Sigma J Eng & Nat Sci 37 (1), 2019, 119-128

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

THE GALERKIN FINITE ELEMENT METHOD FOR ADVECTION DIFFUSION EQUATION

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Received: 08.01.2018 Revised: 12.06.2018 Accepted: 15.07.2018

ABSTRACT

Cubic B-spline Galerkin method, based on second and fourth order single step methods for time integration is used to solve numerically the advection diffusion equation (ADE). Second order single step method is also known as Crank Nicolson method. Two numerical examples are used to validate the proposed method which is found to be accurate and efficient. The effects of the advection and diffusion terms on the solution domain and the absolute error of the numerical solution are studied with the help of graphs. The obtained results show that the proposed fourth order single step method has a high success as a numerical technique for solving the ADE.
Keywords: Cubic B-spline, Galerkin method, fourth order single step method, Crank Nicolson method, advection-diffusion equation.

1. INTRODUCTION

ADE (sometimes called the convection-diffusion equation) is a one dimensional parabolic partial differential equation which describes the process of transport in a medium. Because of the growing surface and subsurface hydro-environment degradation and the air pollution, ADE has been studied especially by civil engineers, mathematical modelers and hydrologists as well as soil physicists, petroleum engineers, chemical engineers and bio-scientists [12]. Numerical methods have been used for many years for numerical solutions of various partial differential equations [7,8,17-19]. In addition to the existence of analytical solutions of ADE, many numerical solutions have been obtained by the finite difference approximations [3,4,13,16], the least-squares B-spline method [1], the differential quadrature method [9-11], Taylor-Galerkin method [2], linear B-spline collocation method [5], cubic B-spline collocation method [4], extended cubic B-spline collocation method [6], trigonometric cubic B-spline collocation method [14].

Since the ADE is simple equation modelling advection and diffusion process, many scientists have proposed new numerical methods for the ADE. In this study, we present another new numerical method which has high order accurate in time. In section 2, firstly the time discretization of ADE is applied by using higher accuracy finite difference method, then a system of algebraic equation is obtained by using a finite element space discretization. In the numerical

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experiment section, proposed methods are tested for two tests and in the last section, a summary of the study is given.

We consider the following one dimensional ADE

$$u_t + \alpha u_x - \mu u_{xx} = 0,\ a \le x \le b \tag{1}$$

with the boundary and initial conditions

$$\begin{matrix} u(a,t) = u(b,t) = 0 \\ u_x(a,t) = u_x(b,t) = 0 \end{matrix} \quad ,t \in [0,T] \tag{2}$$

$$u(x,0) = f(x),\ a \le x \le b \tag{3}$$

in a restricted solution domain over an space/time interval $[a,b]\times[0,T]$. In Eq. (1), α and μ denote the steady uniform fluid velocity and the constant diffusion coefficient, respectively.

2. APPLICATION OF THE METHOD

The solution domain is discretized by grid with the time step Δt and space step h. The exact solution of ADE at the grid points is denoted by

$$u(x_k, t_n) = u_k^n,\ \mathrm{k} = 0,1,\ldots,N; \quad n = 0,1,2,\ldots$$

where $x_k = a + kh$, $t_n = n\Delta t$. The numerical value of u_k^n is denoted by U_k^n.

2.1. Time Discretization

Consider the advection diffusion equation of the form

$$u_t = \mu u_{xx} - \alpha u_x \tag{4}$$

and the following one-step method

$$u^{n+1} = u^n + \theta_1 u_t^{n+1} + \theta_2 u_t^n + \theta_3 u_{tt}^{n+1} + \theta_4^n u_{tt}^n. \tag{5}$$

If we take $\theta_1 = \theta_2 = \Delta t/2$, $\theta_3 = \theta_4 = 0$ in (5), the method is of order 2 (M1) known as Crank-Nicolson method (CN method) and then if we take $\theta_1 = \theta_2 = \Delta t/2$, $\theta_3 = -(\Delta t)^2/12$, $\theta_4 = (\Delta t)^2/12$ the method is of order 4 (M2). Using the (5) and ADE of the form (4), we obtain the following equation

$$\begin{aligned} &u^{n+1} - \theta_1\left(\mu u_{xx} - \alpha u_x\right)^{n+1} - \theta_3\left(\mu^2 u_{xxxx} - 2\alpha\mu u_{xxx} + \alpha^2 u_{xx}\right)^{n+1} \\ &= u^n + \theta_2\left(\mu u_{xx} - \alpha u_x\right)^n + \theta_4\left(\mu^2 u_{xxxx} - 2\alpha\mu u_{xxx} + \alpha u_{xx}\right)^n \end{aligned} \tag{6}$$

for the time discretization of the Eq. (1). Note that order of the proposed method M2 is higher than well known Crank-Nicolson method.

2.2. Space Discretization

Space domain $[a,b]$ of the problem is divided into N equal sized elements as

$$a = x_0 < x_1 < \ldots < x_N = b$$

where $h = x_i - x_{i-1}$, $i = 1,\ldots,N$. Then, the cubic B-splines φ_k, $k = -1,\ldots,N+1$ have the following form:

$$\varphi_k(x) = \frac{1}{h^3}\begin{cases} g\left(x_{k-2}\right)^3 & , x_{k-2} \le x \le x_{k-1} \\ h^3 + 3h^2 g\left(x_{k-1}\right) + 3hg\left(x_{k-1}\right)^2 - 3g\left(x_{k-1}\right)^3 & , x_{k-1} \le x \le x_k \\ h^3 - 3h^2 g\left(x_{k+1}\right) + 3hg\left(x_{k+1}\right)^2 + 3g\left(x_{k+1}\right)^3 & , x_k \le x \le x_{k+1} \\ -g\left(x_{k+2}\right)^3 & , x_{k+1} \le x \le x_{k+2} \\ 0 & , \text{otherwise} \end{cases} \tag{7}$$

where $g(x_k) = x - x_k$.

The set of cubic B-splines $\varphi_k(x)$ forms a basis over the space interval $a \le x \le b$ [13]. The approximate solution $U(x,t)$ can be written as

$$u\left(x,t\right) \approx U(x,t) = \sum_{j=-1}^{N+1} \delta_j(t)\varphi_j(x). \tag{8}$$

Since function (7) and its first two derivatives are continuous, trial solutions (8) have continuity up to second order. Using Eqs. (7) and (8), approximation U, U' and U'' at the knots can be computed as

$$\begin{aligned} U_k &= U(x_k) = \delta_{k-1} + 4\delta_k + \delta_{k+1}, \\ U_k' &= U'(x_k) = \frac{3}{h}\left(\delta_{k+1} - \delta_{k-1}\right), \\ U_k'' &= U''(x_k) = \frac{6}{h^2}\left(\delta_{k-1} - 2\delta_k + \delta_{k+1}\right). \end{aligned} \tag{9}$$

Using transformation $\xi = x - x_k$, $x_k \le x \le x_{k+1}$, the cubic B-spline shape functions in terms of ξ over the element $\left[0,h\right]$ can be written by

$$\begin{aligned} \varphi_{k-1}(\xi) &= \left(1 - \frac{\xi}{h}\right)^3, \\ \varphi_k(\xi) &= 4 - 6\frac{\xi^2}{h^2} + 3\frac{\xi^3}{h^3}, \\ \varphi_{k+1}(\xi) &= 1 + 3\frac{\xi}{h} + 3\frac{\xi^2}{h^2} - 3\frac{\xi^3}{h^3}, \\ \varphi_{k+2}(\xi) &= \frac{\xi^3}{h^3}. \end{aligned} \tag{10}$$

Combination of the element shape functions φ_j together with element time parameters δ_j, $j = k-1,\ldots,k+1$ gives an approximation for the typical element $\left[0,h\right]$

$$U^e = U(\xi,t) = \sum_{j=k-1}^{k+2} \delta_j(t)\varphi_j\left(\xi\right). \tag{11}$$

Applying Galerkin method to Eq. (6) with weight function $W(x)$ leads to the equation:

$$\begin{aligned}&\int_a^b W(x)\Big(u^{n+1}-\theta_1\left(\mu u_{xx}-\alpha u_x\right)^{n+1}-\theta_3\left(\mu^2 u_{xxxx}-2\alpha\mu u_{xxx}+\alpha^2 u_{xx}\right)^{n+1}\Big)dx\\&=\int_a^b W(x)\Big(u^{n}+\theta_2\left(\mu u_{xx}-\alpha u_x\right)^{n}+\theta_4\left(\mu^2 u_{xxxx}-2\alpha\mu u_{xxx}+\alpha u_{xx}\right)^{n}\Big)dx.\end{aligned} \tag{12}$$

If we integrate by parts and use boundary conditions, we find that:

$$\begin{aligned}&\int_a^b\Big(W(x)u^{n+1}-\theta_1 W(x)\left(\mu u_{xx}-\alpha u_x\right)^{n+1}-\theta_3\left(\mu^2 W''(x)u_{xx}+2\alpha\mu W'(x)u_{xx}+\alpha^2 W(x)u_{xx}\right)^{n+1}\Big)dx\\&=\int_a^b\Big(W(x)u^{n}+\theta_2 W(x)\left(\mu u_{xx}-\alpha u_x\right)^{n}+\theta_4\left(\mu^2 W''(x)u_{xx}+2\alpha\mu W'(x)u_{xx}+\alpha W(x)u_{xx}\right)^{n}\Big)dx.\end{aligned} \tag{13}$$

The weight function $W(x)$ and exact solution in (13) are replaced with cubic B-splines shape functions (10) and approximation given by (11), respectively.

Thus we obtain a fully discrete approximation is obtained over the element $[0,h]$ as

$$\begin{aligned}&\sum_{j=k-1}^{k+2}\left[\int_0^h\Big(\varphi_i\varphi_j-\theta_1\varphi_i\left(\mu\varphi_j''-\alpha\varphi_j'\right)-\theta_3\left(\mu^2\varphi_i''\varphi_j''+2\alpha\mu\varphi_i'\varphi_j''+\alpha^2\varphi_i\varphi_j''\right)\Big)d\xi\right]\delta_j^{n+1}\\&=\sum_{j=k-1}^{k+2}\left[\int_0^h\Big(\varphi_i\varphi_j+\theta_2\varphi_i\left(\mu\varphi_j''-\alpha\varphi_j'\right)+\theta_4\left(\mu^2\varphi_i''\varphi_j''+2\alpha\mu\varphi_i'\varphi_j''+\alpha\varphi_i\varphi_j''\right)\Big)d\xi\right]\delta_j^{n}\end{aligned} \tag{14}$$

where i and j take only the values $k-1,\ldots,k+2$ and $k=0,1,\ldots,N-1$ for the typical element $[0,h]$.

(14) can be written matrix form as

$$\begin{aligned}&\left[\boldsymbol{A}^e-\theta_1\left(\mu\boldsymbol{B}^e-\alpha\mathbf{C}^e\right)-\theta_3\left(\mu^2\mathbf{D}^e+2\alpha\mu\boldsymbol{E}^e+\alpha^2\boldsymbol{B}^e\right)\right]\left(\boldsymbol{\delta}^{n+1}\right)^e\\&=\left[\boldsymbol{A}^e+\theta_2\left(\mu\boldsymbol{B}^e-\alpha\boldsymbol{C}^e\right)+\theta_4\left(\mu^2\boldsymbol{D}^e+2\alpha\mu\boldsymbol{E}^e+\alpha^2\boldsymbol{B}^e\right)\right]\left(\boldsymbol{\delta}^{n}\right)^e\end{aligned} \tag{15}$$

where

$$\begin{aligned}&A_{ij}^e=\int_0^h\varphi_i\varphi_j d\xi,\quad B_{ij}^e=\int_0^h\varphi_i\varphi_j'' d\xi,\\&C_{ij}^e=\int_0^h\varphi_i\varphi_j' d\xi,\quad D_{ij}^e=\int_0^h\varphi_i''\varphi_j'' d\xi,\\&E_{ij}^e=\int_0^h\varphi_i'\varphi_j'' d\xi,\quad \left(\delta_j^{n+1}\right)^e=\left(\delta_{k-1},\delta_k,\delta_{k+1},\delta_{k+2}\right)^T.\end{aligned}$$

Assembling contributions from all elements, (15) leads to the following linear system for the time evolution of $\boldsymbol{\delta}$:

$$\begin{aligned}&\left[\boldsymbol{A}-\theta_1\left(\mu\boldsymbol{B}-\alpha\boldsymbol{C}\right)-\theta_3\left(\mu^2\boldsymbol{D}+2\alpha\mu\boldsymbol{E}+\alpha^2\boldsymbol{B}\right)\right]\boldsymbol{\delta}_j^{n+1}\\&=\left[\boldsymbol{A}+\theta_2\left(\mu\boldsymbol{B}-\alpha\mathbf{C}\right)+\theta_4\left(\mu^2\boldsymbol{D}+2\alpha\mu\boldsymbol{E}+\alpha^2\mathbf{B}\right)\right]\boldsymbol{\delta}_j^{n}\end{aligned} \tag{16}$$

The linear system (16) consists of $N+3$ linear equations in $N+3$ unknowns $\left(\delta_{-1}^{n+1},\ldots,\delta_{N+1}^{n+1}\right)$. After the first and last equations are deleted in the system (16), imposition of the boundary conditions $U(a,x)=U(b,x)=0$ at the both ends of the region yields to eliminate δ_{-1}^{n+1} and δ_{N+1}^{n+1} from the above system. Therefore the solution of the linear system with the dimensions

$\left(N+1\right)\times\left(N+1\right)$ is obtained by way of Gauss elimination algorithms. To carry on the iteration of the system (16), the initial parameters $\delta_{-1}^{0},\ldots,\delta_{N+1}^{0}$ must be obtained from the initial condition and the derivatives of the boundary conditions at both ends:

$$U(x_k,0)=\delta_{k+1}^{0}+4\delta_{k}^{0}+\delta_{k-1}^{0},\ \mathrm{k}=0,\ldots,N,$$
$$U'(x_0,0)=U'(x_N,0)=0.$$

3. TEST PROBLEMS

For the test problems, accuracy of the proposed two algorithms is worked out by measuring error norm L_∞

$$L_\infty=\max_m\left|u_m-U_m\right|, \tag{17}$$

and the order of convergence is computed by the formula

$$\text{order}=\frac{\log\left|\left(L_\infty\right)_{h_i}/\left(L_\infty\right)_{h_{i+1}}\right|}{\log\left|h_i/h_{i+1}\right|}, \tag{18}$$

where $\left(L_\infty\right)_{h_i}$ is the error norm L_∞ for space step h_i.

3.1. First Test Problem

By choosing $\mu=0$ in ADE, the pure advection equation has the exact solution

$$u(x,t)=10\exp\left(-\frac{(x-\tilde{x}_0-\alpha t)^2}{2\rho^2}\right). \tag{19}$$

The numerical simulation is accomplished with flow velocity $\alpha=0.5m/s$, initial peak location $\tilde{x}_0=2km$ and $\rho=264$ by the terminating time $t=9600s$. After the program run up to time $t=9600s$, initial solution and waves are depicted in Figure 1 for the M2 with $h=\Delta t=1$. It can be seen from the figure that the wave keeps its initial profile while propagating without any change in its shape with speed $\alpha=0.5m/s$. Therefore the initial condition $u(x,0)$ is spread out in a long channel with no change in shape or size by the time because of the advection effect. Thus, the initial state travels at a distance of $4.8km$ from the initial position, and the peak value of the solution remains constant 10 over time.

Absolute error distribution at time $t=9600s$ is also depicted in Figure 2. Since the maximum error appears at about peak value of the wave at time $t=9600s$, the effect of boundary conditions can be ignored.

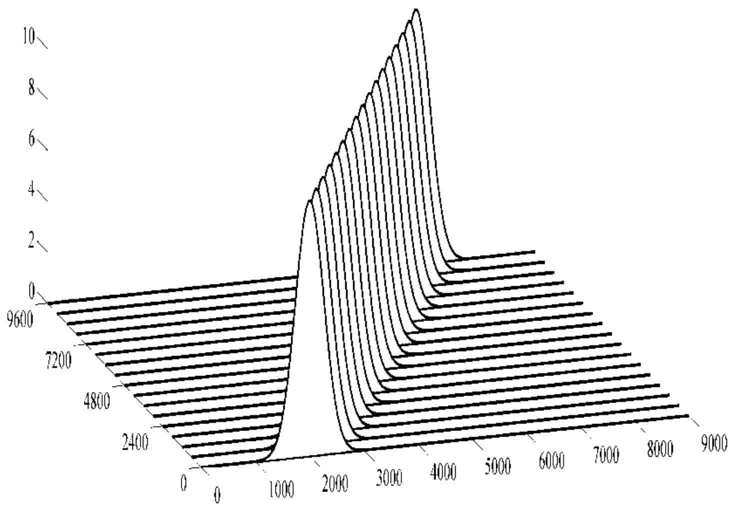

Figure 1. Wave profiles.

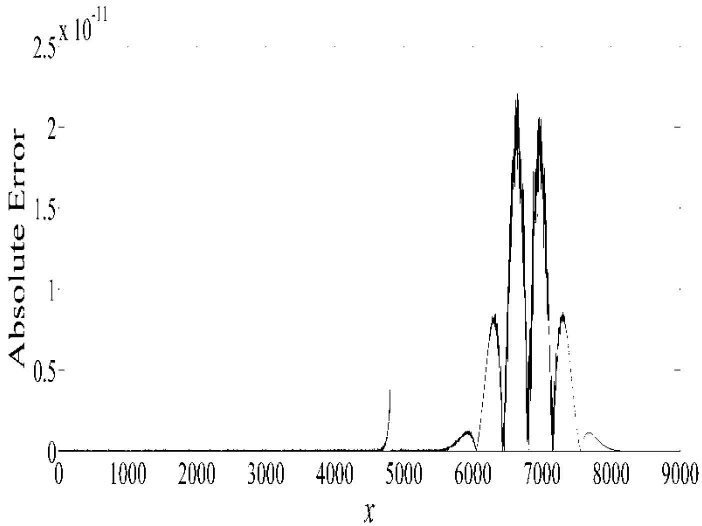

Figure 2. Absolute error for M2 with $h = \Delta t = 1$

The error norms L_∞ and order of convergence are listed in Table 1. According to the Table 1, when time and space steps are reduced from 200 to 1, the error norms decrease for the both algorithms. It can also be seen that the order of convergence is almost two for M1 and almost four for M2. Therefore, the proposed methods especially M2 are quite satisfactory. To compare with the other studies, the error norms L_∞ of the proposed methods are given together with the error norms of the least-squares and the extended cubic B-spline collocation methods in Table 2. It can be seen from Table 2, M2 has more accurate results than the other methods.

Table 1. Error norms L_∞ and rate of convergence at time $t = 9600s$ with $0 \le x \le 9000$ for M1 and M2.

	M1		M2	
$h = \Delta t$	L_∞	Order	L_∞	Order
200	2.32	1.66	1.05×10^{-1}	5.87
100	7.34×10^{-1}	1.95	1.88×10^{-3}	4.01
50	1.90×10^{-1}	2.00	1.17×10^{-4}	4.00
20	3.01×10^{-2}	2.00	3.00×10^{-6}	4.00
10	7.51×10^{-3}	2.00	1.88×10^{-7}	4.00
5	1.88×10^{-3}	2.00	1.17×10^{-8}	4.00
2	3.00×10^{-4}	2.00	3.01×10^{-10}	3.79
1	7.50×10^{-5}		2.21×10^{-11}	

Table 2. Error norms L_∞ at time $t = 9600s$ with $0 \le x \le 9000$.

$h = \Delta t$	M1	M2	[6]	[1]
200	2.32	1.05×10^{-1}	1.29	5.18×10^{-1}
100	7.34×10^{-1}	1.88×10^{-3}	3.25×10^{-1}	3.76×10^{-1}
50	1.90×10^{-1}	1.17×10^{-4}	1.98×10^{-1}	3.73×10^{-1}
10	7.51×10^{-3}	1.88×10^{-7}	7.51×10^{-3}	
1	7.50×10^{-5}	2.21×10^{-11}	7.50×10^{-5}	

3.2. Second Test Problem

The exact solution of ADE is

$$u(x,t) = \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{\left(x-\tilde{x}_0-\alpha t\right)^2}{\mu\left(4t+1\right)}\right) \tag{20}$$

modelling fade out of an initial bell shaped concentration of height 1. This solution corresponds to a wave of magnitude $1/\sqrt{4t+1}$, initially centered on the position $\tilde{x}_0$ propagating towards the right across the interval $[a,b]$ over the up to the time T with a steady velocity α. After the program run up to time $t = 5$ with $\alpha = 0.8m/s$, $\mu = 0.005m^2/s$ and $0 \le x \le 9$, error norms L_∞ and order of convergence for both methods are listed in Table 3.

Table 3. Error norms L_∞ and rate of convergence at time $t = 5$ with $0 \le x \le 9$ for M1 and M2.

	M1		M2	
$h = \Delta t$	L_∞	Order	L_∞	Order
0.1	5.36×10^{-2}	1.93	4.11×10^{-3}	7.81
0.05	1.41×10^{-2}	2.05	2.83×10^{-5}	4.00
0.02	2.17×10^{-3}	2.01	7.32×10^{-7}	4.00
0.01	5.38×10^{-4}	2.00	4.60×10^{-8}	4.00
0.005	1.34×10^{-4}	2.00	2.87×10^{-9}	4.00
0.002	2.15×10^{-5}	2.00	7.36×10^{-11}	3.67
0.001	5.37×10^{-6}		5.80×10^{-12}	

Table 4. Error norms L_∞ at time $t=5$ with $0 \le x \le 9$.

$h=\Delta t$	M1	M2	[10]	[10]
0.1	5.36×10^{-2}	4.11×10^{-3}	6.96×10^{-3}	1.46×10^{-2}
0.05	1.41×10^{-2}	2.83×10^{-5}	1.25×10^{-1}	1.36×10^{-1}

It can be seen from the Table 3 that, when time and space steps are reduced from 0.1 to 0.001, the error norms decrease for the both algorithms. According to the error norms and order of convergence in the table, the M2 generates better results than the M1. Table 4 shows that the numerical solutions obtained by M2 is better than the results existed in [10]. Initial and numerical solutions for the M2 are drawn in Figure 3 for visual view of the solution up to time $t=5$ with $h=\Delta t=0.001$. Because of the advection term effect, initial wave is propagating and due to the diffusion term, the wave decreases and its width increases with time.

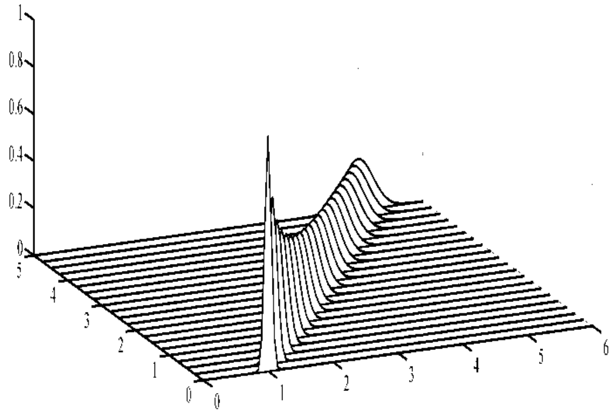

Figure 3. Wave profiles.

The graph of absolute error at time $t=5$ is plotted for M2 with $h=\Delta t=0.001$ in Figure 4. Maximum error is observed near the peak of the amplitude of the final wave.

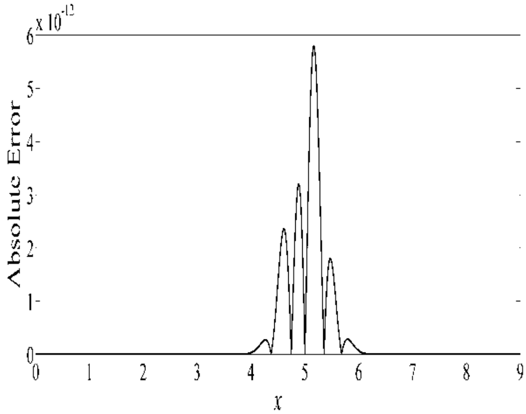

Figure 4. Absolute error for M2 with $h=\Delta t=0.001$.

4. CONCLUSION

Whereas the first method M1 known as Crank-Nicolson method is of order two in time, our proposed new method known as M2 is of order four in time. For the both methods cubic B-spline Galerkin method is used for space discretization. Two test problems including advection effect in the first problem and advection-diffusion effect in the second problem were simulated well with the proposed two algorithms. According to test problems, the proposed two algorithms especially the M2 have produced outstanding results for solving ADE. It is clear from Tables 2 and 4 that the method is of order 4 in time has higher accuracy than other studies in the literature. Thus, the proposed methods are suitable methods for similar physically important equations. Also it can be used for nonlinear problems.

Acknowledgments

This paper was presented at International Congress on Fundamental and Applied Sciences 2016 (ICFAS2016) in Istanbul.

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