Sigma J Eng & Nat Sci 37 (2), 2019, 445-459

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

SOME NOTES ON INTEGRABILITY CONDITIONS, SASAKI METRICS AND OPERATORS ON (1,1)-TENSOR BUNDLE

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Received: 04.10.2018 Revised: 24.02.2019 Accepted: 05.03.2019

ABSTRACT

The main purpose of the present paper is to study integrability conditions by calculating the Nijenhuis Tensors of almost paracomplex structure F on $(1,1)-$Tensor Bundle. Later, we obtain the Lie derivatives applied to Sasakian metrics with respect to the horizontal and vertical lifts of vector and kovector fields, respectively. Finally, we get the results of Tachibana and Vishnevskii operators applied to horizontal and vertical lifts according to structure F on $(1,1)-$Tensor Bundle $T_1^1(M)$.

Keywords: Integrability conditions, Sasaki Metrics, Tachibana operators, Almost Paracomplex structure, $(1,1)-$Tensor Bundle.

1. INTRODUCTION

Let M be a differentiable manifold of class C^∞ and finite dimension n. Then the set $T_1^1(M) = \bigcup_{P\in M} T_1^1(P)$ is, by definition, the tensor bundle of type $(1,1)$ over M, where $\cup$ denotes the disjoint union of the tensor spaces $T_1^1(P)$ for all $P \in M$. For any point $\tilde{P}$ of $T_1^1(M)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi: T_1^1(M) \to M$. The projection π defines the natural differentiable manifold structure of $T_1^1(M)$, that is, $T_1^1(M)$ is a $C^\infty -$manifold of dimension $n + n^2$. A local coordinate a neighborhood $\{(U; x^j,\ j = 1,\dots,n)\}$ in M induces on $T_1^1(M)$ a local coordinate neighborhood $\{\pi^{-1}(U); x^j, x^{\bar{j}} = t_j^i, \quad j = 1,\dots,n\}, \quad \bar{j}: = n + j$ $(\bar{j} = n + 1,\dots,n + n^2)$, where $x^{\bar{j}} = t_j^i$ are the components of the $(1,1)$ tensor field t in each $(1,1)$ tensor space $T_1^1(P)$, $P \in U$ with respect to the natural base [11].

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We denote by $\Im_s^r(M)$ the module over $F(M)$ of all C^∞ tensor fields of type (r,s) on M, where $F(M)$ is thr ring of real-valued C^∞ functions on M. If $\alpha \in \Im_1^1(M)$, it is regarded, by contraction, as a function on $T_1^1(M)$, which we denote by $\iota\alpha$. If α has the local expression $\alpha = \alpha_i^j \, \partial_j \otimes dx^i$ in a coordinate neighborhood $U(x^j) \subset M$, then $\iota\alpha = \alpha(t)$ has the local expression $\iota\alpha = \alpha_i^j t_j^i$ with respect to the coordinates $(x^j, x^{\bar{\bar{j}}})$ in $\pi^{-1}(U)$.Suppose that $A \in \Im_1^1(M)$. Then there is a unique vector field ${}^V A \in \Im_0^1(T_1^1(M))$, such that for $\alpha \in T_1^1(M)$ [8]

$${}^V A(\iota\alpha) = \alpha(A) o \pi = {}^V(\alpha(A)) \tag{1.1}$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in F(M)$ [11]. The vertical lift ${}^V A$ of A

to $T_1^1(M)$ has components

$${}^V A = \begin{pmatrix} {}^V A^j \\ {}^V A^{\bar{\bar{j}}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix} \tag{1.2}$$

with respect to the coordinates $(x^i, x^{\bar{\bar{j}}})$ in $T_1^1(M)$ [4, 11]. Let L_V be the Lie derivation with respect to $V \in \Im_0^1(M)$. We define the complete lift ${}^C V = \bar{L}_V$ of V to $T_1^1(M)$ by

$${}^C V(\iota\alpha) = \iota(L_V \alpha) \tag{1.3}$$

for $\alpha \in \Im_1^1(M)$ [8, 11]. If ${}^C V = {}^C V^k \partial_k + {}^C V^{\bar{\bar{k}}} \partial_{\bar{\bar{k}}}$, from (1.3), we have

$${}^C V^k t_j^i \partial_k \alpha_i^j + {}^C V^{\bar{\bar{k}}} \alpha_h^k = t_j^i (V^k \partial_k \alpha_i^j - (\partial_k V^j)\alpha_i^k + (\partial_i V^k)\alpha_k^j). \tag{1.4}$$

Discussing in the same way as in the case of the vertical lift, from (1.4), we see that the complete lift ${}^C V$ has the components

$${}^C V = \begin{pmatrix} {}^C V^j \\ {}^C V^{\bar{\bar{j}}} \end{pmatrix} = \begin{pmatrix} V^j \\ t_j^m(\partial_m V^i) - t_m^i(\partial_j V^m) \end{pmatrix} \tag{1.5}$$

with respect to the coordinates $(x^i, x^{\bar{\bar{j}}})$ in $T_1^1(M)$ [4, 9, 11].

Let ∇ be a symetric connection on M. The horizontal lift ${}^H V$ of $V \in \Im_0^1(M)$ to $T_1^1(M)$ has the components

$${}^H V = \begin{pmatrix} {}^C V^j \\ {}^C V^{\bar{\bar{j}}} \end{pmatrix} = \begin{pmatrix} V^j \\ V^s(\Gamma_{sj}^m t_m^i - \Gamma_{sm}^i t_j^m) \end{pmatrix} \tag{1.6}$$

with respect to the c oordinates $(x^i, x^{\bar{\bar{j}}})$ in $T_1^1(M)$, where Γ_{ij}^k are the local components of ∇ on M [4, 9, 11, 12].

Let $\varphi \in \Im_1^1(M)$, which are locally represented by $\varphi = \varphi_j^i \frac{\partial}{\partial x^i} \otimes dx^j$. The vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi$ on $T_1^1(M)$ are defined by

$$\gamma\varphi = (t_j^m \varphi_m^i) \frac{\partial}{\partial x^{\bar{\bar{j}}}},$$

$$\tilde{\gamma}\varphi = (t_m^i \varphi_{j_\mu}^m) \frac{\partial}{\partial x^{\bar{\bar{j}}}}$$

with respect to the coordinates $(x^i, x^{\bar{\bar{j}}})$ in $T_1^1(M)$. From (1.2) we easily see that the vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi$ determine respectively global vector fields on $T_1^1(M)$ [4].

Definition 1.1 The bracket operation of vertical and horizontal vector fields is given by the formulas

$$
\begin{aligned}
{}^{V}A,{}^{V}B] &= 0, \\
{}^{H}X,{}^{V}A] &= {}^{V}(\nabla_X A), \\
{}^{H}X,{}^{H}Y] &= {}^{H}[X,Y] + (\tilde{\gamma} - \gamma)R(X,Y),
\end{aligned}
\tag{1.7}
$$

where R denotes the curvature tensor field of the connection ∇, and $\tilde{\gamma} - \gamma : \varphi \to \Im_0^1(T_1^1(M))$ is the operator defined by

$$(\tilde{\gamma} - \gamma)\varphi = \begin{pmatrix} 0 \\ t_m^i \varphi_j^m - t_j^m \varphi_m^i \end{pmatrix}$$

for any $\varphi \in \Im_1^1(M)$ [11].

From (1.2) and (1.6), we have

$$ {}^{H}X_{(j)} = \delta_j^h \partial_h + (-\Gamma_{js}^k t_h^s + \Gamma_{jh}^s t_s^k)\partial_{\bar{\bar{h}}}, \tag{1.8}$$

$$ {}^{V}A^{(\bar{\bar{j}})} = \delta_i^k \delta_h^j \partial_{\bar{\bar{h}}} \tag{1.9}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^H}\} = \{\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^{\bar{\bar{h}}}}\}$ in $T_1^1(M)$, where $x^{\bar{\bar{h}}} = t_h^k$ and δ_i^j is the Kronecker's. These $n + n^2$ vector fields are linearly independent and generate, respectively, the horizontal distribution of ∇ and the vertical distribution of $T_1^1(M)$. We call the set $\{{}^{H}X_{(j)}, {}^{V}A^{(\bar{\bar{j}})}\}$ the frame the adapted to the affine connection ∇ on $\pi^{-1}(U) \subset T_1^1(M)$. Putting

$$e_{(j)} = {}^{H}X_{(j)}, \; e_{(\bar{\bar{j}})} = {}^{V}A^{(\bar{\bar{j}})},$$

we write the adapted frame as $\{e_\beta\} = \{e_{(j)}, e_{(\bar{\bar{j}})}\}$. The indices $\alpha, \beta, \gamma, \ldots$ run over the range $\{1, \ldots, n, n+1, \ldots, n+n^2\}$ and indicate the indices with respect to the adapted frame $\{e_\beta\}$. Using (1.8) and (1.9), we have the components of the lifts ${}^{H}X$ and ${}^{V}A$

$$ {}^{H}X = ({}^{H}X^{\beta}) = \begin{pmatrix} {}^{H}X^{j} \\ {}^{H}X^{\bar{\bar{j}}} \end{pmatrix} = \begin{pmatrix} X^{j} \\ 0 \end{pmatrix}, \tag{1. 10} $$

$$ {}^{V}A = ({}^{V}A^{\beta}) = \begin{pmatrix} {}^{V}A^{j} \\ {}^{V}A^{\bar{\bar{j}}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j}^{i} \end{pmatrix} \tag{1.11} $$

with respect to the adepted frame $\{e_{\beta}\}$, X^{j} and A_{j}^{i} are the local components of X and A on M, respectively [7,11]. For each $P \in M$, the extension of scalar product g (denoted by G) is defined on the tensor space $\pi^{-1}(P) = T_{1}^{1}(P)$ by $G(A,B) = g_{it} g^{jl} A_{j}^{i} B_{l}^{t}$ for all $A, B \in \Im_{1}^{1}(P)$.

Definition 1.2 The Sasaki metric ${}^{S}g$ (or diagonal lift of g) is defined on $T_{1}^{1}(M)$ by the following three equations [11]

$$ {}^{S}g({}^{V}A, {}^{V}B) = {}^{V}(G(A,B)), \tag{1.12} $$

$$ {}^{S}g({}^{V}A, {}^{H}Y) = 0, \tag{1.13} $$

$$ {}^{S}g({}^{H}X, {}^{H}Y) = {}^{V}(g(X,Y)), \tag{1.14} $$

for any $X, Y \in \Im_{0}^{1}(M)$ and $A, B \in \Im_{1}^{1}(M)$. Since any tensor field of type (0,2) on $T_{1}^{1}(M)$ is completely determined by its action on vector fields of type ${}^{H}X$ and ${}^{V}A$ (see [13], p. 280), it follows that ${}^{S}g$ is completely determined by the equations (1.12)-(1-14).

Definition 1.3 The horizontal lift ${}^{H}\nabla$ of any connection ∇ on the tensor bundle $T_{1}^{1}(M)$ is defined by

$$ {}^{H}\nabla_{{}^{V}A}{}^{V}B = 0, \; {}^{H}\nabla_{{}^{V}A}{}^{H}Y = 0 $$

$$ {}^{H}\nabla_{{}^{H}X}{}^{V}B = {}^{V}(\nabla_{X}B), \; {}^{H}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_{X}Y) \tag{1.15} $$

for all $X, Y \in \Im_{0}^{1}(M)$ and $A, B \in \Im_{1}^{1}(M)$ (see [5,7,8,11]).

1.1. Sasaki Metric ${}^{S}g$ and Paracomplex Structure F on $T_{1}^{1}(M)$

Let $(T_{1}^{1}(M), {}^{S}g)$ be the (1,1) tensor bundle with the Sasaki metrik ${}^{S}g$. From the equation (1.12)-(1-14), we easily see that the horizontal distribution H, induced by ∇_{g} and determined by the horizontal lifts, is orthogonal to the fibres of $T_{1}^{1}(M)$.

Let now $E \in \Im_{0}^{1}(M)$ be a nowhere zero vector field on M. For any $X \in \Im_{0}^{1}(M)$ and $\tilde{E} = g \circ E \in \Im_{1}^{0}(M)$, we define the vertical lift ${}^{V}(X \otimes \tilde{E})$ of X with respect to E.

The map $X \rightarrow^{V} (X \otimes \tilde{E})$ is a monomorphism of $\Im_0^1(M) \rightarrow \Im_0^1(T_1^1(M))$. Hence a n-dimensional C^{∞} vertical distribution V^E is defined on $T_1^1(M)$. Let $V^{\perp}$ be the distribution on $T_1^1(M)$ which is the ortogonal to H and V^E. Then H, V^E and $V^{\perp}$ are mutually orthogonal distributions with respect to Sasaki metric ${}^{S}g$. We define a tensor field F of type $(1,1)$ on $T_1^1(M)$ by [11]

$$\begin{cases} F^{H}X =^{V} (X \otimes \tilde{E}), \\ F^{V}(X \otimes \tilde{E}) =^{H} X, \\ F({}^{V}A) =^{V} A \end{cases} \tag{1.16}$$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$, where $\tilde{E} = g \circ E \in \Im_1^0(M)$. The restrictions of F to $H + V^E$ and $V^{\perp}$ are endomorphisms, and hence F a tensor field type $(1,1)$ on $T_1^1(M)$. It is easy to see that $F^2 = I$. In fact, we have by virtue of (1.16)

$$F^2({}^{H}X) = F(F^{H}X) = F({}^{V}(X \otimes \tilde{E})) =^{H} X,$$
$$F^2({}^{V}(X \otimes \tilde{E})) = F(F^{V}(X \otimes \tilde{E})) = F({}^{H}X) =^{V} (X \otimes \tilde{E}),$$
$$F^2({}^{V}A) = F(F^{V}A) = F({}^{V}A) =^{V} A$$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$, which implies $F^2 = I$ [11].

2. MAIN RESULTS

2.1. Integrability Conditions of Almost Complex Structure on $T_1^1(M)$,

Definition 2.1 Let F be an almost complex structure on M, i.e., $F^2 = -I$. We say that F is integrable if the Nijenhuis tensor N_F of F is identically equal to zero. The Nijenhuis tensor N_F is defined by

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y] \tag{2.1}$$

for any $X, Y \in \Im_0^1(M)$ [1, 10].

In addition the structures are called as an almost product structure for $F^2 = I$ and dual structure for $F^2 = 0$. The condition of $N_F(X, Y) = N(X, Y) = 0$ is essential to integrability condition in these structures. The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^{k}\partial_k = (F_i^s \partial_s^k F_j^k - F_j^l \partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k)\partial_k,$$

where $X = \partial_i$, $Y = \partial_j$, $F \in \Im_1^1(M_n)$,i.e., $F^2 = -I$.

Theorem 2.1 Let $N_F({}^{V}A, {}^{V}B)$, $N_F({}^{H}X, {}^{V}B)$ and $N_F({}^{V}A, {}^{V}(X \otimes \tilde{E}))$ be the Nijenhuis tensors of almost paracomplex structure F on $T_1^1(M)$. Then the almost paracomplex structure F on $T_1^1(M)$ is integrable, where F a tensor field type $(1,1)$ on

$T_1^1(M)$,i.e., $F^2 = I, E \in \Im_0^1(M)$ be a nowhere zero vector field on M, ${}^V(X \otimes \tilde{E})$ is the vertical lift of X with respect to E for any $X \in \Im_0^1(M)$ and $\tilde{E} = g \circ E \in \Im_1^0(M)$.

Proof.

$i)$ From (1.7), (1.16) and Definintion 4, we get the following results

$$N_F({}^VA, {}^VB) = [F^VA, F^VB] - F[F^VA, {}^VB] - F[{}^VA, F^VB] + F^2[{}^VA, {}^VB]$$
$$= [{}^VA, {}^VB] - F[{}^VA, {}^VB] - F[{}^VA, {}^VB] + [{}^VA, {}^VB]$$
$$= 0,$$

$ii)$ $N_F({}^HX, {}^VB) = [F^HX, F^VB] - F[F^HX, {}^VB] - F[{}^HX, F^VB] + F^2[{}^HX, {}^VB]$

$$= [{}^V(X \otimes \tilde{E}), {}^VB] - F[{}^V(X \otimes \tilde{E}), {}^VB] - F[{}^HX, {}^VB] + [{}^HX, {}^VB]$$
$$= -F^V(\nabla_X B) + {}^V(\nabla_X B)$$
$$= -{}^V(\nabla_X B) + {}^V(\nabla_X B)$$
$$= 0,$$

$iii)$ $N_F({}^VA, {}^V(X \otimes \tilde{E})) = [F^VA, F^V(X \otimes \tilde{E})] - F[F^VA, {}^V(X \otimes \tilde{E})]$

$$-F[{}^VA, F^V(X \otimes \tilde{E})] + F^2[{}^VA, {}^V(X \otimes \tilde{E})]$$
$$= [{}^VA, {}^HX] - F[{}^VA, {}^V(X \otimes \tilde{E})] - F[{}^VA, {}^HX]$$
$$+[{}^VA, {}^V(X \otimes \tilde{E})]$$
$$= -{}^V(\nabla_X A) + {}^V(\nabla_X A)$$
$$= 0.$$

Theorem 2.2 Let $N_F({}^HX, {}^HY)$, $N_F({}^V(X \otimes \tilde{E}), {}^V(Y \otimes \tilde{E}))$ and $N_F({}^V(X \otimes \tilde{E}), {}^HY)$ be the Nijenhuis tensors of almost paracomplex structure F on $T_1^1(M)$. Then the almost paracomplex structure F on $T_1^1(M)$ is integrable if and only if the following $i)$, $ii)$ and $iii)$ conditions are required.

$i)$ T is the Torsion tensor $T(X, Y) = 0$,
$ii)$ $\nabla E = 0$,
$iii)$ $R = 0$,

where F a tensor field type $(1,1)$ on $T_1^1(M)$,i.e., $F^2 = I, E \in \Im_0^1(M)$ be a nowhere zero vector field on M, ${}^V(X \otimes \tilde{E})$ is the vertical lift of X with respect to E for any $X \in \Im_0^1(M)$ and $\tilde{E} = g \circ E \in \Im_1^0(M)$.

Proof.

$i)$ From (1.7), (1.16) and Definintion 4, we get

$$N_F({}^HX, {}^HY) = [F^HX, F^HY] - F[F^HX, {}^HY] - F[{}^HX, F^HY] + F^2[{}^HX, {}^HY]$$
$$= [{}^V(X \otimes \tilde{E}), {}^V(Y \otimes \tilde{E})] - F[{}^V(X \otimes \tilde{E}), {}^HY]$$
$$-F[{}^HX, {}^V(Y \otimes \tilde{E})] + [{}^HX, {}^HY]$$
$$= {}^H(\nabla_Y X) + {}^V(X \otimes (g \circ (\nabla_Y E))) - {}^H(\nabla_X Y)$$

$$-^V(Y\otimes(g\circ(\nabla_X E)))+^H[X,Y]+(\tilde{\gamma}-\gamma)R(X,Y)$$
$$=-^H(T(X,Y))+^V(X\otimes(g\circ(\nabla_Y E)))$$
$$-^V(Y\otimes(g\circ(\nabla_X E)))+(\tilde{\gamma}-\gamma)R(X,Y)$$

$$ii)N_F(^V(X\otimes\tilde{E}),^V(Y\otimes\tilde{E}))=[F^V(X\otimes\tilde{E}),F^V(Y\otimes\tilde{E})]-F[F^V(X\otimes\tilde{E}),^V(Y\otimes\tilde{E})]$$
$$-F[^V(X\otimes\tilde{E}),F^V(Y\otimes\tilde{E})]+F^2[^V(X\otimes\tilde{E}),^V(Y\otimes\tilde{E})]$$
$$=[^HX,^HY]-F[^HX,^V(Y\otimes\tilde{E})]-F[^V(X\otimes\tilde{E}),^HY]$$
$$+[^V(X\otimes\tilde{E}),^V(Y\otimes\tilde{E})]$$
$$=-^H(T(X,Y))-^V(Y\otimes(g\circ(\nabla_X E)))$$
$$+^V(X\otimes(g\circ(\nabla_Y E)))+(\tilde{\gamma}-\gamma)R(X,Y)$$

$$iii)\ N_F(^V(X\otimes\tilde{E}),^HY)=[F^V(X\otimes\tilde{E}),F^HY]-F[F^V(X\otimes\tilde{E}),^HY]$$
$$-F[^V(X\otimes\tilde{E}),F^HY]+F^2[^V(X\otimes\tilde{E}),^HY]$$
$$=[^HX,^V(Y\otimes\tilde{E})]-F[^HX,^HY]$$
$$-F[^V(X\otimes\tilde{E}),^V(Y\otimes\tilde{E})]+[^V(X\otimes\tilde{E}),^HY]$$
$$=^V((\nabla_X Y)\otimes\tilde{E})+^V(Y\otimes(g\circ(\nabla_X E)))-^V((L_X Y)\otimes\tilde{E})$$
$$-(\tilde{\gamma}-\gamma)R(X,Y)-^V((\nabla_Y X)\otimes\tilde{E})-^V(X\otimes(g\circ(\nabla_Y E)))$$
$$=^V(T(X,Y)\otimes\tilde{E})+^V(Y\otimes(g\circ(\nabla_X E)))$$
$$-^V(X\otimes(g\circ(\nabla_Y E)))-(\tilde{\gamma}-\gamma)R(X,Y)$$

T is the Torsion tensor $T(X,Y)=0$, $\nabla E=0$ and $R=0$

2.2. Lie Derivations of Sasakian metric ${}^Sg$ with respect to horizontal and vertical lifts on $T_1^1(M)$

Definition 2.2 Let M^n be an $n-$dimensional differentiable manifold. Differential transformation $D=L_X$ is called as Lie derivation with respect to vector field $X\in\Im_0^1(M^n)$ if

$$L_X f=Xf,\forall f\in\Im_0^0(M^n), \tag{2.2}$$
$$L_X Y=[X,Y],\forall X,Y\in\Im_0^1(M^n).$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [2, 3, 13]

$$(L_X F)Y=[X,FY]-F[X,Y]. \tag{2.3}$$

Theorem 2.3 Let ${}^Sg$ be Sasakian metric ${}^Sg$, is defined by (1.12),(1.13),(1.14) and L_X the operator Lie derivation with respect to X. From (1.7), (1.15) and Definintion 5, we get the following results

$$i)(L_{^VC}{}^Sg)(^VA,^VB)=0,$$
$$ii)(L_{^VC}{}^Sg)(^VA,^HY)=^V(G(A,\nabla_Y C)),$$

$iii)(L_{{}^{V}C}{}^{S}g)({}^{H}X, {}^{V}B) = {}^{V}(G(\nabla_X C, B)),$
$iv)(L_{{}^{V}C}{}^{S}g)({}^{H}X, {}^{H}Y) = 0,$
$v)(L_{{}^{H}Z}{}^{S}g)({}^{V}A, {}^{V}B) = {}^{V}((\nabla_Z G)(A,B)),$
$vi)(L_{{}^{H}Z}{}^{S}g)({}^{V}A, {}^{H}Y) = {}^{S}g({}^{V}A, (\tilde{\gamma}-\gamma)R(Z,Y)),$
$vii)(L_{{}^{H}Z}{}^{S}g)({}^{H}X, {}^{V}B) = -{}^{S}g((\tilde{\gamma}-\gamma)R(Z,X), {}^{V}B),$
$$viii)(L_{{}^{H}Z}{}^{S}g)({}^{H}X, {}^{H}Y) = {}^{V}((L_Z{}^{S}g)(X,Y)) - {}^{S}g((\tilde{\gamma}-\gamma)R(Z,X), {}^{H}Y) - {}^{S}g({}^{H}X, (\tilde{\gamma}-\gamma)R(Z,Y)),$$

where the horizontal lifts ${}^{H}X \in \Im_0^1(T_1^1M)$ of $X \in \Im_0^1(M)$ and the vertical lift ${}^{V}A \in \Im_0^1(T_1^1M)$ of $A \in \Im_1^1(M)$ defined by (1.10),(1.11), respectively.

Proof. From (1.7), (1.15) and Definintion 5, we get

$$i)\ (L_{{}^{V}C}{}^{S}g)({}^{V}A, {}^{V}B) = L_{{}^{V}C}{}^{S}g({}^{V}A, {}^{V}B) - {}^{S}g(L_{{}^{V}C}{}^{V}A, {}^{V}B) - {}^{S}g(A^{V}, L_{{}^{V}C}{}^{V}B)$$
$$= {}^{V}C^{V}(G(A,B))$$
$$= 0$$

$$ii)\ (L_{{}^{V}C}{}^{S}g)({}^{V}A, {}^{H}Y) = L_{{}^{V}C}{}^{S}g({}^{V}A, {}^{H}Y) - {}^{S}g(L_{{}^{V}C}{}^{V}A, {}^{H}Y) - {}^{S}g({}^{V}A, L_{{}^{V}C}{}^{H}Y)$$
$$= {}^{S}g({}^{V}A, {}^{V}(\nabla_Y C))$$
$$= {}^{V}(G(A, \nabla_Y C))$$

$$iii)\ (L_{{}^{V}C}{}^{S}g)({}^{H}X, {}^{V}B) = L_{{}^{V}C}{}^{S}g({}^{H}X, {}^{V}B) - {}^{S}g(L_{{}^{V}C}{}^{H}X, {}^{V}B) - {}^{S}g({}^{H}X, L_{{}^{V}C}{}^{V}B)$$
$$= {}^{S}g({}^{V}(\nabla_X C), {}^{V}B)$$
$$= {}^{V}(G(\nabla_X C, B))$$

$$iv)\ (L_{{}^{V}C}{}^{S}g)({}^{H}X, {}^{H}Y) = L_{{}^{V}C}{}^{S}g({}^{H}X, {}^{H}Y) - {}^{S}g(L_{{}^{V}C}{}^{H}X, {}^{H}Y) - {}^{S}g({}^{H}X, L_{{}^{V}C}{}^{H}Y)$$
$$= {}^{V}C^{V}(g(X,Y)) + {}^{S}g({}^{V}(\nabla_X C), {}^{H}Y) + {}^{S}g({}^{H}X, {}^{V}(\nabla_Y C))$$
$$= 0$$

$$v)(L_{{}^{H}Z}{}^{S}g)({}^{V}A, {}^{V}B) = L_{{}^{H}Z}{}^{S}g({}^{V}A, {}^{V}B) - {}^{S}g(L_{{}^{H}Z}{}^{V}A, {}^{V}B) - {}^{S}g({}^{V}A, L_{{}^{H}Z}{}^{V}B)$$
$$= {}^{V}(ZG(A,B)) - {}^{S}g({}^{V}(\nabla_Z A), {}^{V}B) - {}^{S}g({}^{V}A, {}^{V}(\nabla_Z B))$$
$$= {}^{V}((\nabla_Z G)(A,B))$$

$$vi)(L_{{}^{H}Z}{}^{S}g)({}^{V}A, {}^{H}Y) = L_{{}^{H}Z}{}^{S}g({}^{V}A, {}^{H}Y) - {}^{S}g(L_{{}^{H}Z}{}^{V}A, {}^{H}Y) - {}^{S}g({}^{V}A, L_{{}^{H}Z}{}^{H}Y)$$

$$= -{}^{S}g({}^{V}(\nabla_Z A), {}^{H}Y) - {}^{S}g({}^{V}A, {}^{H}[Z,Y] + (\tilde{\gamma}-\gamma)R(Z,Y))$$
$$= -{}^{S}g({}^{V}A, {}^{H}[Z,Y]) - {}^{S}g({}^{V}A, (\tilde{\gamma}-\gamma)R(Z,Y))$$
$$= {}^{S}g({}^{V}A, (\tilde{\gamma}-\gamma)R(Z,Y))$$

$$vii)\ (L_{{}^{H}Z}{}^{S}g)({}^{H}X, {}^{V}B) = L_{{}^{H}Z}{}^{S}g({}^{H}X, {}^{V}B) - {}^{S}g(L_{{}^{H}Z}{}^{H}X, {}^{V}B) - {}^{S}g({}^{H}X, L_{{}^{H}Z}{}^{V}B)$$
$$= -{}^{S}g({}^{H}[Z,X] + (\tilde{\gamma}-\gamma)R(Z,X), {}^{V}B) - {}^{S}g({}^{H}X, {}^{V}(\nabla_Z B))$$
$$= -{}^{S}g({}^{H}(L_Z X), {}^{V}B) - {}^{S}g((\tilde{\gamma}-\gamma)R(Z,X), {}^{V}B)$$
$$= -{}^{S}g((\tilde{\gamma}-\gamma)R(Z,X), {}^{V}B)$$

$$viii)\ (L_{{}^HZ}{}^Sg)({}^HX,{}^HY) = L_{{}^HZ}{}^Sg({}^HX,{}^HY) - {}^Sg(L_{{}^HZ}{}^HX,{}^HY) - {}^Sg({}^HX,L_{{}^HZ}{}^HY)$$
$$= {}^HZ^V(g(X,Y)) - {}^Sg({}^H[Z,X] + (\tilde{\gamma}-\gamma)R(Z,X),{}^HY)$$
$$- {}^Sg({}^HX,{}^H[Z,Y] + (\tilde{\gamma}-\gamma)R(Z,Y))$$
$$= {}^V((L_Z{}^Sg)(X,Y)) - {}^Sg((\tilde{\gamma}-\gamma)R(Z,X),{}^HY)$$
$$- {}^Sg({}^HX,(\tilde{\gamma}-\gamma)R(Z,Y))$$

2.3. Tachibana operators applied to ${}^HX$ and ${}^VA$ According to an Almost Paracomplex Structure F on $T_1^1(M)$

Definition 2.3 Let $\varphi \in \Im_1^1(M)$, and $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M)$ be a tensor alebra over R. A map $\phi_\varphi|_{r+s>0} : \overset{*}{\Im}(M) \to \Im(M)$ is called as Tachibana operatör or ϕ_φ operatör on M if

$a)$ ϕ_φ is linear with respect to constant coefficient,

$b)$ $\phi_\varphi : \overset{*}{\Im}(M) \to \Im_{s+1}^r(M)$ for all r and s,

$c)$ $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K, L \in \overset{*}{\Im}(M)$,

$d)$ $\phi_{\varphi X} Y = -(L_Y\varphi)X$ for all $X, Y \in \Im_0^1(M)$, where L_Y is the Lie derivation with respect to Y (see [2, 3, 6]),

$e)$ $(\phi_{\varphi X}\eta)Y = (d(i_Y\eta))(\varphi X) - (d(i_Y(\eta o \varphi)))X + \eta((L_Y\varphi)X)$
$= \phi X(i_Y\eta) - X(i_{\varphi Y}\eta) + \eta((L_Y\varphi)X)$

for all $\eta \in \Im_1^0(M_n)$ and $X, Y \in \Im_0^1(M_n)$, where $i_Y\eta = \eta(Y) = \eta \overset{C}{\otimes} Y$, $\overset{*}{\Im}{}_s^r(M_n)$ the module of all pure tensor fields of type (r,s) on M_n with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction C [10].

Remark 2.1 If $r = s = 0$, then from $c), d)$ and $e)$ of Definition 6 we have $\phi_{\varphi X}(i_Y\eta) = \phi X(i_Y\eta) - X(i_{\varphi Y}\eta)$ for $i_Y\eta \in \Im_0^0(M_n)$, which is not well-defined $\phi_\varphi -$operator. Different choices of Y and η leading to same function $f = i_Y\eta$ do get the same values. Consider $M = R^2$ with standard coordinates x, y. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $i_Y\eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$. Nov the right-hand side of $\phi_{\varphi X}(i_Y\eta) = \phi X(i_Y\eta) - X(i_{\varphi Y}\eta)$ is $(\phi X)1 - 0 = 0$ in the first case, and $(\phi X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$ [10].

Remark 2.2 From $d)$ of Definition 6 we have

$$\phi_{\varphi X} Y = [\varphi X, Y] - \varphi X, Y].$$

By virtue of

$$fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

for any $f, g \in \Im_0^0(M_n)$, we see that $\phi_{\varphi X} Y$ is linear in X, but not Y [10].

On the other hand, let $t \in \overset{*}{\Im}{}_1^1(M_n)$, i.e $to\varphi = \varphi ot$ (t is a pure with respect to φ and conversely φ is also a pure with respect to t). Then $tY \in \Im_0^1(M_n)$ for any $Y \in \Im_0^1(M_n)$ and by $c)$ of Definition 6 we have

$$\begin{aligned}(\phi_\varphi tY)X &= \left((\phi_\varphi t)X\right) \overset{C}{\otimes} Y + t \overset{C}{\otimes} (\phi_\varphi Y)X \\ &= (\phi_\varphi t)(X,Y) + t\left((\phi_\varphi Y)X\right).\end{aligned} \tag{2.4}$$

Using (2.4), we have from $d)$ of Definition 6

$$\begin{aligned}(\phi_\varphi t)(X,Y) &= (\phi_\varphi tY)X - t\left((\phi_\varphi Y)X\right) \\ &= \left(-L_{tY}\varphi + t(L_Y\varphi)\right)X \\ &= [\varphi X, tY] - \varphi[X, tY] - t[\varphi X, Y] + \varphi t[X,Y]\end{aligned} \tag{2.5}$$

Since $t \circ \varphi = \varphi \circ t$ is trivially satisfied for $t = \varphi$, we obtain from (2.5)

$$\begin{aligned}(\phi_\varphi \varphi)(X,Y) &= \left(-L_{\varphi Y}\varphi + \varphi(L_Y\varphi)\right)X \\ &= [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + \varphi^2[X,Y] \\ &= N_\varphi(X,Y).\end{aligned} \tag{2.6}$$

Thus we have the relationship between the Tacibana operator $\phi_\varphi \varphi$ and the Nijenhuis tensor N_φ,constructed from φ [10].

Theorem 2.4 Let F be an almost paracomplex structure on $T_1^1(M)$,i.e., $F^2 = I$ and ϕ_F be the Tachibana operator, defined by Definition 6, then we get the following results

$i)$ $$\phi_{F^H X}{}^H Y = -{}^V((\nabla_Y X) \otimes \tilde{E}) + {}^V((L_X Y) \otimes \tilde{E}) - {}^V(X \otimes g \circ \nabla_Y E]) + (\tilde{\gamma} - \gamma)R(X,Y),$$

$ii)$ $$\phi_{F^V (Y \otimes \tilde{E})}{}^H X = -{}^H(L_X Y) + {}^H(\nabla_X Y) - (\tilde{\gamma} - \gamma)R(X,Y) + {}^V(Y \otimes g \circ (\nabla_X E)]),$$

$iii)$ $$\phi_{F^V (Y \otimes \tilde{E})}{}^V(X \otimes \tilde{E}) = {}^V((\nabla_Y X) \otimes \tilde{E}) + {}^V(X \otimes g \circ (\nabla_Y E)]),$$

$iv)$ $$\phi_{F^H Y}{}^V(X \otimes \tilde{E}) = -{}^H(\nabla_Y X) - {}^V(X \otimes (g \circ (\nabla_Y E))),$$

$v)$ $$\phi_{F^H Y}{}^V A = -{}^V(\nabla_Y A),$$

$vi)$ $$\phi_{F^V (Y \otimes \tilde{E})}{}^V A = {}^V(\nabla_Y A),$$

vii) $\phi_{F^V{}_B}{}^V(X \otimes \tilde{E}) = 0,$

viii) $\phi_{F^V{}_B}{}^H X = 0,$

ix) $\phi_{F^V{}_B}{}^V A = 0,$

where $E \in \Im_0^1(M)$ be a nowhere zero vector field on M, ${}^V(X \otimes \tilde{E})$ is the vertical lift of X with respect to E for any $X \in \Im_0^1(M)$ and $\tilde{E} = g \circ E \in \Im_1^0(M)$,the horizontal lifts ${}^H X \in \Im_0^1(T_1^1 M)$ of $X \in \Im_0^1(M)$ and the vertical lift ${}^V A \in \Im_0^1(T_1^1 M)$ of $A \in \Im_1^1(M)$ defined by (1.10),(1.11), respectively.

Proof.

i)
$$\begin{aligned}
\phi_{F^H{}_X}{}^H Y &= -(L_{{}^H Y}F)^H X \\
&= -L_{{}^H Y} F^H X + F L_{{}^H Y}{}^H X \\
&= -L_{{}^H Y}{}^V(X \otimes \tilde{E}) + F({}^H[X,Y] + (\tilde{\gamma} - \gamma)R(X,Y)) \\
&= -{}^V((\nabla_Y X) \otimes \tilde{E}) - {}^V(X \otimes \nabla_Y \tilde{E}) \\
&\quad + {}^V((L_X Y) \otimes \tilde{E}) + (\tilde{\gamma} - \gamma)R(X,Y) \\
&= -{}^V((\nabla_Y X) \otimes \tilde{E}) + {}^V((L_X Y) \otimes \tilde{E}) \\
&\quad - {}^V(X \otimes g \circ \nabla_Y E]) + (\tilde{\gamma} - \gamma)R(X,Y)
\end{aligned}$$

ii)
$$\begin{aligned}
\phi_{F^V{}_{(Y \otimes \tilde{E})}}{}^H X &= -(L_{{}^H X}F)^V(Y \otimes \tilde{E}) \\
&= -L_{{}^H X} F^V(Y \otimes \tilde{E}) + F L_{{}^H X}{}^V(Y \otimes \tilde{E}) \\
&= -L_{{}^H X}{}^H Y + F^V(\nabla_X Y \otimes \tilde{E}) \\
&= -{}^H(L_X Y) - (\tilde{\gamma} - \gamma)R(X,Y) + {}^H(\nabla_X Y) \\
&\quad + F^V(Y \otimes g \circ (\nabla_X E)]) \\
&= -{}^H(L_X Y) + {}^H(\nabla_X Y) - (\tilde{\gamma} - \gamma)R(X,Y) \\
&\quad + {}^V(Y \otimes g \circ (\nabla_X E)])
\end{aligned}$$

iii)
$$\begin{aligned}
\phi_{F^V{}_{(Y \otimes \tilde{E})}}{}^V(X \otimes \tilde{E}) &= -(L_{{}^V(X \otimes \tilde{E})}F)^V(Y \otimes \tilde{E}) \\
&= -L_{{}^V(X \otimes \tilde{E})} F^V(Y \otimes \tilde{E}) + F L_{{}^V(X \otimes \tilde{E})}{}^V(Y \otimes E) \\
&= {}^V((\nabla_Y X) \otimes \tilde{E}) + {}^V(X \otimes g \circ (\nabla_Y E)])
\end{aligned}$$

iv)
$$\begin{aligned}
\phi_{F^H{}_Y}{}^V(X \otimes \tilde{E}) &= -(L_{{}^V(X \otimes \tilde{E})}F)^H Y \\
&= -L_{{}^V(X \otimes \tilde{E})} F^H Y + F(L_{{}^V(X \otimes \tilde{E})}{}^H Y) \\
&= -F^V((\nabla_Y X) \otimes \tilde{E} + X \otimes ((\nabla_Y g) \circ E + g \circ (\nabla_Y E))) \\
&= -{}^H(\nabla_Y X) - {}^V(X \otimes (g \circ (\nabla_Y E)))
\end{aligned}$$

v)
$$\begin{aligned}
\phi_{F^H{}_Y}{}^V A &= -(L_{{}^V A}F)^H Y \\
&= -L_{{}^V A}{}^V(Y \otimes \tilde{E}) - F^V(\nabla_Y A) \\
&= -{}^V(\nabla_Y A)
\end{aligned}$$

vi) $\phi_{F^V{}_{(Y \otimes \tilde{E})}}{}^V A = -(L_{{}^V A}F)^V(Y \otimes \tilde{E})$

$$
\begin{aligned}
&= -L_{{}^VA} F^V (Y \otimes \tilde{E}) + F L_{{}^VA}{}^V (Y \circ E) \\
&= -L_{{}^VA}{}^H Y \\
&= {}^V (\nabla_Y A)
\end{aligned}
$$

$vii)$
$$
\begin{aligned}
\phi_{F^V B}{}^V (X \otimes \tilde{E}) &= -(L_{{}^V(X \otimes \tilde{E})} F)^V B \\
&= -L_{{}^V(X \otimes \tilde{E})} F^V B + F L_{{}^V(X \otimes \tilde{E})}{}^V B \\
&= 0
\end{aligned}
$$

$viii)$
$$
\begin{aligned}
\phi_{F^V B}{}^H X &= -(L_{{}^HX} F)^V B \\
&= -L_{{}^HX} F^V B + F L_{{}^HX}{}^V B \\
&= -{}^V(\nabla_X B) + {}^V(\nabla_X B) \\
&= 0
\end{aligned}
$$

$ix)$
$$
\begin{aligned}
\phi_{F^V B}{}^V A &= -(L_{{}^VA} F)^V B \\
&= -L_{{}^VA} F^V B + F L_{{}^VA}{}^V B \\
&= -L_{{}^VA}{}^V B \\
&= 0
\end{aligned}
$$

2.4. Vishnevskii operators applied to ${}^H X$ and ${}^V A$ According to an Almost Paracomplex Structure F on $T_1^1(M)$

Definition 2.4 Suppose now that ∇ is a linear connection on M, and let $\varphi \in \Im_1^1(M)$. We can replace the condition $d)$ of defination 6 by

$$
d') \ \psi_{\varphi X} Y = \nabla_{\varphi X} Y - \varphi \nabla_X Y
$$

for any $X, Y \in \Im_0^1(M)$. Then we can consider a new operator by a Vishnevskii operator or ψ_φ −operator on M, we shall mean a map $\psi_\varphi : \overset{*}{\Im}(M) \to \Im(M)$, which satisfies conditions $a), b), c), e)$ of definition 6 and the condition (d') [10].

Let $\omega \in \Im_1^0(M)$. Using Definition 7, we have

$$
\begin{aligned}
(\psi_\varphi \omega)(X, Y) &= (\psi_{\varphi X} \omega) Y \\
&= (\varphi X)(\iota_Y \omega) - X(\iota_{\varphi Y} \omega) - \omega\left(\nabla_{\varphi X} Y - \varphi(\nabla_X Y)\right) \\
&= \left(\nabla_{\varphi X} \omega - \nabla_X (\omega \circ \varphi)\right) Y
\end{aligned} \tag{2.7}
$$

for any $X, Y \in \Im_0^1(M)$,where $(\omega \circ \varphi) Y = \omega(\varphi Y)$. From (2.7) we see that

$$
\psi_{\varphi X} \omega = \nabla_{\varphi X} \omega - \nabla_X (\omega \circ \varphi)
$$

is a 1 −form [10].

Theorem 2.5 Let F be an almost paracomplex structure on $T_1^1(M)$,i.e., $F^2 = I$ and ψ_F be the Vishnevskii operator, defined by Definition 7, then we get the following results

$i)$ $\psi_{F^V{}_{(Y\otimes\tilde{E})}}{}^V(X\otimes\tilde{E}) ={}^V((\nabla_Y X)\otimes\tilde{E})+{}^V(X\otimes(g\circ(\nabla_Y E)))$,

$ii)$ $\psi_{F^H{}_Y}{}^V(X\otimes\tilde{E}) ={}^H(\nabla_Y X)-{}^V(X\otimes(g\circ(\nabla_Y E)))$,

$iii)$ $\psi_{F^V{}_{(Y\otimes\tilde{E})}}{}^V A ={}^V(\nabla_Y A)$,

$iv)$ $\psi_{F^V{}_{(Y\otimes\tilde{E})}}{}^H X ={}^H(\nabla_Y X)$,

$v)$ $\psi_{F^H{}_Y}{}^H X = -{}^V((\nabla_Y X)\otimes\tilde{E})$,

$vi)$ $\psi_{F^H{}_Y}{}^V A = -{}^V(\nabla_Y A)$,

$vii)$ $\psi_{F^V{}_B}{}^H X = 0$,

$viii)$ $\psi_{F^V{}_B}{}^V A = 0$,

$ix)$ $\psi_{F^V{}_B}{}^V(X\otimes\tilde{E}) = 0$,

where $E\in\Im_0^1(M)$ be a nowhere zero vector field on M, ${}^V(X\otimes\tilde{E})$ is the vertical lift of X with respect to E for any $X\in\Im_0^1(M)$ and $\tilde{E}=g\circ E\in\Im_1^0(M)$,the horizontal lifts ${}^H X\in\Im_0^1(T_1^1M)$ of $X\in\Im_0^1(M)$ and the vertical lift ${}^V A\in\Im_0^1(T_1^1M)$ of $A\in\Im_1^1(M)$ defined by (1.10),(1.11), respectively.

Proof.

$i)$ From Definition 3, we get

$$\begin{aligned}\psi_{F^V{}_{(Y\otimes\tilde{E})}}{}^V(X\otimes\tilde{E}) &={}^H\nabla_{F^V{}_{(Y\otimes\tilde{E})}}{}^V(X\otimes\tilde{E}) - F\,{}^H\nabla_{{}^V(Y\otimes\tilde{E})}{}^V(X\otimes E)\\ &={}^H\nabla_{{}^H Y}{}^V(X\otimes\tilde{E})\\ &={}^V((\nabla_Y X)\otimes\tilde{E}+X\otimes(\nabla_Y g)\circ E+g\circ(\nabla_Y E)])\\ &={}^V((\nabla_Y X)\otimes\tilde{E})+{}^V(X\otimes(g\circ(\nabla_Y E)))\end{aligned}$$

$ii)$
$$\begin{aligned}\psi_{F^H{}_Y}{}^V(X\otimes\tilde{E}) &={}^H\nabla_{F^H{}_Y}{}^V(X\otimes\tilde{E}) - F\,{}^H\nabla_{{}^H Y}{}^V(X\otimes\tilde{E})\\ &={}^H\nabla_{{}^V(Y\otimes\tilde{E})}{}^V(X\otimes\tilde{E}) - F\,{}^V(\nabla_Y(X\otimes\tilde{E}))\\ &= -{}^H(\nabla_Y X) - F\,{}^V(X\otimes((\nabla_Y g)\circ E+g\circ(\nabla_Y E)))\\ &={}^H(\nabla_Y X)-{}^V(X\otimes(g\circ(\nabla_Y E)))\end{aligned}$$

$iii)$
$$\begin{aligned}\psi_{F^V{}_{(Y\otimes\tilde{E})}}{}^V A &={}^H\nabla_{F^V{}_{(Y\otimes\tilde{E})}}{}^V A - F\,{}^H\nabla_{{}^V(Y\otimes E)}{}^V A\\ &={}^H\nabla_{{}^H Y}{}^V A\\ &={}^V(\nabla_Y A)\end{aligned}$$

$iv)$
$$\begin{aligned}\psi_{F^V{}_{(Y\otimes\tilde{E})}}{}^H X &={}^H\nabla_{F^V{}_{(Y\otimes\tilde{E})}}{}^H X - F\,{}^H\nabla_{{}^V(Y\otimes\tilde{E})}{}^H X\\ &={}^H\nabla_{{}^H Y}{}^H X\\ &={}^H(\nabla_Y X)\end{aligned}$$

$v)$
$$\begin{aligned}\psi_{F^H{}_Y}{}^H X &={}^H\nabla_{F^H{}_Y}{}^H X - F\,{}^H\nabla_{{}^H Y}{}^H X\\ &={}^H\nabla_{{}^V(Y\otimes\tilde{E})}{}^H X - F\,{}^H(\nabla_Y X)\\ &= -{}^V((\nabla_Y X)\otimes\tilde{E})\end{aligned}$$

$$vi)\ \psi_{F^H Y}{}^V A = {}^H\nabla_{F^H Y}{}^V A - F^H \nabla_{{}^H Y}{}^V A$$
$$= {}^H\nabla_{{}^V(Y\otimes\tilde{E})}{}^V A - F^V(\nabla_Y A)$$
$$= -{}^V(\nabla_Y A)$$

$$vii)\ \psi_{F^V B}{}^H X = {}^H\nabla_{F^V B}{}^H X - F^H \nabla_{{}^V B}{}^H X$$
$$= {}^H\nabla_{{}^V B}{}^H X - F^H \nabla_{{}^V B}{}^H X$$
$$= 0$$

$$viii)\ \psi_{F^V B}{}^V A = {}^H\nabla_{F^V B}{}^V A - F^H \nabla_{{}^V B}{}^V A$$
$$= {}^H\nabla_{{}^V B}{}^V A$$
$$= 0$$

$$ix)\ \psi_{F^V B}{}^V(X\otimes\tilde{E}) = {}^H\nabla_{F^V B}{}^V(X\otimes\tilde{E}) - F^H \nabla_{{}^V B}{}^V(X\otimes\tilde{E})$$
$$= {}^H\nabla_{{}^V B}{}^V(X\otimes\tilde{E})$$
$$= 0$$

3. CONCLUSION

In this paper, firstly, we study integrability conditions by calculating the Nijenhuis Tensors of almost paracomplex structure F on $(1,1)-$Tensor Bundle. Later, we obtain the Lie derivatives applied to Sasakian metrics with respect to the horizontal and vertical lifts of vector and kovector fields, respectively. Finally, we get the results of Tachibana and Vishnevskii operators applied to horizontal and vertical lifts according to structure F on $(1,1)-$Tensor Bundle $T_1^1(M)$.

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