|_2^2+\|Av\|_2^2+\|\Delta wt\|_2^2++\|\Delta Vt\|_2^2+\|\zeta 1t\|_{D(A)}^2+\|\zeta 2t\|_{D(A)}^2\leq N$$

Completion of the proof of Theorem 4.1

By collecting Lemma 4.1 and Lemma 4.2, we get $(U0,V0, U1, V1, \phi 10, \phi 20)\in H1$ and

$$\|Aw\|_2^2+\|Av\|_2^2+\|\Delta wt\|_2^2++\|\Delta Vt\|_2^2+\|\zeta 1t\|_{D(A)}^2+\|\zeta 2t\|_{D(A)}^2\leq N$$

As $u(t,x)$ satisfies (7) − (9) with initial data $(U0,V0, U1, V1, \phi 10, \phi 20)$, we easily obtain

$$\|(U0,V0, U1, V1, \phi 10, \phi 20)\|_{H0}\leq N$$

Thus A is a bounded subset of H1.

Which completes the proof.

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REFERENCES

[1] I. Chueshov, I. Lasiecka, *Von Karman Evolution Equations: Well-Posedness and Long-Time Dynamics*, Springer Monographs in Mathematics, Springer, New York, 2010.

[2] C. M. Dafermos, *Asymptotic stability in viscoelasticity,* Arch. Ration. Mech. Anal. (1970)37, pp. 204-229.

[3] F. Gazzola, *Mathematical Models for Suspension Bridges Nonlinear Structural Instability,* Springer international publishing Switzerland 2015.

[4] J.R. Kang, *Asymptotic behavior of the thermoelastic suspension bridge equation with linear memory,* Bound Value Probl (2016) 206. 1-18.

[5] A.C. Lazer, P.J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis,* SIAM Rev. 32 1990; 4: 537-578.

[6] S. Liu, Q. Ma, *Global attractors and regularity for the extensible suspension bridge equations with past history,* Korean J. Math. 24 (2016), No. 3, pp. 375-395.

[7] Q. Ma, B. Wang, *Existence of pullback attractors for the coupled suspension bridge equations,* Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 16, pp. 1-10.

[8] S. A. Messaoudi, A. Bonfoh, S. E. Mukiawa, C. D. Enyid, *The global attractor for a suspension bridge with memory and partially hinged boundary conditions,* Z. Angew. Math. Mech 97(2)(2017) 1-14.

[9] J.Y. Park, J.R. Kang, *Pullback D-attractors for non-autonomous suspension bridge equations,* Nonlinear Anal.71(2009), pp. 4618-4623.

[10] J.Y. Park, J.R. Kang, *Global attractors for the suspension bridge equations with non-linear damp ing,* Quart.Appl.Math.69(2011), pp. 465-475.

[11] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* Appl. Math. Sci, Vol. 68, Springer-Verlag, New York, 1988.

[12] M. Como, S. Del Ferraro, A. Grimaldi, *A parametric analysis of the flutter instability for long span suspension bridges*, Wind and Structures 8, 1-12 (2005).
[13] Feng.B, Pelicer.M.L, Andrade. D *Long-time behavior of a semilinear wave equation with memory*, Boundary Value Problems 37(2016).
[14] Feng. B *Long-time dynamics of a plate equation with memory and time delay*, Bull. Braz. Math. Society, 49 1-24(2018).
[15] Monica.C, Pelin.G.G *Existence of smooth global attractors for nonlinear viscoelastic equations with memory*, J.Evol.Equa 15 533-558(2015).