Turk. J. Math. Comput. Sci.
14(1)(2022) 166–173
© MatDer
DOI : 10.47000/tjmcs.1010770

The Properties of Projective, Concircular and Conharmonic Curvature Tensor Fields on a Complex Sasakian Manifold

Aysel Turgut Vanli

Department of Mathematics, Faculty of Science, Gazi University, 06500, Ankara, Turkey.

Received: 16-10-2021 • Accepted: 25-05-2022

Abstract. In this article, the properties of projective, concircular and conharmonic curvature tensor fields on a complex Sasakian manifold are investigated.

2010 AMS Classification: 53C15, 53C25, 53D10

Keywords: Complex Sasakian manifolds, projective, concircular, conharmonic.

1. Introduction

Complex contact manifolds has an old history same as real contact manifolds but because of some difficulties and restrictions researchers could not give their attention to subject. The first work on complex contact manifolds was done by Kobayashi [6] in 1959. In 1980, Ishihara and Konishi [4,5] showed the presence of almost contact structure on complex contact manifolds, obtained the Hermitian metric associated with the contact structure, and gave the definition of complex almost contact metric manifold.

Although there are many articles on real Sasakian manifolds, studies on complex Sasakian manifolds were not available until 2000. In 2000, Foreman published his first work on complex Sasakian manifolds [3]. In 2006, Fetcu studied harmonic map between complex Sasakian manifolds [2]. In 2021, present author et al. define a complex Sasakian manifold by considering the real case, and curvature, Ricci curvature and $\mathcal{GH}-$sectional curvature of complex Sasakian manifolds are studied [9].

Properties of many tensors and curvatures on real Sasakian manifolds have been studied, and there are many articles in the literature on this subject. Complex Sasakian manifolds are very difficult to work with, the calculations are complex and long. For this reason, obtaining the properties of tensor fields and curvature tensor fields on a complex Sasakian manifold requires a very laborious work. The best known of the curvature tensor fields are the projective, concircular and conharmonic curvature tensor fields. So, these tensors are studied on a complex Sasakian manifold.

In this paper, firstly the basic concepts of complex Sasakian manifolds are given, then the definitions of projective, conharmonic and concircular curvature tensors on a complex Sasakian manifold are given, and the properties of these tensors are investigated.

Email address: avanli@gazi.edu.tr (A. Turgut Vanlı)

2. Preliminaries

For details of complex contact manifolds we recommend [1].

Definition 2.1. Let N be a complex manifold of odd complex dimension $2n+1$ covered by an open covering $C = \{\mathcal{A}_i\}$ consisting of coordinate neighborhoods. If there is a holomorphic 1-form η_i on each $\mathcal{A}_i \in C$ in such a way that for any $\mathcal{A}_i, \mathcal{A}_j \in C$ and for a holomorphic function f_{ij} on $\mathcal{A}_i \cap \mathcal{A}_j \neq \varnothing$

$$\eta_i \wedge (d\eta_j)^n \neq 0 \text{ in } \mathcal{A}_i,$$

$$\eta_i = f_{ij}\eta_j, \ \mathcal{A}_i \cap \mathcal{A}_j \neq \varnothing,$$

then N is called a complex contact manifold.

The complex contact structure determines a non-integrable distribution H_i by the equation $\eta_i = 0$ such as

$$H_i = \{W_p : \eta_i(W_p) = 0, W_p \in T_pN\}$$

and a holomorphic vector field ξ_i is defined by

$$\eta_i(\xi_i) = 1$$

and a complex line bundle is defined by $E_i = \mathrm{Span}\{\xi_i\}$.
Let $T^c(N)$ be complexified of tangent bundle of (N, J, η_i) and let define vector fields

$$U_i = \xi_i + \bar{\xi}_i, \qquad V_i = -i(\xi_i + \bar{\xi}_i)$$

and 1-forms

$$u_i = \frac{1}{2}(\eta_i + \bar{\eta}_i), \qquad v_i = \frac{1}{2}i(\eta_i - \bar{\eta}_i).$$

The complexified H_i and E_i is defined by

$$\begin{aligned} H_i^c &= \{W \in T^c(N) | u(W) = v(W) = 0\}, \\ E_i^c &= \mathrm{Span}\{U, V\}. \end{aligned}$$

We use notation $\mathcal{H}$ and $\mathcal{V}$ for the union of H_i^c and E_i^c, respectively. $\mathcal{H}$ is called horizontal distribution and $\mathcal{V}$ is called vertical distribution and we have

$$TN = \mathcal{H} \oplus \mathcal{V}.$$

Definition 2.2. Let N be a complex manifold of odd complex dimension $2n+1$ with complex structure J, Hermitian metric g and $C = \{\mathcal{A}_i\}$ be open covering of N with coordinate neighbourhoods $\{\mathcal{A}_i\}$. If N satisfies the following two conditions, then it is called a *complex almost contact metric manifold*:

(i) In each $\mathcal{A}_i$ there exist 1-forms u_i and $v_i = u_i \circ J$, with dual vector fields U_i and $V_i = -JU_i$ and $(1,1)$ tensor fields G_i and $H_i = G_iJ$ such that

$$H_i^2 = G_i^2 = -I + u_i \otimes U_i + v_i \otimes V_i,$$

$$G_iJ = -JG_i, \quad GU_i = 0,$$

$$g(W, G_iZ) = -g(G_iW, Z),$$

where W, Z are vector fields on N.

(ii) On $\mathcal{A}_i \cap \mathcal{A}_j \neq \varnothing$ we have

$$\begin{aligned} u_j &= au_i - bv_i, \quad v_j = bu_i + av_i, \\ G_j &= aG_i - bH_i, \quad H_j = bG_i + aH_i \end{aligned}$$

where a and b are functions on $\mathcal{A}_i \cap \mathcal{A}_j$ with $a^2 + b^2 = 1$ [4,6].

By direct computation we have

$$\begin{aligned}
H_iG_i &= -G_iH_i = J_i + u_i \otimes V_i - v_i \otimes U_i,\\
J_iH_i &= -H_iJ_i = G_i,\\
G_iU_i &= H_iU_i = H_iV_i = 0,\\
u_iG_i &= v_iG_i = u_iH_i = v_iH_i = 0,\\
J_iV_i &= U_i,\ g(U_i, V_i) = 0,\\
g(H_iW, Z) &= -g(W, H_iZ).
\end{aligned}$$

Define fundamental 2-form $\tilde{G}$ and $\tilde{H}$ on N by

$$\begin{aligned}
\tilde{G}(W, Z) &= du(W, Z) - (\sigma \wedge v)(W, Z),\\
\tilde{H}(W, Z) &= dv(W, Z) + (\sigma \wedge u)(W, Z),
\end{aligned}$$

where $\sigma(W) = g(\nabla_W U, V)$, ∇ being the Levi-Civita connection of g [4].

Ishihara and Konishi [4] defined local tensors

$$\begin{aligned}
S(W, K) &= [G, G](W, K) + 2g(W, GK)U - 2g(W, HK)V\\
&\quad +2(v(K)HW - v(W)HK) + \sigma(GK)HW\\
&\quad -\sigma(GW)HK + \sigma(W)GHK - \sigma(K)GHW,
\end{aligned}$$

$$\begin{aligned}
T(W, K) &= [H, H](W, K) - 2g(W, GK)U + 2g(W, HK)V\\
&\quad +2(u(K)GW - u(W)GK) + \sigma(HW)GK\\
&\quad -\sigma(HK)GW + \sigma(W)GHK - \sigma(K)GHW,
\end{aligned}$$

where

$$[G, G](W, K) = (\nabla_{GW}G)K - (\nabla_{GK}G)W - G(\nabla_W G)K + G(\nabla_K G)W$$

is the Nijenhuis torsion of G, and K is vector fields on N.

Definition 2.3. A complex almost contact metric manifold is called IK-Normal if $S = T = 0$.

Korkmaz gave a weaker definition for normality in [7].

Definition 2.4. A complex almost contact metric manifold is called normal if
$S(W, Z) = T(W, Z) = 0$ for all W, Z in $\mathcal{H}$ and $S(W, U) = T(W, V) = 0$ for all W.

Complex Sasakian manifold are normal due to Korkmaz's definition.

Definition 2.5. Let $(N, G, H, J, U, V, u, v, g)$ be a normal complex contact metric manifold. If fundemental 2- forms $\widetilde{G}$ and $\widetilde{H}$ is defined by

$$\widetilde{G}(W, Z) = du(W, Z) \text{ and } \widetilde{H}(W, Z) = dv(W, Z),$$

then N is called complex Sasakian manifold [9].

Thus, we get following result;

Theorem 2.6. *[9] A normal complex contact metric manifold N is complex Sasakian if and only if*

$$(\nabla_W G)Z = -2v(W)HGZ - u(Z)W - v(Z)JW + g(W, Z)U + g(JW, Z)V,$$

$$(\nabla_W H)Z = -2u(W)HGZ + u(Z)JW - v(Z)W - g(JW, Z)U + g(W, Z)V.$$

On a complex Sasakian manifold, we get

$$\nabla_W U = -GW, \quad \nabla_W V = -HW.$$

In [9], the properties of the curvature tensor of a complex Sasakian manifold are studied. The properties of the curvature tensor of a complex Sasakian manifold are as follows

$$\begin{aligned}
R(U,V)V &= R(V,U)U = 0,\\
R(W,U)U &= W + u(W)U + v(W)V, \qquad (2.1)\\
R(W,V)V &= W - u(W)U - v(W)V,\\
R(W,U)V &= -3JW - 3u(W)V + 3v(W)U,\\
R(W,V)U &= 0,\\
R(W,Z)U &= v(W)JZ - v(Z)JW + 2v(W)u(Z)V - 2v(Z)u(W)V \qquad (2.2)\\
&\quad + u(Z)W - u(W)Z - 2g(JW,Z)V,\\
R(W,Z)V &= 3u(W)JZ - 3u(Z)JW - 2u(W)v(Z)U + 2u(Z)v(W)U\\
&\quad + v(Z)W - v(W)Z + 2g(JW,Z)U,\\
R(U,V)W &= JW + u(W)V - v(W)U,\\
R(W,U)Z &= -2v(Z)v(W)U + 2u(Z)v(W)V - g(Z,W)U \qquad (2.3)\\
&\quad + u(Z)W + g(JZ,W)V,\\
R(W,V)Z &= 3u(Z)JW + 2u(Z)u(W)V + 3g(JZ,W)U\\
&\quad - 2v(Z)u(W)U - g(Z,W)V + v(Z)W - 2u(W)JZ,
\end{aligned}$$

where $W,Z \in \Gamma(TN)$. Present author and Unal [8] presented properties of Ricci curvature tensor of a normal complex contact metric manifold. For complex Sasakian case we have the following relations;

$$\begin{aligned}
\rho(U,U) &= \rho(V,V) = 4n, \quad \rho(U,V) = 0, \qquad (2.4)\\
\rho(W,U) &= 4nu(W), \quad \rho(W,V) = 4nv(W), \qquad (2.5)\\
\rho(W,Z) &= \rho(GW,GZ) + 4n\left(u(W)u(Z) + v(W)v(Z)\right),\\
\rho(W,Z) &= \rho(HW,HZ) + 4n\left(u(W)u(Z) + v(W)v(Z)\right),
\end{aligned}$$

where $W,Z \in \Gamma(TN)$.

3. The Properties of Projective, Concircular and Conharmonic Curvature Tensor Fields on a Complex Sasakian Manifold

In this section, we will investigate the properties of projective concircular and conharmanic curvature tensors, which are important curvature tensor fields, on a complex Sasakian manifold. First, the definition of the projective curvature tensor field of a complex Sasakian manifold is given.

Definition 3.1. The projective curvature tensor is defined as follow on a complex Sasakian manifold N;

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{4n+1}[\rho(Y,Z)X - \rho(X,Z)Y], \qquad (3.1)$$

where $X,Y,Z \in \Gamma(TN)$.

The properties we get from the definition of the projective curvature tensor field are expressed in the following theorem.

Theorem 3.2. *The projective curvature tensor field of a complex Sasakian manifold N provides the following properties:*

$$\begin{aligned}
\mathcal{P}(U,W)U &= -\frac{1}{4n+1}W,\\
\mathcal{P}(Y,W)U &= -2g(JY,W)V,\\
\mathcal{P}(QY,W)U &= -2g(JQY,W)V,\\
\mathcal{P}(U,V)U &= \frac{4n}{4n+1}V,\\
\mathcal{P}(U,U)U &= 0,\\
\mathcal{P}(U,QU)U &= -\frac{1}{4n+1}QU - \frac{1}{4n+1}\rho(QU,U)U,\\
\mathcal{P}(U,W)Y &= -g(JY,W)V + g(Y,W)U - \frac{1}{4n+1}\rho(W,Y)U,\\
\mathcal{P}(U,W)QY &= -g(JQY,W)V + \rho(Y,W)U - \frac{1}{4n+1}\rho(W,QY)U,
\end{aligned}$$

where Q is Ricci operator of N, U,V vertical and W,Y horizontal vector fields on N.

Proof. If we take $X = U$, $Y = W$ and $Z = U$ in eq. (3.1) we have

$$\mathcal{P}(U,W)U = R(U,W)U - \frac{1}{(4n+1)}[\rho(W,U)U - \rho(U,U)W].$$

From eq. (2.4) and (2.5), we get

$$\mathcal{P}(U,W)U = -R(W,U)U + \frac{4n}{4n+1}W.$$

Using eq. (2.1), we have

$$\mathcal{P}(U,W)U = -W + \frac{4n}{4n+1}W = -\frac{1}{4n+1}W.$$

Similarly, by taking $X = Y$, $Y = W$ and $Z = U$ in eq. (3.1) we have

$$\mathcal{P}(Y,W)U = R(Y,W)U - \frac{1}{(4n+1)}[\rho(W,U)Y - \rho(Y,U)W].$$

From eq. (2.2) and (2.5) we get

$$\mathcal{P}(Y,W)U = -2g(JY,W)V.$$

On the other hand, by taking $X = QY$, $Y = W$ and $Z = U$ in eq. (3.1) we get

$$\mathcal{P}(QY,W)U = R(QY,W)U - \frac{1}{(4n+1)}[\rho(W,U)QY - \rho(QY,U)W].$$

From eq. (2.2) and (2.5) we have

$$\mathcal{P}(QY,W)U = -2g(JQY,W)V.$$

We choose $X = U$, $Y = V$ and $Z = U$ in eq. (3.1). Then,

$$\mathcal{P}(U,V)U = R(U,V)U - \frac{1}{(4n+1)}[\rho(V,U)U - \rho(U,U)V].$$

From eq. (2.1) and (2.4) we get

$$\mathcal{P}(U,V)U = \frac{4n}{4n+1}V.$$

We choose $X = U$, $Y = U$ and $Z = U$ in eq. (3.1), we have

$$\mathcal{P}(U,U)U = R(U,U)U - \frac{1}{(4n+1)}[\rho(U,U)U - \rho(U,U)U],$$

and from eq. (2.1) and (2.4)

$$\mathcal{P}(U,U)U = 0.$$

We choose $X = U$, $Y = QU$ and $Z = U$ in eq. (3.1). Then,

$$\mathcal{P}(U,QU)U = R(U,QU)U - \frac{1}{(4n+1)}[\rho(QU,U)U - \rho(U,U)QU],$$

and from eq. (2.1) and (2.4)

$$\mathcal{P}(U,QU)U = -\frac{1}{4n+1}QU - \frac{1}{4n+1}\rho(QU,U)U.$$

We take $X = U$, $Y = W$ ve $Z = Y$ in eq. (3.1). So, we get

$$\mathcal{P}(U,W)Y = R(U,W)Y - \frac{1}{(4n+1)}[\rho(W,Y)U - \rho(U,Y)W],$$

and from eq. (2.3) and (2.5) we have

$$\mathcal{P}(U,W)Y = -g(JY,W)V + g(Y,W)U - \frac{1}{4n+1}\rho(W,Y)U.$$

We take $X = U$, $Y = W$ ve $Z = QY$ in eq. (3.1). Then, we obtain

$$\mathcal{P}(U,W)QY = R(U,W)QY - \frac{1}{(4n+1)}[\rho(W,QY)U - \rho(U,QY)W],$$

and from eq. (2.3) and (2.5)

$$\mathcal{P}(U,W)QY = -g(JQY,W)V + \rho(Y,W)U - \frac{1}{4n+1}\rho(W,QY)U.$$ □

Definition 3.3. The concircular curvature tensor is defined as follow on a complex Sasakian manifold

$$\mathcal{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{(4n+2)(4n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.2}$$

where $X, Y, Z \in \Gamma(TN)$ and τ is scalar curvature of N.

Theorem 3.4. *The concircular curvature tensor field of M provides the following properties:*

$$\mathcal{Z}(U,Y)W = -g(JW,Y)V + \frac{(4n+2)(4n+1)-\tau}{(4n+2)(4n+1)}g(Y,W)U,$$

$$\mathcal{Z}(U,Y)U = \frac{\tau-(4n+2)(4n+1)}{(4n+2)(4n+1)}Y, \tag{3.3}$$

$$\mathcal{Z}(U,Y)QW = -g(JQW,Y)V + \rho(W,Y)U - \frac{\tau}{(4n+2)(4n+1)}\rho(Y,W)U, \tag{3.4}$$

where U, V vertical and W, Y horizontal vector fields on N.

Proof. If we take $X = U$, $Y = W$ and $Z = W$ in eq. (3.2), and using eq. (2.3), we have

$$\begin{aligned}\mathcal{Z}(U,Y)W &= R(U,Y)W - \frac{\tau}{(4n+2)(4n+1)}[g(Y,W)U - g(U,W)Y]\\ &= -g(JW,Y)V + g(W,Y)U - \frac{\tau}{(4n+2)(4n+1)}g(Y,W)U\\ &= -g(JW,Y)V + \frac{(4n+2)(4n+1)-\tau}{(4n+2)(4n+1)}g(Y,W)U.\end{aligned}$$

In order to find eq. (3.3), write $X = U$, $Z = U$ in eq. (3.2) and replacing eq. (2.1) to get

$$\begin{aligned}\mathcal{Z}(U,Y)U &= R(U,Y)U - \frac{\tau}{(4n+2)(4n+1)}[g(Y,U)U - g(U,U)Y]\\ &= \frac{\tau-(4n+2)(4n+1)}{(4n+2)(4n+1)}Y.\end{aligned}$$

To find eq. (3.4), write $X = U$, $Z = QW$ in eq. (3.2), and using eq. (2.3) we have

$$\begin{aligned}\mathcal{Z}(U,Y)QW &= R(U,Y)QW - \frac{\tau}{(4n+2)(4n+1)}[g(Y,QW)U - g(U,QW)Y]\\ &= -g(JQW,Y)V + \rho(W,Y)U - \frac{\tau}{(4n+2)(4n+1)}\rho(Y,W)U.\end{aligned}$$ □

Definition 3.5. The conharmonic curvature tensor is defined as follow on a complex Sasakian manifold

$$\mathcal{K}(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[\rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{3.5}$$

where $X, Y, Z \in \Gamma(TN)$.

Theorem 3.6. *The conharmonic curvature tensor field of M provides the following properties:*

$$\mathcal{K}(U,W)U = -\frac{QW}{4n},$$

$$\mathcal{K}(U,U)U = 0,$$

$$\mathcal{K}(U,QU)U = -QU - \frac{1}{4n}\rho(QU,U)U + \frac{1}{4n}Q(QU), \tag{3.6}$$

$$\mathcal{K}(U,V)U = V + \frac{QV}{4n}, \tag{3.7}$$

$$\mathcal{K}(Y,W)U = -2g(JY,W)V, \tag{3.8}$$

$$\mathcal{K}(U,W)Y = -g(JY,W)V + g(Y,W)U - \frac{1}{4n}[\rho(W,Y)U + g(W,Y)QU],$$

$$\mathcal{K}(U,Y)U = \frac{1}{4n}QY, \tag{3.9}$$

$$\mathcal{K}(QY,W)U = -2g(JQY,W)V, \tag{3.10}$$

$$\begin{aligned}\mathcal{K}(U,W)QY = -g(JQY,W)V + \rho(Y,W)U - \frac{1}{4n}[\rho(W,QY)U\\ + \rho(U,QY)W + \rho(W,Y)QU],\end{aligned} \tag{3.11}$$

where U, V vertical and W, Y horizontal vector fields on N.

Proof. If we take $X = U$, $Y = W$ and $Z = U$ in eq. (3.5), then we have

$$\mathcal{K}(U,W)U = R(U,W)U - \frac{1}{4n}[\rho(W,U)U - \rho(U,U)W + g(W,U)QU - g(U,U)QW].$$

From eq. (2.1),(2.4), and (2.5) we get

$$\mathcal{K}(U,W)U = -W - \frac{1}{4n}[-4nW + QW] = -\frac{QW}{4n}.$$

We take $X = U$, $Y = U$ and $Z = U$ in eq. (3.5). Thus, we have

$$\mathcal{K}(U,U)U = R(U,U)U - \frac{1}{4n}[\rho(U,U)U - \rho(U,U)U + g(U,U)QU - g(U,U)QU].$$

Using eq. (2.1) and (2.4) we get

$$\mathcal{K}(U,U)U = 0.$$

If $X = U$, $Y = W$ and $Z = U$ are written in eq. (3.5) and eq. (2.1) and (2.4) are used, eq. (3.6) is obtained.
If $X = U$, $Y = V$ and $Z = U$ are written in eq. (3.5) and eq. (2.1) and (2.4) are used, eq. (3.7) is obtained.
If $X = Y$, $Y = W$ and $Z = U$ are written in eq. (3.5) and eq. (2.2) and (2.5) are used, eq. (3.8) is obtained.
If we take $X = U$, $Y = W$ and $Z = Y$ in eq. (3.5), then we have

$$\begin{aligned}\mathcal{K}(U,W)Y = R(U,W)Y - \frac{1}{4n}[\rho(W,Y)U - \rho(U,Y)W\\ + g(W,Y)QU - g(U,Y)QW].\end{aligned}$$

From eq. (2.3) and (2.5), we get

$$\mathcal{K}(U,W)Y = -g(JY,W)V + g(Y,W)U - \frac{1}{4n}[\rho(W,Y)U + g(W,Y)QU].$$

If $X = Y$ and $Z = U$ are written in eq. (3.5) and eq. (2.1), (2.5), (2.4) are used, eq. (3.9) is obtained.
If $X = U$ and $Z = U$ are written in eq. (3.5) and eq. (2.1), (2.5), (2.4) are used, eq. (3.10) is obtained.
If $X = U$, $Y = W$ and $Z = QY$ are written in eq. (3.5) and eq. (2.3) and (2.5) are used, eq. (3.11) is obtained. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

References

[1] Blair, D.E., Riemannian Geometry of Contact and Symplectic Manifolds, 2nd edn., Birkhäuser, Boston, 2010.
[2] Fetcu, D., *Harmonic maps between complex Sasakian manifolds*, Rend. Semin. Mat. Univ. Politec. Torino, **64**(3)(2006), 319–329.
[3] Foreman, B., *Complex contact manifolds and hyperkähler geometry*, Kodai Math. J., **23**(1)(2000), 12–26.
[4] Ishihara, S., Konishi, M., *Complex almost contact manifolds*, Kodai Math. J., **3**(1980), 385–396.
[5] Ishihara, S., Konishi, M., *Complex almost contact structures in a complex contact manifold*, Kodai Math. J., **5**(1982), 30–37.
[6] Kobayashi, S.,*Remarks on complex contact manifolds*, Proc. Amer. Math. Soc., **10**(1959), 164–167.
[7] Korkmaz, B., *Normality of complex contact manifolds*, Rocky Mountain J. Math., **30**(2000), 1343–1380.
[8] Turgut Vanli, A., Unal, I., *Ricci semi-symmetric normal complex contact metric manifolds*, Italian Journal of Pure and Applied Mathematics, **43**(2020), 477–491.
[9] Turgut Vanli, A., Unal, I., Avcu, K., *On complex Sasakian manifold*, Afrika Matematika, **32**(2021), 507–516.