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Research Article

MULTIPLICATIVELY HARMONICALLY *P*-FUNCTIONS AND SOME RELATED INEQUALITIES

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ABSTRACT

In this study, we introduce a new class of functions called as multiplicatively harmonically P-function. Some new Hermite-Hadamard type inequalities are obtained for this class of functions.
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1. PRELIMINARIES

The following double inequality is well known as the Hadamard inequality in the literature.

Theorem 1 [1] *$f:[a,b]\to\mathbb{R}$ be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality.

Definition 1 [2] *We say that a function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ belongs to the class $P(I)$ (or called P-function) if it is nonnegative and for all $x,y\in I$ and $\lambda\in[0,1]$ satisfies the following inequality*

$$f(\lambda x+(1-\lambda)y)\leq f(x)+f(y)$$

holds.

Note that $P(I)$ contain all nonnegative monotone convex and quasi-convex functions.

In [2], Dragomir et al. proved the following inequality of Hadamard type for class of P-functions.

Theorem 2 *Let $f\in P(I)$, $a,b\in I$ with $a<b$ and $f\in L[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2}{b-a}\int_a^b f(x)dx\leq 2[f(a)+f(b)].$$

Both inequalities are the best possible.

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In [4], İşcan gave the definition of harmonically convexity as follows:

Definition 2 *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f: I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.1}$$

for all $x, y \in I$ and $t \in 0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

Example 1 *Let $f: (0,\infty) \to \mathbb{R}$, $f(x) = x$, and $g: (-\infty, 0) \to \mathbb{R}$, $g(x) = x$, then f is a harmonically convex function and g is a harmonically concave function.*
The following proposition is obvious from this example:

Proposition 1 *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval and $f: I \to \mathbb{R}$ is a function, then ;*

- if $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is harmonically convex.
- if $I \subset (0,\infty)$ and f is harmonically convex and nonincreasing function then f is convex.
- if $I \subset (-\infty, 0)$ and f is harmonically convex and nondecreasing function then f is convex.
- if $I \subset (-\infty, 0)$ and f is convex and nonincreasing function then f is a harmonically convex.

The following result of the Hermite-Hadamard type holds.

Theorem 3 *Let $f: I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

The above inequalities are sharp.

In [4], İşcan used the following lemma to prove Theorems.

Lemma 1 *Let $f: I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt.$$

Definition 3 [3] *A function $f: I \subseteq (0,\infty) \to \mathbb{R}$ is said to be harmonically P-function on I or belong to the class $HP(I)$ if it is nonnegative and,*

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq f(x) + f(y),$$

for any $x, y \in I$ and $t \in [0,1]$.

Proposition 2 [3] *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$. If f is P-function and nondecreasing, then $f \in HP(I)$.*

Proposition 3 [3] *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$. If $f \in HP(I)$ and nonincreasing, then f is P-function on I.*

Hermite-Hadamard's inequalities can be represented for harmonically P-function as follows.

Theorem 4 [3] *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically P-function on $[a,b]$, then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{2ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \leq 2[f(a)+f(b)]. \tag{1.3}$$

Recently, Kadakal gave a new definition called as multiplicatively P-function as follows.

Definition 4 *Let $I \neq \emptyset$ be an interval in $\mathbb{R}\backslash\{0\}$. The function $f: I \to 0,\infty)$ is said to be multiplicatively P-function, if the inequality*

$$f(tx+(1-t)y) \leq f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [5], Kadakal also gave the following Hermite Hadamard type inequalities for this class of functions.

Theorem 5 *Let the function $f: I \subseteq \mathbb{R} \to 1, \infty)$,be a multiplicatively P-function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequalities hold:*

i) $f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx \leq [f(a) f(b)]^2$

ii) $f\left(\frac{a+b}{2}\right) \leq f(a) f(b) \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a) f(b)]^2$.

The main purpose of this paper is to give a new concept called as multiplicatively harmonically P-function, compare other function classes with this class of functions, establish Hermite-Hadamard type inequalities for functions multiplicatively harmonically P-function. Ideas of this paper may stimulate further research.

2. MULTIPLICATIVELY HARMONICALLY *P*-FUNCTIONS

In this section, we begin by setting the definition of multiplicatively harmonically P-function and some algebraic properties for this class of functions.

Definition 5 *Let $I \neq \emptyset$ be an interval in $\mathbb{R} \backslash \{0\}$. The function $f: I \to [0, \infty)$ is said to be multiplicatively harmonically P-function, if the inequality*

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq f(x) f(y) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

We will denote by $MHP(I)$ the class of all multiplicatively harmonically P-functions on interval I.

Remark 1 *If $f \in MHP(I)$, the range of f is greater than or equal to* 1.

Proof. In the inequality (2.1), for $t = 1$;

$$f(x) \leq f(x) f(y) \Longrightarrow f(x)[1 - f(y)] \leq 0.$$

Since $f(x) \geq 0$ for all $x \in I$, we obtain $f(y) \geq 1$, for all $y \in I$. Also, since for $t = 0$,

$$f(y) \leq f(x) f(y) \Longrightarrow f(y)[1 - f(x)] \leq 0,$$

and $f(y) \geq 0$ for all $x \in I$, we get $f(x) \geq 1$, for all $x \in I$.

Remark 2 *i.) Let $f: I \subseteq (0, \infty) \to [1, \infty)$ be a function. Then, f is multiplicatively harmonically P-function if and only if lnf is harmonically P-function. So, a multiplicatively harmonically P-function $f: I \subseteq (0, \infty) \to [1, \infty)$ can be called as log-harmonically P-function.*

ii.) If $f: I \subseteq (0, \infty) \to [1, \infty)$ is a harmonically P-function, then f is also a multiplicatively harmonically P-function. Since we have

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq f(x) + f(y) \leq f(x) f(y).$$

Example 2 *The function $f: [1, \infty) \to [1, \infty), f(x) = x$ is a multiplicatively harmonically P-function. Really, for any $x, y \in 1, \infty)$ with $x < y$, we have*

$$f\left(\frac{xy}{ty+(1-t)x}\right) = \frac{xy}{tx+(1-t)y} \leq y \leq xy = f(x) f(y).$$

Example 3 *i.) The function $f: (0, \infty) \to (1, \infty), f(x) = e^x$ is a multiplicatively harmonically P-function. Since, for any $x, y \in (0, \infty)$ with $x < y$, we have*

$$f\left(\frac{xy}{ty+(1-t)x}\right) = e^{\frac{xy}{ty+(1-t)x}} \leq e^y \leq e^x e^y = f(x) f(y).$$

ii.) The function $f: (-\infty, 0) \to (1, \infty), f(x) = e^{-x}$ is a multiplicatively harmonically P-function.

Example 4 *The function $f:[e,\infty)\to[1,\infty), f(x)=lnx$ is a multiplicatively harmonically P-function. Since, for any $x,y\in 0,\infty)$ with $x<y$, we have*

$$f\left(\frac{xy}{ty+(1-t)x}\right)=ln\left(\frac{xy}{ty+(1-t)x}\right)=lny+ln\left(\frac{x}{ty+(1-t)x}\right)$$
$$\leq lny\leq lny.lnx=f(x)f(y).$$

Proposition 4 *Let $f:I\subset\mathbb{R}\backslash\{0\}\to[1,\infty)$ be a function and $g:\left\{\frac{1}{x},x\in I\right\}\to I$, $g(x)=1/x$. f is multiplicatively harmonically P-function on the interval I if and only if $f\circ g$ is multiplicatively P-function on the interval $g^{-1}(I)=\left\{\frac{1}{x},x\in I\right\}$.*

Proof. Let f be a multiplicatively harmonically P-function on the interval I. If we take arbitrary $x,y\in g^{-1}(I)$, then there exist $u,v\in I$ such that $x=1/u$ and $y=1/v$

$$(f\circ g)(tx+(1-t)y)=f\left(\frac{uv}{tv+(1-t)u}\right)\leq f(u)f(v)=(f\circ g)(x)(f\circ g)(y)$$

Conversely, if $f\circ g$ is multiplicatively P-function on the interval $g^{-1}(I)$ then it is easily seen that f is multiplicatively harmonically P-function on the interval I by a similar procedure. The details are omitted.

Proposition 5 *Let $I\subseteq\mathbb{R}\backslash\{0\}$ be a real interval and $f:I\to[1,\infty)$ is a function, then ;*

- if f is harmonically convex, then f is also harmonically multiplicatively P-function.
- if $I\subseteq(0,\infty)$ and f is multiplicatively P-function and nondecreasing function then f is harmonically multiplicatively P-function.
- if $I\subseteq(0,\infty)$ and f is harmonically multiplicatively P-function and nonincreasing function then f ismultiplicatively P-function.
- if $I\subseteq(-\infty,0)$ and f is harmonically multiplicatively P-function and nondecreasing function then f is multiplicatively P-function.
- if $I\subseteq(-\infty,0)$ and f is multiplicatively P-function and nonincreasing function then f is a harmonicallymultiplicatively P-function.

Proof. i.) Since

$$f\left(\frac{xy}{ty+(1-t)x}\right)\leq tf(x)+(1-t)f(y)\leq f(x)f(y),$$

f is also multiplicatively P-function.

ii.) Since for any $x,y\in I\subseteq(0,\infty)$ and $t\in[0,1]$

$$\frac{xy}{ty+(1-t)x}\leq tx+(1-t)y, \tag{2.2}$$

and f is nondecreasing and multiplicatively P-function we have

$$f\left(\frac{xy}{ty+(1-t)x}\right)\leq f(tx+(1-t)y)\leq f(x)f(y).$$

iii.) By the inequality (2.2) and since f is nonincreasing and harmonically multiplicatively P-function we have

$$f(tx+(1-t)y)\leq f\left(\frac{xy}{ty+(1-t)x}\right)\leq f(x)f(y).$$

for any $x,y\in I\subseteq(0,\infty)$ and $t\in[0,1]$

iv.) Since for any $x,y\in I\subseteq(-\infty,0)$ and $t\in[0,1]$

$$\frac{xy}{ty+(1-t)x}\geq tx+(1-t)y, \tag{2.3}$$

and f is nondecreasing and harmonically multiplicatively P-function we have

$$f(tx+(1-t)y) \leq f\left(\frac{xy}{ty+(1-t)x}\right) \leq f(x)f(y).$$

v.) By the inequality (2.3) and since f is nonincreasing and multiplicatively P-function we have

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq f(tx+(1-t)y) \leq f(x)f(y).$$

Theorem 6 *Let $f,g: I \subseteq \mathbb{R}\backslash\{0\} \to [1,\infty)$. If f and g are multiplicatively harmonically P-function, then fg are multiplicatively harmonically P-function.*

Proof. For $x,y \in I$ and $t \in [0,1]$, we have

$$\begin{aligned}(fg)\left(\frac{xy}{ty+(1-t)x}\right) &= f\left(\frac{xy}{ty+(1-t)x}\right) g\left(\frac{xy}{ty+(1-t)x}\right)\\ &\leq [f(x)f(y)][g(x)y(y)]\\ &= [f(x)g(x)][f(y)g(y)]\\ &= [(fg)(x)][(fg)(y)]\end{aligned}$$

This completes the proof of theorem.

Theorem 7 *Let $f,g: I \subseteq \mathbb{R}\backslash\{0\} \to [1,\infty)$. If f is multiplicatively P-function and nonincreasing and g is harmonically convex function, then fog is multiplicatively harmonically P-function.*

Proof. For $x,y \in I$ and $t \in [0,1]$, we obtain

$$\begin{aligned}(fog)\left(\frac{xy}{ty+(1-t)x}\right) &= f\left(g\left(\frac{xy}{ty+(1-t)x}\right)\right)\\ &\leq f(tg(x)+(1-t)g(y))\\ &\leq f(g(x))f(g(y))\\ &= (fog)(x)(fog)(y).\end{aligned}$$

This completes the proof of theorem.

3. HERMITE-HADAMARD TYPE INEQUALITIES

The goal of this paper is to develop concepts of the multiplicatively harmonically P-functions and to establish some inequalities of Hermite-Hadamard type for these classes of functions.

Theorem 8 *Let the function $f: I \subseteq \mathbb{R}\backslash\{0\} \to [1,\infty)$,be a multiplicatively harmonically P-function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

i) $f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)f([a^{-1}+b^{-1}-x^{-1}]^{-1})}{x^2}dx \leq [f(a)f(b)]^2$

ii) $f\left(\frac{2ab}{a+b}\right) \leq f(a)f(b)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq [f(a)f(b)]^2$.

Proof. i) Since the function f is a multiplicatively harmonically P-function, we write the following inequality:

$$f\left(\frac{2ab}{a+b}\right) = f\left(\frac{2\left[\frac{ab}{ta+(1-t)b}\right]\left[\frac{ab}{tb+(1-t)a}\right]}{\left[\frac{ab}{ta+(1-t)b}\right]+\left[\frac{ab}{tb+(1-t)a}\right]}\right) \leq f\left(\frac{ab}{ta+(1-t)b}\right) f\left(\frac{ab}{tb+(1-t)a}\right)$$

By integrating this inequality on $[0,1]$ and changing the variable as $x = \frac{ab}{ta+(1-t)b}$, then

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)f([a^{-1}+b^{-1}-x^{-1}]^{-1})}{x^2}dx.$$

Moreover, a simple calculation give us that

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) f\left(\frac{ab}{tb+(1-t)a}\right) dt \leq [f(a)f(b)]^2.$$

So, we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)f([a^{-1}+b^{-1}-x^{-1}]^{-1})}{x^2}dx \leq [f(a)f(b)]^2.$$

ii) Similarly, as f is a multiplicatively harmonically P-function, we write the following:

$$f\left(\frac{2ab}{a+b}\right) \leq f\left(\frac{ab}{ta+(1-t)b}\right) f\left(\frac{ab}{tb+(1-t)a}\right) \leq f(a)f(b)f\left(\frac{ab}{tb+(1-t)a}\right)$$

Here, by integrating this inequality on $[0,1]$ and changing the variable as $x = \frac{ab}{tb+(1-t)a}$, then, we have

$$f\left(\frac{2ab}{a+b}\right) \leq f(a)f(b)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx.$$

Since,

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right) dt \leq f(a)f(b),$$

we obtain

$$f\left(\frac{2ab}{a+b}\right) \leq f(a)f(b)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq [f(a)f(b)]^2.$$

This completes the proof of theorem.

Remark 3 *Above Theorem (i) and (ii) can be written together as follows:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)f([a^{-1}+b^{-1}-x^{-1}]^{-1})}{x^2}dx \leq f(a)f(b)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq [f(a)f(b)]^2. \quad (3.1)$$

Then by (2.2) we get required inequalities.

Remark 4 *By helping Theorem 5 and Proposition 4, the proof of Theorem 8 can also be given as follows :*

Since $f: I \subseteq \mathbb{R}\backslash\{0\} \rightarrow 1, \infty)$ is a multiplicatively harmonically P-function, $f \circ g$ is multiplicatively P-function on the interval $[1/b, 1/a]$ for $a, b \in I$ with $a < b$ So, by Theorem 5 we have

i) $(f\circ g)\left(\frac{1/a+1/b}{2}\right) \leq \frac{1}{1/b-1/a}\int_{1/a}^{1/b} (f\circ g)(u)(f\circ g)(1/a+b-u)du$
$\leq [(f\circ g)(1/a)(f\circ g)(1/b)]^2$

ii) $(f\circ g)\left(\frac{1/a+1/b}{2}\right) \leq (f\circ g)(1/a)(f\circ g)(1/b)\frac{1}{1/b-1/a}\int_{1/a}^{1/b} (f\circ g)(u)du$
$\leq \left[(f\circ g)\left(\frac{1}{a}\right)(f\circ g)\left(\frac{1}{b}\right)\right]^2.$

In the last inequalities, if we put $g(x) = 1/x$ and change the variable as $u = 1/x$ in the integrals, then we obtain the inequalities in Theorem 8.

By using Theorem 4 and Remark 2, we can give the following integral inequalities for multiplicatively harmonically P-functions.

Theorem 9 *Let the function $f: I \subseteq (0,\infty) \rightarrow [1,\infty)$,be a multiplicatively harmonically P-function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq exp\left\{\frac{2ab}{b-a}\int_a^b \frac{lnf(u)}{u^2}du\right\} \leq [f(a)f(b)]^2. \quad (3.2)$$

Proof. The proof of inequalities are easily seen that by using Theorem 4 and Remark 2. We ommited the detailes.

For finding some new inequalities of Hermite-Hadamard type for functions whose derivatives are multiplicatively harmonically P-function, we need Lemma 1.

Theorem 10 *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is multiplicatively harmonically P-function on $[a,b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)|f'(a)||f'(b)|}{2}\left[\frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right)\right] \tag{3.3}$$

Proof. From Lemma 1 and using the Power-mean integral inequality, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right| \left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right| dt$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right| \left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Hence, by being multiplicatively harmonically P-function of $|f'|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)|f'(a)||f'(b)|}{2}\left(\int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^2}dt\right)$$

It is easily check that

$$\int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^2}dt = \frac{1}{ab} - \frac{2}{(b-a)^2} ln\left(\frac{(a+b)^2}{4ab}\right).$$

Theorem 11 *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is multiplicatively harmonically P-function on $[a,b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)|f'(a)||f'(b)|}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} L_{-2q}^{-2}(a,b), \tag{3.4}$$

where $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$ is the p-logarithmic mean.

Proof. From Lemma 1, Hölder's inequality and since $|f'|^q$ is the multiplicatively harmonically P-function on $[a,b]$,we have,

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)|f'(a)||f'(b)|}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}dt\right)^{\frac{1}{q}},$$

where an easy calculation gives

$$\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}dt = \frac{b^{-2q+1}-a^{-2q+1}}{(-2q+1)(b-a)},$$

which completes the proof.

4. SOME APPLICATIONS FOR SPECIAL MEANS

Let us recall the following special means of two nonnegative number a, b with $b > a$:

1. The arithmetic mean: $A = A(a,b) := \frac{a+b}{2}$.
2. The geometric mean: $G = G(a,b) := \sqrt{ab}$.
3. The harmonic mean: $H = H(a,b) := \frac{2ab}{a+b}$.
4. The Logarithmic mean $L = L(a,b) := \frac{b-a}{\ln b - \ln a}$.
5. The p-Logarithmic mean: $L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\}$.
6. The Identric mean: $I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$.

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature: $H \leq G \leq L \leq I \leq A$.

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 6 *Let* $1 \leq a < b$. *Then we have the following inequality*

$$A^{-1} \leq H.L^{-1} \leq G^2.L^{-1} \leq G^2.$$

Proof. The assertion follows from the inequality (3.1), for $f: [1,\infty) \to \mathbb{R},\ f(x) = x$.

Proposition 7 *Let* $1 \leq a < b$ *and* $q > 1$. *Then we have the following inequality*

$$\left|A\left(a^{1+1/q}, b^{1+1/q}\right) - G^2 L_{1/q-1}^{1/q-1}\right| \leq \frac{(q+1)(b-a)G^{2(1+1/q)}}{2q}\left[G^{-2} - \frac{4}{(b-a)^2}\ln\frac{A}{G}\right].$$

Proof. The assertion follows from the inequality (3.3) for $f: [1,\infty) \to \mathbb{R},\ f(x) = \frac{q}{q+1}x^{1+1/q}$.

Proposition 8 *Let* $0 < a < b$ *and* $q > 1$. *Then we have the following inequality*

$$\left|A\left(a^{1+1/q}, b^{1+1/q}\right) - G^2 L_{1/q-1}^{1/q-1}\right| \leq \frac{(q+1)(b-a)G^{2(1+1/q)}}{2q}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} L_{-2q}^{-2}(a,b).$$

Proof. The assertion follows from the inequality (3.4) for $f: [1,\infty) \to \mathbb{R},\ f(x) = f(x) = \frac{q}{q+1}x^{1+1/q}$.

Proposition 9 *Let* $0 < a < b$. *Then we have the following inequality* $H.L \leq 2G^2 \leq 2A.L$

Proof. The assertion follows from the inequality (3.2) for $f: (0,\infty) \to \mathbb{R},\ f(x) = e^x$.

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