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Research Article

LACUNARY I_σ-STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCE

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ABSTRACT

This study introduces the concepts of lacunary I_σ-statistical convergence of complex uncertain sequences: lacunary I_σ-statistical convergence almost surely ($\mathrm{S}_\theta(I_\sigma)$.a.s.), lacunary I_σ-statistical convergence in measure, lacunary I_σ-statistical convergence in mean, lacunary I_σ-statistical convergence in distribution and lacunary I_σ-statistically convergence uniformly almost surely ($\mathrm{S}_\theta(I_\sigma)$.u.a.s). In addition, decomposition theorems and relationships among them are discussed.

Keywords: Lacunary convergence, invariant, uncertainty theory, complex uncertain variable, ideal convergence.

1. INTRODUCTION

Freedman and Sember introduced the concept of lower asymptotic density and defined the concept of convergence in density, in [1]. Taking this definition, we can give the definition of statistical convergence which has been formally introduced by Fast [2]. Schoenberg reintroduced this concept independently [3]. A number sequence (x_k) is statistically convergent to L provided that for every $\varepsilon > 0$, $d(\{k \in N: |x_k - L| \geq \varepsilon\}) = 0$ or equivalently there exists a subset $K \subset N$ with $d(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0(\varepsilon)$ and $k \in K$ imply that $|x_k - L| < \varepsilon$. In this case we write $st - lim x_k = L$. From the definition, we can easily show that any convergent sequence is statistically convergent, but not conversely.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

The concept of lacunary statistical convergence was defined by Fridy and Orhan [4]. A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$, $lim_r \frac{1}{h_r} |\{k \in I_r: |x_k - L| \geq \varepsilon\}| = 0$. In this case we write $S_\theta - lim x_k = L$ or $x_k \to L(S_\theta)$.

The concept of I-convergence of real sequences is a generalization of statistical convergence which is based on the structure of an ideal I of subsets of the set of natural numbers. P. Kostyrko et al. [5] introduced the concept of I-convergence of sequences in a metric space and studied some properties of this convergence. Later, it was further studied by Salát et al. ([6], [7]) and many

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others. Recently, Das et al. [8] introduced some new notions, namely *I*-statistical convergence and *I*-lacunary statistical convergence by using ideals.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \left(\sigma^{m-1}(n)\right), m = 1,2,3, \ldots$. A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

(1) $\Phi(x) \geq 0$, for all sequences $x = (x_n)$ with $x_n \geq 0$ for all *n*;

(2) $\Phi(e) = 1$, where $e = (1,1,1, \ldots)$;

(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x)$ for all $x \in l_\infty$.

The mapping Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers *n* and *m*, where $\sigma^m(n)$ denotes the *m*th iterate of the mapping σ at *n*. Thus, Φ extends the limit functional on *c*, the space of convergent sequences, in the sense that $\Phi(x) = lim x$, for all $x \in c$.

Savaş and Nuray [9] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Pancaroğlu and Nuray [10] defined the concept of lacunary invariant summability and *p*-strongly lacunary invariant summability. The concept of lacunary strongly σ-convergence was introduced by Savaş [11].
Let $A \subset \mathbb{N}$ and

$$s_m := min_n |\mathrm{A} \cap \{\sigma(\mathrm{n}), \sigma^2(\mathrm{n}), \ldots, \sigma^m(\mathrm{n})\}|$$
$$S_m := max_n |\mathrm{A} \cap \{\sigma(\mathrm{n}), \sigma^2(\mathrm{n}), \ldots, \sigma^m(\mathrm{n})\}|$$

If the following limits exist

$$\underline{V}(A) := \lim_{m\to\infty} \frac{s_m}{m}, \overline{V}(A) := \lim_{m\to\infty} \frac{S_m}{m}$$

then, they are called a lower and an upper σ-uniform density of the set *A*, respectively. If $V(A) = \underline{V}(A) = \overline{V}(A)$, then *V(A)* is called the σ-uniform density of *A*.

Denote by I_σ the class of all $A \subset \mathbb{N}$ with $V(A) = 0$.

In [12], the concept of σ-uniform density of subsets *A* of the set *N* of positive integers and corresponding I_σ-convergence were introduced. A sequence $x = (x_n)$ is said to be I_σ-convergent to the number *L* if for every $\varepsilon > 0$, $A(\varepsilon) := \{k : |x_k - L| \geq \varepsilon\}$ belongs to I_σ; i.e., $V(A_\varepsilon) = 0$. In this case we write I_σ- $lim x_k = L$.

Let $\theta = \{k_r\}$ be a double lacunary sequence, $A \subset \mathbb{N}$ and

$$p_r := min_m \left|\mathrm{A} \cap \{\sigma^k(\mathrm{m}) : \mathrm{k} \in I_r\}\right|$$

and

$$P_r := max_m \left|\mathrm{A} \cap \{\sigma^k(\mathrm{m}) : \mathrm{k} \in I_r\}\right|$$

If the following limit exist

$$\underline{V}^\theta(A) := \lim_{r\to\infty} \frac{p_{,r}}{h_r}, \overline{V}^\theta(A) := \lim_{r\to\infty} \frac{P_r}{h_r}$$

then they are called a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set *A*, respectively. If $\underline{V}^\theta(A) = \underline{V}^\theta(A)$, then $V^\theta = \underline{V}^\theta(A) = \underline{V}^\theta(A)$ is called lacunary σ-uniform density of *A*.

Throughout the paper, we take $I^{\sigma\theta}$ as a strongly admissible ideal in ℕ.

Recently, the concept of lanunary σ-uniform density of the set $A \subset \mathbb{N}$, lacunary I_σ-convergence, lacunary I_σ^*-convergence, lacunary I_σ-Cauchy sequences, lacunary I_σ^*-Cauchy sequences of real numbers were defined by Ulusu and Nuray [13].

Recently, some other authors studied ideal convergence and invariant convergence ([14-21]).

In [22], Demirci exten ded the concepts of statistical limit superior and inferior to *I*-limit superior and *I*-limit inferior and gave some *I*-analogue of properties of statistical limit superior and inferior.

However, in our daily life, we often encounter the case that there are lack of or no observed data about the events, not only for economic reasons or technical difficulties, but also for influence of unexpected events.

In order to deal with belief degree, an uncertainty theory was founded by Liu [23] in 2007, and redefined by Liu [24] in 2011 which based on an uncertain measure which satisfies normality, duality, subadditivity, and product axioms. In 2007, Liu [23] first introduced convergence in measure, convergence in mean, convergence almost surely (a.s.) and convergence in distribution and their relationships were also discussed. Thereafter, a concept of uncertain variable was proposed to represent the uncertain quantity and a concept of uncertainty distribution to describe uncertain variables. Up to now, uncertainty theory has successfully been applied to uncertain programming (Liu [25], Liu and Chen [26]), uncertain risk analysis and uncertain reliability analysis (Liu [27]), uncertain logic (Liu [28]), uncertain differential equations (Yao and Chen [29]), uncertain graphs (Gao and Gao [30], Zhang and Peng [31]), uncertain finance (Chen [32], Liu [33]), etc.

In real life, uncertainty not only appears in real quantities but also in complex quantities. In order to model complex uncertain quantities, Peng [34] presented the concepts of complex uncertain variable and complex uncertainty distribution, and also the expected value was proposed to measure a complex uncertain variable in 2012. Since sequence convergence plays an important role in the fundamental theory of mathematics, there are also many convergence concepts in uncertainty theory. You [35] introduced another type of convergence named convergence uniformly almost surely and showed the relationships among those convergence concepts. Zhang [36] proved some theorems on the convergence of uncertain sequence. After that, Guo and Xu [37] gave the concept of convergence in mean square for uncertain Tripathy and Nath [38] introduced statistical convergence of complex uncertain sequences. Kişi and Ünal [39] defined lacuanary statistical convergence and Kişi [40] introduced *I*-lacunary statistical convergence of complex uncertain sequences. Inspired by these, we study the convergence concepts of lacunary I_σ-statistically convergence of complex uncertain sequences and discuss the relationships among them in this study.

2. MAIN RESULTS

Definition 1. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary I_σ-statistically convergent almost surely $(S_\theta(\,I_\sigma).a.s)$ to ζ if for every $\varepsilon,\delta>0$ there exists an event Λ with $M(\Lambda)=1$ such that

$$\left\{r\in \mathrm{N}:\ \frac{1}{h_r}\left|\left\{k\in I_r\ :\ \left\|\zeta_k(\gamma)-\zeta(\gamma)\right\|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathtt{I}_\sigma,$$

for every $\gamma\in\Lambda$. In this case we write $\zeta_n\to\zeta$ ($S_\theta(\,I_\sigma)$.*a.s.*).

Definition 2. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary I_σ-statistically convergent in measure to ζ if

$$\left\{r\in \mathrm{N}:\ \frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\left\|\zeta_k-\zeta\right\|\geq\varepsilon\right)\geq\delta\right\}\right|\geq\vartheta\right\}\in\mathtt{I}_\sigma,$$

for every $\varepsilon,\delta,\vartheta>0$.

Definition 3. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary I_σ-statistically convergent in mean to ζ if

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \,:\, E\left(\left\|\zeta_k - \zeta\right\|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathbf{I}_\sigma,$$

for every $\varepsilon, \delta > 0$.

Definition 4. Let $\Phi, \Phi_1, \Phi_2, \ldots$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, \ldots$, respectively. We say the complex uncertain sequence $\{\zeta_n\}$ be lacunary I_σ-statistically converges in distribution to ζ if for every $\varepsilon, \delta > 0$,

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \,:\, \left\|\Phi_k(c) = \Phi(c)\right\| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathbf{I}_\sigma,$$

for all c at which $\Phi(c)$ is continuous.

Definition 5. The complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary I_σ-statistically convergent uniformly almost surely $(S_\theta(\, I_\sigma).u.a.s)$ to ζ if for every $\varepsilon, \sigma > 0$, $\exists \delta > 0$ and a sequence of events $\{E_k{}'\}$ such that

$$\left\{r \in \mathrm{N}: \tfrac{1}{h_r}\left|\left\{k \in I_r \,:\, \left|M\left(E_k^{'}\right) - 0\right| \geq \varepsilon\right\}\right| \geq \sigma\right\} \in \mathbf{I}_\sigma$$

$$\Rightarrow \left\{r \in \mathrm{N}: \tfrac{1}{h_r}\left|\left\{k \in I_r \,:\, \left|\zeta_k(x) - \zeta(x)\right| \geq \delta\right\}\right| \geq \sigma\right\} \in \mathbf{I}_\sigma.$$

Definition 6. A complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary I_σ-statistically bounded or $S_\theta(\, I_\sigma)$-bounded if there exists a real number *K>0* such that

$$q = \left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \,:\, \left\|\zeta_k\right\| \geq K\right\}\right| \geq \varepsilon\right\} \in \mathbf{I}_\sigma,$$

for every $\varepsilon > 0$, *i.e.*, $\delta^{I_{\sigma\theta}}(q) = 0$.

Definition 7. A complex uncertain sequence $\{\zeta_n\}$ is said to be lacunary I_σ-statistically convergent to ζ if for every $\varepsilon, \delta > 0$,

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \,:\, \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathbf{I}_\sigma,$$

for every $\gamma \in \Lambda$.

Now, we give the relationships among the convergence concepts of complex uncertain sequences.

Theorem 1. If the complex uncertain sequence $\{\zeta_n\}$ lacunary I_σ-statistically converges in mean to ζ, then, $\{\zeta_n\}$ lacunary I_σ-statistically converges in measure to ζ.

Proof. It follows from the Markov inequality that for any given $\varepsilon, \delta, \vartheta > 0$, we have

$$\left\{r \in \mathrm{N}: \tfrac{1}{h_r}\left|\left\{k \in I_r \,:\, M\left(\left\|\zeta_k - \zeta\right\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \vartheta\right\}$$

$$\subseteq \left\{r \in \mathrm{N}: \tfrac{1}{h_r}\left|\left\{k \in I_r \,:\, \left(\frac{E\left(\left\|\zeta_k - \zeta\right\|\right)}{\varepsilon}\right) \geq \delta\right\}\right| \geq \vartheta\right\} \in \mathbf{I}_\sigma.$$

Thus, $\{\zeta_n\}$ lacunary I_σ-statistically converges in measure to ζ and the theorem is thus proved.

Remark 1. Converse of above theorem is not true. i.e., $\{\zeta_n\}$ lacunary I_σ-statistical convergence in measure does not imply $\{\zeta_n\}$ lacunary I_σ-statistical convergence in mean. Following example illustrates this.

Example 1. Consider the uncertaintly space (Γ, L, M) to be $\gamma_1, \gamma_2, \ldots$ with

$$M\{\Lambda\}=\begin{cases}\sup_{\gamma_n\in\Lambda}\frac{1}{n+1}, & \text{if } \sup_{\gamma_n\in\Lambda}\frac{1}{n+1}<0.5,\\ 1-\sup_{\gamma_n\in\Lambda^c}\frac{1}{n+1}, & \text{if } \sup_{\gamma_n\in\Lambda^c}\frac{1}{n+1}<0.5,\\ 0.5, & \text{otherwise,}\end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma)=\begin{cases}(n+1)i, & \text{if } \gamma=\gamma_n,\\ 0, & \text{otherwise.}\end{cases}$$

for $n=1,2,\ldots$ and $\zeta\equiv 0$. For some small numbers $\varepsilon,\delta,\vartheta>0$ and $n\geq 2$, we have

$$\left\{r\in\mathrm{N}:\frac{1}{h_r}\left|\left\{k\in I_r : M\left(\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|\geq\vartheta\right\}$$

$$=\left\{r\in\mathrm{N}:\frac{1}{h_r}\left|\left\{k\in I_r : M\left(\gamma : \|\zeta_k(\gamma)-\zeta(\gamma)\|\geq\varepsilon\right)\geq\delta\right\}\right|\geq\vartheta\right\}$$

$$=\left\{r\in\mathrm{N}:\frac{1}{h_r}\left|\left\{k\in I_r : M\{\gamma_n\}\geq\delta\right\}\right|\geq\vartheta\right\}\in\mathbf{I}_\sigma.$$

Thus, the sequence $\{\zeta_n\}$ lacunary I_σ-statistically converges in measure to ζ.

However, for $n\geq 2$, we have the uncertainty distribution of uncertain variable $\|\xi_n-\xi\|=\|\xi_n\|$.

$$\Phi_n(x)=\begin{cases}0, & \text{if } x<0,\\ 1-\frac{1}{n+1}, & \text{if } 0\leq x<n+1,\\ 1, & x\geq n+1.\end{cases}$$

Hence, for each $n\geq 2$, and for every $\varepsilon,\delta>0$, we have

$$\left\{r\in\mathrm{N}:\frac{1}{h_r}\left|\left\{k\in I_r : E\left(\|\zeta_k-\zeta\|-1\right)\geq\varepsilon\right\}\right|\geq\delta\right\}$$

$$=\left\{r\in\mathrm{N}:\frac{1}{h_r}\left|\left\{k\in I_r : \left(\left[\int_0^{n+1}1-\left(1-\frac{1}{n+1}\right)dx\right]-1\right)\geq\varepsilon\right\}\right|\geq\delta\right\},$$

which is impossible. That is, the sequence $\{\zeta_\mathrm{n}\}$ does not lacunary I_σ-statistically converges in mean to ζ.

Theorem 2. Assume complex uncertain sequence $\{\zeta_n\}$ with real part $\{\xi_n\}$ and imaginary part $\{\gamma_n\}$, respectively, for $n=1,2,\ldots$. If uncertain sequences $\{\xi_n\}$ and $\{\gamma_n\}$ lacunary I_σ-statistically convergent in measure to ξ and γ, respectively, then, complex uncertain sequence $\{\zeta_n\}$ lacunary I_σ-statistically convergent in measure to $\zeta=\xi+i\gamma$.

Proof. It follows from the definition of lacunary I_σ-statistically convergence in measure of uncertain sequence that for any small numbers $\varepsilon,\delta,\vartheta>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\xi_k-\xi\|\geq\frac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|\geq\vartheta\right\}\in\mathbf{I}_\sigma$$

and

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\gamma_k-\gamma\|\geq\frac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|\geq\vartheta\right\}\in\mathbf{I}_\sigma.$$

Note that $\|\zeta_n-\zeta\|=\sqrt{|\xi_n-\xi|^2+|\gamma_n-\gamma|^2}$. Thus, we have

$$\left\{\|\zeta_n-\zeta\|\geq\varepsilon\right\}\subset\left\{\|\xi_n-\xi\|\geq\frac{\varepsilon}{\sqrt{2}}\cup\|\gamma_n-\gamma\|\geq\frac{\varepsilon}{\sqrt{2}}\right\}.$$

Using the subadditivity axiom of uncertain measure, we obtain

$$\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|\geq\vartheta\right\}$$

$$\subseteq\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\xi_k-\xi\|\geq\tfrac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|\geq\vartheta\right\}$$

$$\cup\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\gamma_k-\gamma\|\geq\tfrac{\varepsilon}{\sqrt{2}}\right)\geq\delta\right\}\right|\geq\vartheta\right\}\in\mathbf{I}_\sigma.$$

Hence, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\right\}\right|\geq\vartheta\right\}\in\mathbf{I}_\sigma.$$

That is, $\{\zeta_n\}$ lacunary I_σ-statistically converges in measure to ζ.

Theorem 3. Assume complex uncertain sequence $\{\zeta_n\}$ with real part $\{\xi_n\}$ and imaginary part $\{\gamma_n\}$, respectively, for *n=1,2,…*. If uncertain sequences $\{\xi_n\}$ and $\{\gamma_n\}$ lacunary I_σ-statistically convergent in measure to ξ and γ, respectively, then, complex uncertain sequence $\{\zeta_n\}$ lacunary I_σ-statistically convergent in distribution to $\zeta=\xi+i\gamma$.

Proof. Let $c=a+ib$ be a given continuity point of the complex uncertainty distribution Φ. On the other hand, for any $\alpha>a,\beta>b$, we have

$$\begin{aligned}\left\{\xi_n\leq a,\gamma_n\leq b\right\}&=\left\{\xi_n\leq a,\gamma_n\leq b,\xi\leq\alpha,\gamma\leq\beta\right\}\cup\left\{\xi_n\leq a,\gamma_n\leq b,\xi>\alpha,\gamma>\beta\right\}\\&\quad\cup\left\{\xi_n\leq a,\gamma_n\leq b,\xi\leq\alpha,\gamma>\beta\right\}\cup\left\{\xi_n\leq a,\gamma_n\leq b,\xi>\alpha,\gamma\leq\beta\right\}\\&\subset\left\{\xi\leq a,\gamma\leq b\right\}\cup\left\{|\xi_n-\xi|\geq\alpha-a\right\}\cup\left\{|\gamma_n-\gamma|\geq\beta-b\right\}.\end{aligned}$$

It follows from the subadditivity axiom that

$$\Phi_n(c)=\Phi_n(a+ib)\leq\Phi(\alpha+i\beta)+M\left\{|\xi_n-\xi|\geq\alpha-a\right\}+M\left\{|\gamma_n-\gamma|\geq\beta-b\right\}.$$

Since $\{\xi_n\}$ and $\{\gamma_n\}$ lacunary I_σ-statistically converge in measure to ξ and γ, respectively, hence, for any small numbers $\varepsilon,\delta>0$, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r\ :\ M\left(\|\xi_k-\xi\|\geq\alpha-a\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathbf{I}_\sigma$$

and

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \, : M\left(\left\|\gamma_k - \gamma\right\| \geq \beta - b\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathtt{I}_\sigma.$$

Thus, we obtain

$$I_\sigma\text{-}\limsup_{n\to\infty} \Phi_n(c) \leq \Phi(\alpha + i\beta)$$

for any $\alpha > a, \beta > b$. Letting $\alpha + i\beta \to a + ib,$ we get

$$\mathtt{I}_\sigma\text{-}\limsup_{n\to\infty} \Phi_n(c) \leq \Phi(c). \tag{1}$$

On the other hand, for any $x < a, y < b$ we have

$$\begin{aligned}\left\{\xi \leq x, \gamma \leq y\right\} &= \left\{\xi_n \leq a, \gamma_n \leq b, \xi \leq x, \gamma \leq y\right\} \cup \left\{\xi_n \leq a, \gamma_n \leq b, \xi \leq x, \gamma \leq y\right\} \\ &\cup \left\{\xi_n > a, \gamma_n \leq b, \xi \leq x, \gamma \leq y\right\} \cup \left\{\xi_n > a, \gamma_n > b, \xi \leq x, \gamma \leq y\right\} \\ &\subset \left\{\xi_n \leq a, \gamma_n \leq b\right\} \cup \left\{\left|\xi_n - \xi\right| \geq a - x\right\} \cup \left\{\left|\gamma_n - \gamma\right| \geq b - y\right\}.\end{aligned}$$

This implies,

$$\Phi(x+iy) \leq \Phi_n(a+ib) + M\left\{\left\|\xi_n - \xi\right\| \geq a - x\right\} + M\left\{\left\|\gamma_n - \gamma\right\| \geq b - y\right\}.$$

Since

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \, : M\left(\left\|\zeta_k - \zeta\right\| \geq a - x\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathtt{I}_\sigma$$

and

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \, : M\left(\left\|\gamma_k - \gamma\right\| \geq b - y\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathtt{I}_\sigma,$$

we obtain

$$\Phi(x+iy) \leq \mathtt{I}_\sigma\text{-}\liminf_{n\to\infty} \Phi_n(a+ib)$$

for any $x < a, y < b.$ Taking $x + iy \to a + ib,$ we get

$$\Phi(c) \leq \mathtt{I}_\sigma\text{-}\liminf_{n\to\infty} \Phi_n(c) \tag{2}$$

It follows from *(1)* and *(2)* that $\Phi_n(c) \to \Phi(c)$ as $n \to \infty$. That is the complex uncertain sequence $\{\zeta_n\}$ is lacunary I_σ-statistically convergent in distribution to $\zeta = \xi + i\gamma$.

Remark 2. Converse of the above theorem is not necessarily true. *i.e.* lacunary I_σ-statistically convergence in distribution does not imply lacunary I_σ-statistically convergence in measure. Following example illustrates this.

Example 2. Consider the uncertaintly space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M(\gamma_1) = M(\gamma_2) = \frac{1}{2}$.

We define a complex uncertain variable as

$$\zeta(\gamma)=\begin{cases} i, & \text{if } \gamma=\gamma_1, \\ -i, & \text{if } \gamma=\gamma_2. \end{cases}$$

We also define $\zeta_n = -\zeta$ for *n=1,2,…*. Then, ζ_n and ζ have the same distribution

$$\Phi_n(c)=\Phi_n(a+ib)=\begin{cases} 0, & \text{if } a<0, -\infty<b<+\infty, \\ 0, & \text{if } a\geq 0, b<-1, \\ \frac{1}{2}, & \text{if } a\geq 0, -1\leq b<1, \\ 1, & \text{if } a\geq 0, b\geq 1. \end{cases}$$

Then, $\{\zeta_n\}$ is lacunary I_σ-statistically in distribution to ζ. However, for a given *ε,δ>0*, we have

$$\left\{r\in \mathrm{N}: \frac{1}{h_r}\left|\left\{k\in I_r : M\left(\left\|\xi_k-\xi\right\|\geq\varepsilon\right)\geq 1\right\}\right|\geq\delta\right\}$$

$$\left\{r\in \mathrm{N}: \frac{1}{h_r}\left|\left\{k\in I_r : M\left(\gamma : \left\|\xi_k(\gamma)-\xi(\gamma)\right\|\geq\varepsilon\right)\geq 1\right\}\right|\geq\delta\right\}\in \mathbf{I}_\sigma.$$

That is the sequence $\{\zeta_n\}$ does not lacunary I_σ-statistically convergence in measure to ζ. *By Theorem 3,* the real part and imaginary part of $\{\zeta_n\}$ also not lacunary I_σ-statistically convergent in measure. In addition, since $\zeta_n = -\zeta$ *for n=1,2,…*, the sequence $\{\zeta_n\}$ does not is lacunary I_σ-statistically convergence a.s to ζ. This completes the proof.

Lacunary I_σ-statistically convergence a.s. does not imply is lacunary I_σ-statistically convergence in measure.

Example 3. Consider the uncertaintly space (Γ, L, M) to be $\gamma_1, \gamma_2, \ldots$ with

$$M\{\Lambda\}=\begin{cases} \sup_{\gamma_n\in\Lambda}\frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n\in\Lambda}\frac{n}{(2n+1)}<0.5, \\ 1-\sup_{\gamma_n\in\Lambda^c}\frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n\in\Lambda^c}\frac{n}{(2n+1)}<0.5, \\ 0.5, & \text{otherwise.} \end{cases}$$

and we define a complex uncertain variable as

$$\zeta_n(\gamma)=\begin{cases} in, & \text{if } \gamma=\gamma_n, \\ 0, & \text{otherwise.} \end{cases}$$

for *n=1,2,…* and $\zeta\equiv 0$. Then, the sequence $\{\zeta_n\}$ lacunary I_σ-statistically converges *a.s. to* ζ. However for some small numbers $\varepsilon, \delta > 0$, we have

$$\left\{r\in \mathrm{N}: \frac{1}{h_r}\left|\left\{k\in I_r : M\left(\left\|\xi_k-\xi\right\|\geq\varepsilon\right)\geq \frac{1}{2}\right\}\right|\geq\delta\right\}$$

$$=\left\{r\in \mathrm{N}: \frac{1}{h_r}\left|\left\{k\in I_r : M\left(\gamma : \left\|\zeta_k(\gamma)-\zeta(\gamma)\right\|\geq\varepsilon\right)\geq \frac{1}{2}\right\}\right|\geq\delta\right\}$$

$$=\left\{r\in \mathrm{N}: \frac{1}{h_r}\left|\left\{k\in I_r : M\{\gamma_n\}\geq \frac{1}{2}\right\}\right|\geq\delta\right\},$$

That is the sequence $\{\zeta_n\}$ does not lacunary I_σ-statistically converge in measure to ζ.

Remark 3. Lacunary I_σ-statistically convergence in measure does not imply lacunary I_σ-statistically convergence *a.s.*

Example 4. Consider the uncertaintly space (Γ, L, M) to be $[0,1]$ with Borel algebra and Lebesque measure. For any positive integer *n*, there is an integer *p* such that $n = 2^p + k$, where *k* is an integer between *0* and $2^p - 1$. Then, we define a complex uncertain variable by

$$\zeta_n(\gamma) = \begin{cases} i, & \text{if } \frac{k}{2^p} \le \gamma \le \frac{k+1}{2^p}, \\ 0, & \text{otherwise.} \end{cases}$$

for *n=1,2,...*and $\zeta \equiv 0$. For some small numbers $\varepsilon, \delta, \vartheta > 0$ and $n \ge 2$, we have

$$\left\{ r \in \mathrm{N} : \tfrac{1}{h_r} \left| \left\{ k \in I_r : M\left(\left\| \xi_k - \xi \right\| \ge \varepsilon \right) \ge \delta \right\} \right| \ge \vartheta \right\}$$

$$= \left\{ r \in \mathrm{N} : \tfrac{1}{h_r} \left| \left\{ k \in I_r : M\left(\gamma : \left\| \zeta_k(\gamma) - \zeta(\gamma) \right\| \ge \varepsilon \right) \ge \delta \right\} \right| \ge \vartheta \right\}$$

$$= \left\{ r \in \mathrm{N} : \tfrac{1}{h_r} \left| \left\{ k \in I_r : M\{\gamma_n\} \ge \delta \right\} \right| \ge \vartheta \right\},$$

Thus, the sequence $\{\zeta_n\}$ lacunary I_σ-statistically converges in measure to ζ. In addition for every $\varepsilon, \delta > 0$, we have

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : E\left(\left\| \zeta_k - \zeta \right\| \right) \ge \varepsilon \right\} \right| \ge \delta \right\} \in \mathbf{I}_\sigma.$$

Hence, the sequence $\{\zeta_n\}$ also lacunary I_σ-statistically converges in mean to ζ. However, for any $\gamma \in [0,1]$, there is an infinite number of intervals of the form $\left[\frac{k}{2^p}, \frac{k+1}{2^p}\right]$ containing γ. Thus, $\zeta_n(\gamma)$ does not lacunary I_σ-statistically converge to *0*. In other words, the sequence $\{\zeta_n\}$ does not lacunary I_σ-statistically converge *a.s.* to ζ. This completes the proof.

Lacunary I_σ-statistically convergence *a.s.* does not imply lacunary I_σ-statistically convergence in mean.

Example 5. Consider the uncertaintly space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with

$$M\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{3^n}.$$

The complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} i3^n, & \text{if } \gamma = \gamma_n, \\ 0, & \text{otherwise.} \end{cases}$$

for *n=1,2,...*and $\zeta \equiv 0$. Then, the sequence $\{\zeta_n\}$ lacunary I_σ-statistically converges *a.s.* to ζ. However, the uncertaintly distributions of $\|\zeta_n\|$ are

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - \frac{1}{3^n}, & \text{if } 0 \le x < 3^n, \\ 1, & \text{if } x \ge 3^n, \end{cases}$$

for *n=1,2,...,* respectively. Then, we have

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : E\left(\left\| \zeta_k - \zeta \right\| \right) \geq 1 \right\} \right| \geq \delta \right\} \in \mathbf{I}.$$

Therefore, the sequence $\{\zeta_n\}$ does not lacunary I_σ-statistically converge in mean to ζ.

From the example *5*, we can obtain that lacunary I_σ-statistically convergence in mean does not imply lacunary I_σ-statistically converge *a.s.*

Proposition 1. Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then, $\{\zeta_n\}$ lacunary I_σ-statistical converges *a.s* to ζ if and only if for any $\varepsilon, \delta, \vartheta > 0$, we have

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : M\left(\bigcup_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\| \zeta_n - \zeta \right\| \geq \varepsilon \right) \geq \delta \right\} \right| \geq \vartheta \right\} \in \mathbf{I}_\sigma .$$

Proof. By the definition of lacunary I_σ-statistical converges *a.s*, we have that there exists an event Λ with $M(\Lambda) = 1$ such that

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \zeta_k - \zeta \right\| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathbf{I}_\sigma$$

for every $\varepsilon, \delta > 0$. Then, for any $\varepsilon, \vartheta > 0$, there exists a number *k* such that $\|\xi_n - \xi\| < \varepsilon$ where *n*>*k* and for any $\gamma \in \Lambda$, that is equivalent to

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : M\left(\bigcup_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\| \zeta_n - \zeta \right\| < \varepsilon \right) \geq 1 \right\} \right| \geq \vartheta \right\} \in \mathbf{I}_\sigma .$$

It follows from the duality axiom of uncertain measure that

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : M\left(\bigcup_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\| \zeta_n - \zeta \right\| \geq \varepsilon \right) \geq \delta \right\} \right| \geq \vartheta \right\} \in \mathbf{I}_\sigma .$$

Proposition 2. Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then, $\{\zeta_n\}$ lacunary I_σ-statistical converges uniformly *a.s* to ζ if and only if for any $\varepsilon, \delta, \vartheta > 0$, we have

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : M\left(\bigcup_{n=k}^{\infty} \left\| \zeta_k - \zeta \right\| \geq \varepsilon \right) \geq \delta \right\} \right| \geq \vartheta \right\} \in \mathbf{I}_\sigma .$$

Proof If $\{\zeta_n\}$ lacunary I_σ-statistical converges uniformly *a.s* to ζ, then, for any $\vartheta > 0$ there exists a number K such that $M\{K\} < \vartheta$ and $\{\zeta_n\}$ lacunary I_σ-statistical converges to ζ on $\Gamma - K$. Thus, for any $\varepsilon > 0$, there exists a number *k*>*0* such that $\|\xi_n - \xi\| < \varepsilon$; where *n*>*k* and for any $\gamma \in \Gamma - K$. That is

$$\bigcup_{n=k}^{\infty} \left\{ \left\| \zeta_n - \zeta \right\| \geq \varepsilon \right\} \subset K.$$

It follows from the subadditivity axiom that

$$\left\{ r \in \mathrm{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left(\bigcup_{n=k}^{\infty} \left\| \zeta_n - \zeta \right\| \geq \varepsilon \right) \right\} \right| \geq \delta \right\} \subseteq \delta^{\mathbf{I}_{\sigma\theta}} \left(M\left\{ K \right\} \right) \subseteq \vartheta .$$

Then,

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \, : M\left(\bigcup_{n=k}^{\infty}\|\zeta_n - \zeta\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \vartheta\right\} \in \mathtt{I}_\sigma.$$

On the contrary, if

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \, : M\left(\bigcup_{n=k}^{\infty}\|\zeta_n - \zeta\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \vartheta\right\} \in \mathtt{I}_\sigma,$$

for any $\varepsilon, \delta, \vartheta > 0$, then for given $\delta > 0$ and $m \geq 1$, there exists m_k such that

$$\delta^{\mathtt{I}_{\sigma\theta}}\left(M\left(\bigcup_{n=m_k}^{\infty}\|\zeta_n - \zeta\| \geq \frac{1}{m}\right)\right) < \frac{\delta}{2^m}.$$

Let

$$K = \bigcup_{m=1}^{\infty}\bigcup_{n=m_k}^{\infty}\left\{\|\zeta_n - \zeta\| \geq \tfrac{1}{m}\right\}.$$

Then

$$\delta^{\mathtt{I}_{\sigma\theta}}\left(M\{K\}\right) \leq \sum_{m=1}^{\infty}\delta^{\mathtt{I}_{\sigma\theta}}\left(M\left(\bigcup_{n=m_k}^{\infty}\left\{\|\zeta_n - \zeta\| \geq \frac{1}{m}\right\}\right)\right) \leq \sum_{m=1}^{\infty}\frac{\delta}{2^m}.$$

In addition, we get

$$\mathtt{I}_\sigma\text{-}\sup_{\gamma \in \Gamma - K}\|\zeta_n - \zeta\| < \frac{1}{m}$$

for any *m=1,2,3,...* and $n > m_k$. The proposition is thus proved.

Theorem 4 If the complex uncertain sequence $\{\zeta_n\}$ lacunary I_σ-statistical converges uniformly *a.s* to ζ, then, $\{\zeta_n\}$ lacunary I_σ-statistical converges to ζ.

Proof. It follows from above Proposition that $\{\zeta_n\}$ lacunary I_σ-statistical converges uniformly *a.s* to ζ, then

$$\left\{r \in \mathrm{N}: \frac{1}{h_r}\left|\left\{k \in I_r \, : M\left(\bigcup_{n=k}^{\infty}\|\zeta_n - \zeta\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \vartheta\right\} \in \mathtt{I}_\sigma.$$

Since

$$\delta^{\mathtt{I}_{\sigma\theta}}\left(M\left(\bigcup_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\|\zeta_n - \zeta\| \geq \varepsilon\right)\right) \leq \delta^{\mathtt{I}_{\sigma\theta}}\left(M\left(\bigcup_{n=k}^{\infty}\|\zeta_n - \zeta\| \geq \varepsilon\right)\right)$$

taking the limit as $n \to \infty$ on both side of above inequality, we obtain

$$\delta^{\mathtt{I}_{\sigma\theta}}\left(M\left(\bigcup_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\|\zeta_n - \zeta\| \geq \varepsilon\right)\right) = 0.$$

By the first proposition, $\{\zeta_n\}$ lacunary I_σ-statistical converges to ζ.

Theorem 5 If the complex uncertain sequence $\{\zeta_n\}$ lacunary I_σ-statistical converges uniformly *a.s* to ζ, then, $\{\zeta_n\}$ lacunary I_σ-statistical converges in measure to ζ.

Proof. If the complex uncertain sequence $\{\zeta_n\}$ lacunary I_σ-statistical converges uniformly *a.s* to ζ, then, from proposition above we have

$$\left\{ r \in \mathrm{N}: \frac{1}{h_r} \left| \left\{ k \in I_r \ : M\left(\bigcup_{n=k}^{\infty} \left\| \zeta_n - \zeta \right\| \geq \varepsilon \right) \geq \delta \right\} \right| \geq \vartheta \right\} \in \mathtt{I}_\sigma,$$

and

$$\delta^{\mathtt{I}_{\sigma\theta}}\left(M\left(\left\| \zeta_n - \zeta \right\| \geq \varepsilon \right) \right) \leq \delta^{\mathtt{I}_{\sigma\theta}}\left(M\left(\bigcup_{n=k}^{\infty} \left\| \zeta_n - \zeta \right\| \geq \varepsilon \right) \right).$$

Letting $n \to \infty$, we can obtain $\{\zeta_n\}$ lacunary I_σ-statistical converges in measure to ζ.

3. CONCLUSION

In this paper, we give lacunary I_σ-statistically convergence of complex uncertain sequence. In further studies, the lacunary I_σ-statistically convergence by using double sequences can be defined and examined for complex uncertain sequence.

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