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**Research Article**

# THE SOLUTION OF THE GOVERNING EQUATION OF THE BEAM ON LINEAR SPRING FOUNDATION MODELED BY A DISCONTINUITY FUNCTION

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**ABSTRACT**

The structural engineering researches have attracted considerable attention by many scientist for several decades. Determining the dynamical behaviors of structural elements with some discontinuous is of great importance in many engineering applications. The mentioned structures can be modelled two different ways. In the first approximation so-called the classical approach, a fourth order differential equation are written for each part of beam separated in the distinct discontinuity locations. Therefore, we obtain a system of equation containing $n+1$ number of the differential equation with boundary and transient conditions. Secondly, the real problem can be reformulated by only one differential equation having discontinuity function. In this study, we introduce the method of multiple scales as the solution technique. Since we encountered by the differential equation with discontinuity function in the part of order discretization during the perturbative solution, we have used a numerical technique for the solution. The mentioned technique is applied on the beam model lying on lineer spring foundation called as Winkler type foundation.
**Keywords:** Discontinuous structural elements, finite differences method, method of multiple scales, linear spring foundation, Winkler type foundation.

## 1. INTRODUCTION

In this study, the beam lying on a elastic Winkler foundation is considered as Euler-Bernoulli beam with discontinuity function. Winkler type foundation is one of those having widely application among elastic foundations. Winkler type model consists of uncorrelated elastic springs added to each node of the structural element. These models has drawn a lot of attention many researchers in structural engineering. Winkler assumption reveals reasonable performance and it is easily practicable [1]. Wang et. al. [2] obtain the governing equation of the beam lying on Winkler foundation thanks to extended Hamilton principle. Hayir [3] presents dynamic behavior of an elastic beam on a Winkler foundation under a moving load.

Generally, the continuous beam models are considered in the literature. However, the strucure elements can have the discontinuities arising from springs, concentrated masses and cracks. The

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significance of this kind of structures is well recognized due to its engineering applications. Friswell and Penny [4] introduce the nonlinear model of cracked beam with discontinuous stiffness. Failla [5] investigates the behavior of viscoelastic discontinuous beams using the theory of generalized functions to treat the discontinuities of the response variables. Failla and Santini [6] consider a solution method based on appropriate Green's functions for stepped Euler-Bernoulli discontinuous beams with internal translational and rotational springs. Also, the implicit solutions based on Green's functions are developped by them [7] for the bending problem of Euler-Bernoulli discontinuous beams. Li [8] proposes an exact approach for free vibration analysis of a non-uniform beam with $n$ cracks and concentrated masses.

Recently, some authors focus on the various beams model resting elastic foundation. Dinev [9] introduce analitical solution of the beam on elastic foundation modeled by singularity function. Basu and Kameswara Rao [10] investigate the displacement, bending moment, shear force and contact pressure of the infinite beam resting on visco-elastic foundation. Attar et. al. [11] consider the free vibration analysis of deformable beam subject to two parameter elastic foundation.

We analyze dynamic behaviors of beams under partly foundation. The foundation is modeled by using linear spring element which is called Winkler type foundation. The mathematical model of the problem may be presented in two different ways. At the first, we may write two differential equations; one of them is model for part of the beam without foundation and the other is for the part of the beam with foundation. The perturbative solution of the set of differential equations are obtained in conventional ways. The other mathematical model of real problem has the discontinuity function which is called Heaviside step function. In the perturbative solution technique, the differential equation at the first order has a discontinuity function. Then, the analytical solution of this equation is generally difficult. Therefore, we need a numerical method. In this study, we prefer the finite differences method as a numerical technique. After application of solvability condition in the perturbative solution, we calculate numerically mode shapes. For this purpose, Simpson method being one of the numerical integration techniques is used. The results obtained by both techniques are compared with each other.

## 2. GOVERNING EQUATION

We consider the transverse vibrations of the beam lying on the Winkler foundation. We supplement the external force and damping term additional to the mathematical model in [12]. Then, the governing equation becomes

$$EI\,\hat{y}^{iv} + \hat{k}_0\,H\left(\hat{x}-\hat{\eta}\right)\hat{y} + \varepsilon\,\hat{\mu}\,\dot{\hat{y}} + m\,\ddot{\hat{y}} = \hat{f}\left(\hat{x}\right)\cos\hat{\Omega}\hat{t}\,, \tag{1}$$

$$\hat{y}\left(0,t\right) = EI\,\hat{y}''\left(0,t\right) = 0 \text{ and } \hat{y}\left(L,t\right) = EI\,\hat{y}''\left(L,t\right) = 0 \tag{2}$$

where $\hat{y}(\hat{x},\hat{t})$ represents the transverse displacement, $E$ is Young's modulus of the beam material, $I$ is the moment of inertia. $\hat{k}_0$ denotes Winkler spring constant. $H$ represents Heaviside step function, $\varepsilon$ is dimensionless small parameter, $\hat{\mu}$ is linear viscose damping coefficient, $m$ is unit mass. $\hat{\eta}$ describes location of discontinuity. $\hat{f}$ and $\hat{\Omega}$ are the external excitation force and frequency, respectively. $\hat{x}$ and $\hat{t}$ denote space and time variables, respectively. The dot describes differentiation with respect to time $\hat{t}$. The prime denotes differentiation with respect to space.

Let us introduce the dimensionless parameters

$$y = \frac{\hat{y}}{L}, \quad x = \frac{\hat{x}}{L}, \; t = \frac{\hat{t}}{L^2}\sqrt{\frac{EI}{m}}, \tag{3.a}$$

$$\mu = \hat{\mu}\frac{L^2}{\sqrt{EIm}}, \; k_0 H(x-\eta) = \frac{L^4}{EI}\hat{k}_0 H(\hat{x}-\hat{\eta}), \; \varepsilon f = \frac{L^4}{EI}\hat{f}(\hat{x}), \; \Omega = \hat{\Omega}L^2\sqrt{\frac{m}{EI}} \tag{3.b}$$

where $L$ is length between two supports. Then, the governing equation becomes

$$\frac{\partial^4 y}{\partial x^4} + \varepsilon\mu\frac{\partial y}{\partial t} + k_0 H(x-\eta)y + \frac{\partial^2 y}{\partial t^2} = \varepsilon f(x)\cos\Omega t, \tag{4}$$

$$y(0,t) = y''(0,t) = 0 \text{ and } y(1,t) = y''(1,t) = 0. \tag{4.a}$$

The resulting equation and boundary conditions are obtained as nondimensional.

## 3. THE METHOD OF MULTIPLE SCALES

The method of multiple scales is used for the solution of the equation of motion. This method is applied directly to the dimensionless equation. The perturbative series expansion is assumed as

$$y(x,T_0,T_1;\varepsilon) \cong y_0(x,T_0,T_1) + \varepsilon y_1(x,T_0,T_1) + \ldots \tag{5}$$

where $T_n$ is various time scales in the form of $\mathrm{T}_n = \varepsilon^n t$. Thus, the derivatives based on the new time scales are given by

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \cdots, \qquad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots \tag{6}$$

where $D_i = \partial / \partial T_i$.

## 4. THE FINITE DIFFERENCES METHOD

After the perturbative technique has been directly applied to Eq. (4), the differential equation with a discontinuity function in first order is obtained. For the solution of the resulting equation, we need to a numerical technique. Therefore, we prefer the well-known finite differences method as a solution procedure. There are three different finite differences schemes: forward differences, backward differences and central differences. For small truncation error, the central difference is chosen. Then, first four derivatives are given as follows:

$$X_i' \approx \frac{X_{i+1} - X_{i-1}}{2\Delta x}, \tag{7.a}$$

$$X_i'' \approx \frac{X_{i+1} - 2X_i + X_{i-1}}{\Delta x^2}, \tag{7.b}$$

$$X_i''' = \frac{X_{i+2} - 2X_{i+1} + 2X_{i-1} - X_{i-2}}{2\Delta x^3}, \tag{7.c}$$

$$X_i^{iv} = \frac{X_{i+2} - 4X_{i+1} + 6X_i - 4X_{i-1} + X_{i-2}}{\Delta x^4} \tag{7.d}$$

where $\Delta x = 1/N$. $N$ is total number of short segments into system. In these discretized forms, the subscript indicates spatial node.

## 5. SOLUTION PROCEDURE

We directly apply the method of multiple scales to the governing equation. Substituting Eqs. (5) and (6) into Eq. (4) and seperating to order of $\varepsilon$ yield

$$O(1): y_0^{iv} + k_0 H(x-\eta) y_0 + D_0^2 y_0 = 0 \tag{8}$$

$$y_0(0,T_0,T_1) = 0,\ y_0''(0,T_0,T_1) = 0 \tag{9.a}$$

$$y_0(1,T_0,T_1) = 0,\ y_0''(1,T_0,T_1) = 0 \tag{9.b}$$

$$O(\varepsilon): y_1^{iv} + k_0 H(x-\eta) y_1 + D_0^2 y_1 = -\mu D_0 y_0 - 2D_0 D_1 y_0 + f(x)\cos\Omega T_0 \tag{10}$$

$$y_1(0,T_0,T_1) = 0\ ,\ y_1''(0,T_0,T_1) = 0 \tag{11.a}$$

$$y_1(1,T_0,T_1) = 0\ ,\ y_1''(1,T_0,T_1) = 0\ . \tag{11.b}$$

The general solution in the first order is assumed as

$$y_0(x,\mathrm{T}_0,\mathrm{T}_1) = \left[A_n(\mathrm{T}_1)e^{i\omega_n \mathrm{T}_0} + \bar{A}_n(\mathrm{T}_1)e^{-i\omega_n \mathrm{T}_0}\right]X_n(x);\ n = 1,2,3,\ldots \tag{12}$$

where $A_n$ and $\bar{A}_n$ are the complex amplitude and conjugate, respectively. If the relation (12) is the solution of Eq. (8), then it should provide the equation. Thus, the differential equation in the following is obtained

$$X_n^{iv} + \left[k_0 H(x-\eta) - \omega_n^2\right] X_n = 0 \tag{13}$$

$$X_n(0) = X_n''(0) = 0 \text{ and } X_n(1) = X_n''(1) = 0\,. \tag{14}$$

A numerical approach is needed to determine the natural frequency and mode shape. Substituting Eq. (7.d) into Eq. (13) yields the discretized equation at $i$ th spatial node as

$$b_{4,i} X_{n,i+2} + b_{3,i} X_{n,i+1} + b_{2,i} X_{n,i} + b_{1,i} X_{n,i-1} + b_{0,i} X_{n,i-2} = 0 \tag{15}$$

where

$$b_{0,i} = 1\,,\ b_{1,i} = -4\,,\ b_{2,i} = 6 + \left[k_0 H(x-\eta) - \omega_n^2\right]\Delta x^4\,,\ b_{3,i} = -4\,, b_{4,i} = 1 \tag{16}$$

and Heaviside step function $H$ is defined

$$H(x-\eta) = \begin{cases} 0 & x < \eta \\ 1 & x \geq \eta \end{cases}\ . \tag{17}$$

The coefficient $b_{2,i}$ can be written as

$$\overline{b}_{2,i} = 6 - \omega_n^2 \Delta x^4, \quad \underline{b}_{2,i} = 6 + \left(k_0 - \omega_n^2\right)\Delta x^4 \tag{18}$$

where $\overline{b}_{2,i}$ denotes the part which there is not the soil and $\underline{b}_{2,i}$ describes the part which there is the soil. For pinged-pinged support, the boundary conditions applied finite differences is obtained as

$$X_{n,0} = 0,\ X_{n,-1} = -X_{n,1},\ X_{n,N} = 0,\ X_{n,N+1} = -X_{n,N-1}\ . \tag{19}$$

Substituting these conditions into Eq. (15), the obtained algebraic equation system is reduced to the matrix form in the following

$$\begin{bmatrix} \overline{b}_{2,i}-1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & & & & \\ -4 & \overline{b}_{2,i} & -4 & 1 & 0 & 0 & 0 & 0 & & 0 & & \\ 1 & -4 & \overline{b}_{2,i} & -4 & 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & -4 & \overline{b}_{2,i} & -4 & 1 & 0 & 0 & & & & \\ & \ddots & & & & \ddots & & & & \ddots & & \\ & & & & 0 & 0 & 1 & -4 & \underline{b}_{2,i} & -4 & 1 & 0 \\ & & & & 0 & 0 & 0 & 1 & -4 & \underline{b}_{2,i} & -4 & 1 \\ & 0 & & & 0 & 0 & 0 & 0 & 1 & -4 & \underline{b}_{2,i} & -4 \\ & & & & 0 & 0 & 0 & 0 & 0 & 1 & -4 & \underline{b}_{2,i}-1 \end{bmatrix} \begin{Bmatrix} X_{n,1} \\ X_{n,2} \\ X_{n,3} \\ \vdots \\ \vdots \\ \vdots \\ X_{n,N-3} \\ X_{n,N-2} \\ X_{n,N-1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{20}$$

For nontrivial solutions, determinant of the matrix of the coefficients must be equal to zero. Thus, the natural frequency of the system can be approximately found and the mode shapes $X_n$ are numerically obtained. As a result, the solution $y_0$ is determined. Substituting Eq. (12) into right side of Eq. (10) gives

$$\begin{aligned} y_1^{iv} + D_0^2 y_1 + k_0 H(x-\eta) y_1 = &-\left[\mu i\omega_n A(T_1) X_n(x) + 2i\omega_n X_n(x) D_1 A(T_1)\right] e^{i\omega_n T_0} \\ &+ \frac{f(x)}{2} e^{i\Omega T_0} + c.c. \end{aligned} \tag{21}$$

where $c.c.$ represents the complex conjugates. Then, the solution of Eq. (21) is in the form of

$$y_1 = \varphi_n\left(x, T_1\right) e^{i\omega_n T_0} + W_1\left(x, T_0, T_1\right) + c.c. \tag{22}$$

where the first and second term are related to secular and nonsecular terms, respectively.

## 6. CASE STUDIES

In this section, two different cases arise by depending on the numerical values of natural frequency.

### 6.1. Case 1: $\Omega$ away from $\omega_n$

Assuming that $\Omega \neq \omega_n$, the Eq. (21) becomes

$$y_1^{iv} + D_0^2 y_1 + k_0 H(x-\eta) y_1 = -\left[\mu i\omega_n A(T_1) X_n(x) + 2i\omega_n X_n(x) D_1 A(T_1)\right] e^{i\omega_n T_0} + c.c. + NST \quad (23)$$

where $NST$ denotes non-secular terms. Applying the solvability condition [13] to the Eq. (23) yields

$$D_1 A_n + \alpha_{1n} A_n = 0 \quad (24)$$

where the coefficient $\alpha_{1n}$ is

$$\alpha_{1n} = \frac{\mu}{2} \quad (25)$$

and the normalization is

$$\int_0^{\eta} X_n^2 \, dx = 1. \quad (26)$$

($\eta$ represents the length of span). Then, the solution of the Eq. (24) is obtained as

$$A_n = \kappa e^{-\alpha_{1n} T_1} \quad (27)$$

where $\kappa$ denotes an arbitrary constant. Since $\alpha_{1n}$ is real and positive in the solution (27), the amplitude of the system exponentially decreases and the solution is stable.

### 6.2. Case 2: $\Omega$ closed to $\omega_n$

In this section, the parametric resonance occuring in case the frequency of external forcing force equals or closes to one of natural frequencies of the system is analysed. We assume that

$$\Omega = \omega_n + \varepsilon\sigma \quad (28)$$

where $\sigma$ is detuning parameter. Then, the Eq. (21) is obtained as

$$\begin{aligned} y_1^{iv} + D_0^2 y_1 + k_0 H(x-\eta) y_1 = -\left[\mu i\omega_n A(T_1) X_n(x) + 2i\omega_n X_n(x) D_1 A(T_1)\right] e^{i\omega_n T_0} \\ + \frac{f(x)}{2} e^{i\Omega T_0} + c.c. + NST \end{aligned} . \quad (29)$$

From the solvability condition [13], the amplitude equation is obtained as follows:

$$D_1 A_n + \alpha_{1n} A_n = \frac{1}{2} i\alpha_{2n} e^{i\sigma T_1} \quad (30)$$

where

$$\alpha_{2n} = \frac{1}{2\omega_n} \int_{\eta}^{L} f(x) X_n \, dx \tag{31}$$

(the integral (31) is numerically calculated by Simpson method). We assume the polar form of $A_n$ as

$$A_n = \frac{1}{2} a_n (T_1) e^{i \beta_n (T_1)} \tag{32}$$

Substituting the Eq. (32) into Eq. (30) and separating real and imaginary parts, the resulting equation is found as

$$\text{Re: } \frac{da_n}{dT_1} + \alpha_{1n} a_n = -\alpha_{2n} \sin\gamma \tag{33}$$

$$\text{Im: } \sigma a_n - \frac{d\gamma}{dT_1} a_n = \alpha_{2n} \cos\gamma \tag{34}$$

where $\gamma = \sigma T_1 - \beta$. Since $da_n / dT_1$ and $d\gamma / dT_1$ should be equal to zero for steady state solutions, we obtain

$$\sigma = \mp \frac{1}{a_n} \sqrt{\alpha_{2n}^2 - a_n^2 \alpha_{1n}^{\;\;2}} \; . \tag{35}$$

## 7. CLASSICAL TECHNIQUE

We consider well-known classical technique for the problem. Then, the equation is seperately written for each span in the classical approximation. Thus, the dimensionless equations of motion are obtained as

$$y_1^{iv} + \varepsilon \mu \dot{y}_1 + \ddot{y}_1 - \varepsilon f(x) \cos \Omega t = 0 \tag{36.a}$$

$$y_2^{iv} + \varepsilon \mu \dot{y}_2 + k_0 y_2 + \ddot{y}_2 - \varepsilon f(x) \cos \Omega t = 0 \tag{36.b}$$

where the boundary conditions are

$$y_1(0,t) = 0, \; y_1''(0,t) = 0 \text{ and } y_2(1,t) = 0, \; y_2''(1,t) = 0 \tag{37}$$

and the transient conditions are

$$y_1(\eta,t) = y_2(\eta,t) \tag{38.a}$$

$$y_1'(\eta,t) = y_2'(\eta,t) \tag{38.b}$$

$$y_1''(\eta,t) = y_2''(\eta,t) \tag{38.c}$$

$$\left(EI\, y_1''(\eta,t)\right)' = \left(EI\, y_2''(\eta,t)\right)' \tag{38.d}$$

where $\eta$ represents the length of span. We introduce the perturbative series expansion as

$$y_1\left(x,T_0,T_1;\varepsilon\right)\cong y_{10}\left(x,T_0,T_1\right)+\varepsilon y_{11}\left(x,T_0,T_1\right)+\dots, \tag{39.a}$$

$$y_2\left(x,T_0,T_1;\varepsilon\right)\cong y_{20}\left(x,T_0,T_1\right)+\varepsilon y_{21}\left(x,T_0,T_1\right)+\dots \tag{39.b}$$

Substituting Eqs. (39) and the expansions of derivative (6) into Eq. (36)-(38) and seperating to order of $\varepsilon$ yield

$$O\left(1\right): y_{10}^{iv}+D_0^2 y_{10}=0, \tag{40.a}$$

$$y_{20}^{iv}+k_0 y_{20}+D_0^2 y_{20}=0 \tag{40.b}$$

where the boundary and transient conditions are

$$y_{10}\left(0,T_0,T_1\right)=0, EI\ y_{10}''\left(0,T_0,T_1\right)=0 \text{ and } y_{20}\left(1,T_0,T_1\right)=0,\ EI\ y_{20}''\left(1,T_0,T_1\right)=0 \tag{41}$$

and

$$y_{10}\left(\eta,T_0,T_1\right)=y_{20}\left(\eta,T_0,T_1\right) \tag{42.a}$$

$$y_{10}'\left(\eta,T_0,T_1\right)=y_{20}'\left(\eta,T_0,T_1\right) \tag{42.b}$$

$$EI\ y_{10}''\left(\eta,T_0,T_1\right)=EI\ y_{20}''\left(\eta,T_0,T_1\right) \tag{42.c}$$

$$\left(EI\ y_{10}''\left(\eta,T_0,T_1\right)\right)'=\left(EI\ y_{20}''\left(\eta,T_0,T_1\right)\right)', \tag{42.d}$$

respectively.

$$O\left(\varepsilon\right): y_{11}^{iv}+D_0^2 y_{11}=-\mu D_0 y_{10}-2D_0 D_1 y_{10}+f(x)\cos\Omega T_0, \tag{43.a}$$

$$y_{21}^{iv}+D_0^2 y_{21}+k_0 y_{21}=-\mu D_0 y_{20}-2D_0 D_1 y_{20}+f(x)\cos\Omega T_0 \tag{43.b}$$

where the boundary conditions (BC) are

$$y_{11}\left(0,T_0,T_1\right)=0,\ EI\ y_{11}''\left(0,T_0,T_1\right)=0 \text{ and } y_{21}\left(1,T_0,T_1\right)=0,\ EI\ y_{21}''\left(1,T_0,T_1\right)=0 \tag{44}$$

and the transient conditions (TC) are

$$y_{11}\left(\eta,T_0,T_1\right)=y_{21}\left(\eta,T_0,T_1\right) \tag{45.a}$$

$$y_{11}'\left(\eta,T_0,T_1\right)=y_{21}'\left(\eta,T_0,T_1\right) \tag{45.b}$$

$$EI\ y_{11}''\left(\eta,T_0,T_1\right)=EI\ y_{21}''\left(\eta,T_0,T_1\right) \tag{45.c}$$

$$\left(EI\ y_{11}''\left(\eta,T_0,T_1\right)\right)'=\left(EI\ y_{21}''\left(\eta,T_0,T_1\right)\right)'. \tag{45.d}$$

We assume that the solution in the first order is

$$y_{10}=\left(A\left(T_1\right)e^{i\omega_n T_0}+\bar{A}\left(T_1\right)e^{-i\omega_n T_0}\right)X_{1n}\left(x\right), \tag{46.a}$$

$$y_{20}=\left(A\left(T_1\right)e^{i\omega_n T_0}+\bar{A}\left(T_1\right)e^{-i\omega_n T_0}\right)X_{2n}\left(x\right) \tag{46.b}$$

Substituting the Eq. (46) into the Eq. (40), the resulting equation is

$$X_{1n}^{iv} - \omega_n^2 X_{1n} = 0 \tag{47}$$

$$X_{2n}^{iv} + (k_0 - \omega_n^2) X_{2n} = 0 \tag{48}$$

**BC:** $X_{1n}(0) = 0, EI\, X_{1n}''(0) = 0$ and $X_{2n}(1) = 0, EI\, X_{2n}''(1) = 0$ (49)

**TC:** $X_{1n}(\eta) = X_{2n}(\eta)$ (50.a)

$$X_{1n}'(\eta) = X_{2n}'(\eta) \tag{50.b}$$

$$EI\, X_{1n}''(\eta) = EI\, X_{2n}''(\eta) \tag{50.c}$$

$$\left(EI\, X_{1n}''(\eta)\right)' = \left(EI\, X_{2n}''(\eta)\right)' \tag{50.d}$$

Then, the solution is obtained as

$$X_{1n}(x) = c_1 \cos(\sqrt{\omega_n}\, x) + c_2 \sin(\sqrt{\omega_n}\, x) + c_3 \cosh(\sqrt{\omega_n}\, x) + c_4 \sinh(\sqrt{\omega_n}\, x) \tag{51}$$

$$X_{2n}(x) = d_1 \cos(\sqrt{\sqrt{\omega_n{}^2 - k_0}}\, x) + d_2 \sin(\sqrt{\sqrt{\omega_n{}^2 - k_0}}\, x) + d_3 \cosh(\sqrt{\sqrt{\omega_n{}^2 - k_0}}\, x) + d_4 \sinh(\sqrt{\sqrt{\omega_n{}^2 - k_0}}\, x) \tag{52}$$

where $c_i$ and $d_i$ $(i = 1, \ldots, 4)$ are the constants. Applying the boundary conditions, the critical axial load and natural frequency are calculated by depending on the coefficient of spring and its location. Then, the mode shapes are found by determining arbitrary constants. Substituting the Eqs. (46) into the Eqs. (43) yields

$$y_{11}^{iv} + D_0^2 y_{11} = -\left[\mu i\omega_n A(T_1) X_{1n}(x) + 2i\omega_n X_{1n}(x) D_1 A(T_1)\right] e^{i\omega_n T_0} + \frac{f(x)}{2} e^{i\Omega T_0} + c.c. \tag{53}$$

$$y_{21}^{iv} + D_0^2 y_{21} + k_0 y_{21} = -\left[\mu i\omega_n A(T_1) X_{2n}(x) + 2i\omega_n X_{2n}(x) D_1 A(T_1)\right] e^{i\omega_n T_0} + \frac{f(x)}{2} e^{i\Omega T_0} + c.c. \tag{54}$$

Then, the solution in the order $\varepsilon$ is

$$y_{11} = \varphi_1(x, T_1) e^{i\omega_n T_0} + W_1(x, T_0, T_1) + c.c. \tag{55}$$

$$y_{21} = \varphi_2(x, T_1) e^{i\omega_n T_0} + W_2(x, T_0, T_1) + c.c. \tag{56}$$

Proceeding the perturbative solution, two case reveal where $\Omega$ away from $\omega_n$ and close to $\omega_n$, respectively. The amplitude for classical solution procedure is obtained as

$$A_n = Ce^{-\alpha_{1n} T_1} \tag{57}$$

where the coefficient $\alpha_{1n}$ in the Case 1 in the Section 4 is

$$\alpha_{1n} = \frac{\mu}{2} \tag{58}$$

and the normalization

$$\int_0^{\eta} X_{1n}^2(x)\,dx + \int_{\eta}^{L} X_{2n}^2(x)\,dx = 1 \tag{59}$$

Then, the detuning parameter is found as

$$\sigma = \mp\frac{1}{a_n}\sqrt{\alpha_{2nI}^2 - a_n^2\alpha_{1n}^2} \tag{60}$$

where the coefficient $\alpha_{2n}$ in the Case 2 is

$$\alpha_{2n} = \frac{1}{2\omega_n}\left(\int_0^{\eta} f(x)X_{1n}dx + \int_{\eta}^{L} f(x)X_{2n}(x)\,dx\right) \tag{61}$$

The perturbation method and the finite differences method is used in the present approximation. In the expansion of the finite difference, *N* denotes total number of short segments into system. In the tables in the following, the comparison of the natural frequency and the critical load is given for the different values of the coefficient of spring $k_0$ and the location of spring $\eta$.

**Table 1.** The comparison of the natural frequency and the critical load with the classical method and **the present method (bold)** for $N = 200$ and $f(x) = 5$.

| $\eta$ | $k_0 = 50$ | | | | $k_0 = 100$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\omega_n$ | | $\alpha_{2n}$ | | $\omega_n$ | | $\alpha_{2n}$ | |
| 20 | 12.1279 | **12.1414** | 2.6247 | **2.6215** | 14.0271 | **14.0466** | 2.2694 | **2.2659** |
| 50 | 11.9506 | **11.9589** | 2.6644 | **2.6626** | 13.7160 | **13.7210** | 2.3223 | **2.3209** |
| 100 | 11.0497 | **11.0718** | 2.8804 | **2.8743** | 12.0898 | **12.1198** | 2.6318 | **2.6251** |

**Table 2.** The comparison of the natural frequency and the critical load with the classical method and **the present method (bold)** for $N = 200$ and $f(x) = x$.

| $\eta$ | $k_0 = 50$ | | | | $k_0 = 100$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\omega_n$ | | $\alpha_{2n}$ | | $\omega_n$ | | $\alpha_{2n}$ | |
| 20 | 12.1279 | **12.1433** | 52.4824 | **52.4315** | 14.0271 | **14.0429** | 45.3679 | **45.3391** |
| 50 | 11.9506 | **11.9541** | 53.1476 | **53.1306** | 13.7160 | **13.7270** | 46.1972 | **46.1982** |
| 100 | 11.0497 | **11.0664** | 57.1944 | **57.1477** | 12.0898 | **12.1071** | 51.8812 | **51.8427** |

## 8. CONCLUSIONS

We analyze the dynamical behavior of the Euler-Bernoulli beam having the discontinuity lying on a linear spring foundation also called as Winkler-type foundation. We use both perturbation method and finite differences method for solving this equation. This approach provides an advantage in the numerical solution of the mathematical model of structural element containing any discontinuity and also in its dynamical analysis by perturbation method. This technique indicates that solving one equation with discontinuity function has an advantage over solving the system of equation arising classical approach. The performed comparisons show that the results obtained by the classical method are very close to those obtained by the present technique. Thus, it is seen that an appreciable reduction of computational effort is achieved as compared to alternative the solutions in the literature.

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### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

All authors read and approved the final manuscript.

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