Sigma J Eng & Nat Sci 37 (1), 2019, 143-154

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

ON SPACES OF IDEAL CONVERGENT FIBONACCI DIFFERENCE SEQUENCE DEFINED BY ORLICZ FUNCTION

Vakeel A. KHAN*[1], Kamal M. A. S. ALSHLOOL[2], Abdullah A. H. MAKHARESH[3], Sameera A. A. ABDULLAH[4]

[1]*Department of Mathematics, Aligarh Muslim University, Aligarh-INDIA;* ORCID: 0000-0002-4132-0954
[2]*Department of Mathematics, Aligarh Muslim University, Aligarh-INDIA;* ORCID: 0000-0003-0029-2405
[3]*Department of Mathematics, Aligarh Muslim University, Aligarh-INDIA;* ORCID: 0000-0002-7225-3512
[4]*Department of Mathematics, Aligarh Muslim University, Aligarh-INDIA;* ORCID: 0000-0003-0029-2405

Received: 17.08.2018 Revised: 02.11.2018 Accepted: 13.11.2018

ABSTRACT

In this paper, we introduce some new Fibonacci difference sequence spaces $c_0^I(M,\hat{F})$, $c^I(M,\hat{F})$, $\ell_\infty^I(M,\hat{F})$ and $\ell_\infty(M,\hat{F})$ by using the idea of Orlicz function and the Fibonacci difference matrix $\hat{F}$ defined by Fibonacci sequence. We study some topological and algebraic properties on these spaces. Furthermore, we study some inclusion relations concerning these spaces.

Keywords: Fibonacci difference matrix, Fibonacci $I-$convergence, Fibonacci $I-$ Cauchy, Fibonacci $I-$bounded, Orlicz function.

1. INTRODUCTION

Let ℕ and ℝ, denote the sets of all natural and real numbers, respectively. By ω, we denote the vector space of all real sequences. Any vector subspace of ω is called a sequence space. Recall that in [18], let X be a nonempty set, then a family $I \subset 2^X$ (the class of all subsets of X) is said to be an ideal in X if and only if (i) $\varnothing \in I$, (ii) for each $A, B \in I$ we have $A \cup B \in I$, (iii) for each $A \in I$ and $B \subset A$ we have $B \in I$. An ideal $I \subset 2^X$ is said to be nontrivial if $I \neq \varnothing$ and $X \notin I$ and a nontrivial ideal I is said to be an admissible ideal in X if $I \subset \{\{x\} : x \in X\}$. A nonempty of sets $\mathrm{F} \subset 2^X$ is a filter on X if and only if (i) $\varnothing \notin \mathrm{F}$, (ii) $\mathrm{A}, B \in \mathrm{F}$ implies that $\mathrm{A} \cup B \in \mathrm{F}$, (iii) for each $\mathrm{A} \in \mathrm{F}$ and $B \supset A$ we have $B \in \mathrm{F}$. For each ideal I there is a filter $\mathrm{F}(I)$ corresponding to I (filter associate with

* Corresponding Author: e-mail: vakhanmaths@gmail.com, tel: (+) 9411803618

ideal I), that is, $\mathrm{F}(I)=\{K\subseteq X : K^c\in I\}$, where $K^c = X\setminus K$. Depending on the structure of admissible ideals of subsets ℕ, Kostyrko et al. [18] defined the notion of $I-$ convergence as generalization of the notion of statistical convergence which was introduced by Fast [8] and Steinhaus [26] independently. Later on, the notion of $I-$ convergence was further investigated from the sequence space point of view and linked with the summability theory by Salat et al. [24]. Throug hout the paper, by c_0^I, c^I and ℓ_∞^I we denote for the spaces of all $I-$ null, $I-$convergent and $I-$bounded sequences, respectively.

The Fibonacci sequence (f_n) for $n\in\{0,1,...\}$ defined by the linear recurrence equalities $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, $n\geq 2$. Some basic properties of Fibonacci numbers are given in [17]. The Fibonacci sequence was firstly used in the theory of sequence spaces by Kara and Basarir [9]. Later, Kara [10] has introduced a new band matrix $\hat{F}=(f_{nk})$ by using the Fibonacci sequence (f_n) and defined the sequence space

$$\ell_\infty\left(\hat{F}\right)=\left\{x=(x_n)\in\omega : \sup_{n\in\mathbb{N}}\left|\frac{f_n}{f_{n+1}}x_n-\frac{f_{n+1}}{f_n}x_{n-1}\right|<\infty\right\}$$

which is derived by the matrix domain of $\hat{F}$ in the sequence space ℓ_∞ (the class of all bounded sequence), where the matrix $\hat{F}=(f_{nk})$ defined as follows:

$$\hat{F}=(f_{nk})=\begin{cases}-\dfrac{f_{n+1}}{f_n}, (k=n-1)\\ \dfrac{f_n}{f_{n+1}}, (k=n)\\ 0,\ 0\leq k<n-1 \text{ or } k>n,\end{cases}$$

for all $n,k\in$ ℕ. Afterward, the Fibonacci matrix $\hat{F}$ was used to define some spaces of Fibonacci difference null and convergent sequence by Basarir et al. [1]. Quite recently, Khan et al. [15] have introduced and examined the new Fibonacci difference sequence spaces by means of the infinite Fibonacci matrix domain of $\hat{F}$ and the notion of $I-$convergence. That are,

$$X\left(\hat{F}\right)=\left\{x=(x_n)\in\omega : \hat{F}_n(x)\in X\right\}$$

for $X=\left\{\ell_\infty^I, c_0^I, c^I\right\}$ where the sequence $\left(\hat{F}_n(x)\right)$ for $n\in\{0,1,...\}$ which is frequently used as the $\hat{F}-$transform of a sequence $x=(x_n)\in\omega$, defined as follows:

$$\hat{F}_n(x)=\begin{cases}\frac{f_0}{f_1}x_0=x_0,\ n=0\\ \frac{f_n}{f_{n+1}}x_n-\frac{f_{n+1}}{f_n}x_{n-1},\ n\geq 1 \text{ for all } n\in\mathbb{N}.\end{cases} \tag{1.2}$$

In order to give a fuller knowledge to the readers on the Fibonacci sequence spaces see the references [2,3,5,7,11,16]. Recall in [20] that an Orlicz function is a function $M:[0,\infty)\to[0,\infty)$, which is continuous, convex and non-decreasing with $M(0)=0, M(x)>0$, and $M(x)\to\infty$ as $x\to\infty$. If the convexity of an Orlicz function is replaced by $M(x+y)\leq M(x)+M(y)$ then this function is called Modulus function, which was introduced by Nakano [22] and it was further investigated with applications to sequences by Maddox [27].

Remark.1.1:

It is well known if M is a convex function and $M(0)=0$, then $M(\lambda x)\leq\lambda M(x)$ for all λ with $0<\lambda<1$.

An Orlicz function M is said to satisfy Δ_2- condition for all values of u if there exists a constant $K>0$ such that $M(Lu)\leq KLM(u)$ for all values $L>1$ (see [20]). Lindenstrauss and Tzafriri [21] used the idea of Orlicz function to define the sequence space

$$l_M=\left\{x=(x_n)\in\omega:\sum_{n=1}^{\infty}M\left(\frac{|x_n|}{\rho}\right)<\infty \text{ for some } \rho>0\right\},$$

which is called an Orlicz sequence space. This space is a Banach space with the norm

$$\|x\|=\inf\left\{\rho>0,\sum_{n=1}^{\infty}M\left(\frac{|x_n|}{\rho}\right)\leq 1\right\}$$

and the space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(t)=t^p$ for $1\leq p<\infty$. Orlicz function was used to define sequence spaces by Parashar and Choudhary [23], Khan et al. [12]. Later on, with the help of the notion of $I-$convergence, Tripathy and Hazarika [28] introduced some new sequence spaces defined by Orlicz function and further studied by Khan et al. [4,13,14] and many others.

Now, we recall some definitions and lemmas that will be used throughout the article.

Definition.1.1: [8,26] A sequence $(x_n)\in\omega$ is said to be statistically convergent to a number $\ell\in\mathbb{R}$ ℝ if, for every $\varepsilon>0$, the natural density of the set $\{n\in\mathbb{N}:|x_n-\ell|\geq\varepsilon\}$ ℕ equal zero. And we write $st-\lim x_n=\ell$. In case $\ell=0$ then $(x_n)\in\omega$ is said to be $st-$null.

Definition.1.2: [24] A sequence $(x_n) \in \omega$ is said to be $I-$ convergent to a number $\ell \in \mathbb{R}$ if, for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in I$. And we write $I - \lim x_n = \ell$. In case $\ell = 0$ then $(x_n) \in \omega$ is said to be $I-$null.

Definition.1.3: [18] A sequence $(x_n) \in \omega$ is said to be $I-$Cauchy if, for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that the set $\{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\} \in I$.

Definition.1.4: [14] A sequence $(x_n) \in \omega$ is said to be $I-$bounded if there exists $K > 0$, such that, the set $\{n \in \mathbb{N} : |x_n| \geq K\} \in I$.

Definition.1.5: [14] Let $(x_n) \in \omega$ and $(z_n) \in \omega$ be two sequences. We say that $x_n = z_n$ for almost all n relative to I (in short $a.a.n.r.I$) if the set $\{n \in \mathbb{N} : x \neq z\} \in I$.

Definition.1.6: [24] A sequence space E is said to be solid or normal, if $(\alpha_n x_n) \in E$ whenever $(x_n) \in E$ and for any sequence of scalars $(\alpha_n) \in \omega$ with $|\alpha_n| < 1$, for every $n \in \mathbb{N}$.

Definition.1.7: [14] A sequence space E is said to be sequence algebra, if $(x_n) * (z_n) = (x_n \cdot z_n) \in E$ whenever $(x_n), (z_n) \in E$.

Definition.1.8: [24] Let $K = \{n_i \in \mathbb{N} : n_1 < n_2 < \ldots\} \subseteq \mathbb{N}$ and E be a sequence space. A $K-$step space of E is a sequence space

$$\lambda_K^E = \left\{ (x_{n_i}) \in \omega : (x_n) \in E \right\}$$

A canonical preimage of a sequence $(x_{n_i}) \in \lambda_K^E$ is a sequence $(y_n) \in \omega$ defined as follows:

$$y_n = \begin{cases} x_n, \text{if } n \in K \\ 0, \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E. i.e., y is in canonical preimage of λ_K^E iff y is canonical preimage of some element $x \in \lambda_K^E$.

Definition.1.9: [24] A sequence space E is said to be monotone, if it is contains the canonical preimages of it is step space. (i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(x_n) \in E$ the sequence $(\alpha_n x_n) \in E$, where $\alpha_n = 1$ for $n \in K$ and $\alpha_n = 0$ otherwise, belongs to E).

Lemma.1.1: [24] Every solid space is monotone.

Lemma.1.2: [25] Let $K \in \mathrm{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then

$$M \cap K \notin I.$$

Lemma.1.3: [19] If $I \subset 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.

2. MAIN RESULTS

In this section, by using the idea of Orlicz function and the Fibonacci matrix $\hat{F}$ we introduce some new Fibonacci difference sequence spaces. Further, we present some inclusion theorems and study some topological and algebraic properties on these resulting. Throughout the article, we suppose that the sequence $x = (x_n) \in \omega$ and $\left(\hat{F}_n(x)\right)$ are connected with the relation (1.2) and I be an admissible ideal of subset of ℕ.

$$c^I\left(M,\hat{F}\right) := \left\{ x=(x_n)\in\omega : \left\{ n\in\mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x)-\ell\right|}{\rho}\right) \geq \varepsilon, \text{for some } \ell\in\mathbb{R} \right\} \in I \right\}, \quad (2.1)$$

$$c_0^I\left(M,\hat{F}\right) := \left\{ x=(x_n)\in\omega : \left\{ n\in\mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x)\right|}{\rho}\right) \geq \varepsilon \right\} \in I \right\}, \quad (2.2)$$

$$\ell_\infty^I\left(M,\hat{F}\right) := \left\{ x=(x_n)\in\omega : \exists K>0 \text{ s.t} \left\{ n\in\mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x)\right|}{\rho}\right) \geq K \right\} \in I \right\}, \quad (2.3)$$

$$\ell_\infty\left(M,\hat{F}\right) := \left\{ x=(x_n)\in\omega : \sup_n \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x)\right|}{\rho}\right) < \infty \right\}, \quad (2.4)$$

We write

$$m_0^I\left(M,\hat{F}\right) := c_0^I\left(M,\hat{F}\right) \cap \ell_\infty\left(M,\hat{F}\right), \quad (2.5)$$

and

$$m^I\left(M,\hat{F}\right) := c^I\left(M,\hat{F}\right) \cap \ell_\infty\left(M,\hat{F}\right). \quad (2.6)$$

Theorem.2.1: For any Orlicz function M, the classes of sequences $c^I\left(M,\hat{F}\right)$, $c_0^I\left(M,\hat{F}\right)$ $m^I\left(M,\hat{F}\right)$ and $m_0^I\left(M,\hat{F}\right)$ are linear spaces.

Proof: We shall prove the result only for the space $c^I\left(M,\hat{F}\right)$. The others can be treated similarly. Let $x=(x_n)$, $y=(y_n)$ be two arbitrary elements of the space $c^I\left(M,\hat{F}\right)$ and α,β are scalars. Then for given $\varepsilon>0$, there exist positive numbers ρ_1 and ρ_2 such that

$$A_1 := \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x) - \ell_1\right|}{\rho_1} \right) \geq \frac{\varepsilon}{2} \right\} \in I, \text{ for some } \ell_1 \in \mathbb{R} \tag{2.7}$$

and

$$A_2 := \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x) - \ell_2\right|}{\rho_2} \right) \geq \frac{\varepsilon}{2} \right\} \in I, \text{ for some } \ell_2 \in \mathbb{R}. \tag{2.8}$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non- decreasing and convex function, we have

$$M\left(\frac{\left|\alpha\hat{F}_n(x) - \beta\hat{F}_n(y) - (\alpha\ell_1 - \beta\ell_2)\right|}{\rho_3} \right) \leq M\left(\frac{|\alpha|\left|\hat{F}_n(x) - \ell_1\right|}{\rho_3} + \frac{|\beta|\left|\hat{F}_n(y) - \ell_2\right|}{\rho_3} \right)$$

$$\leq M\left(\frac{|\alpha|\left|\hat{F}_n(x) - \ell_1\right|}{\rho_1} \right) + M\left(\frac{|\beta|\left|\hat{F}_n(y) - \ell_2\right|}{\rho_2} \right).$$

Now, from (2.7) and (2.8), we have

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\alpha\hat{F}_n(x) - \beta\hat{F}_n(y) - (\alpha\ell_1 + \beta\ell_2)\right|}{\rho_3} \right) \geq \varepsilon \right\} \subset A_1 \cup A_2 \in I.$$

Therefore, $(\alpha x_n + \beta y_n) \in c^I(M, \hat{F})$. Hence $c^I(M, \hat{F})$ is a linear space.

Theorem.2.2. The spaces $m_0^I(M, \hat{F})$ and $m^I(M, \hat{F})$ are Banach spaces normed by

$$\left\|\hat{F}_n(x)\right\| = \inf\left\{ \rho > 0 : \sup_n \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x)\right|}{\rho} \right) \leq 1 \right\}, \tag{2.9}$$

Proof: The proof of this result is easy, so omitted.

Theorem.2.3: Let M_1, M_2 be Orlicz functions that satisfy the Δ_2 - condition. Then

(a). $W(M_2, \hat{F}) \subseteq W(M_1 \cdot M_2, \hat{F})$,

(b). $W(M_1, \hat{F}) \cap W(M_2, \hat{F}) = W(M_1 + M_2, \hat{F}) \text{ for } W = c^I, c_0^I, m^I, m_0^I$.

Proof: (a). Let $(x_n) \in c_0^I(M_2, \hat{F})$. Then there exists $\rho > 0$ such that

$$\left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} M_2\left(\frac{\left|\hat{F}_n(x)\right|}{\rho}\right) \geq \varepsilon\right\} \in I. \tag{2.10}$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write

$$\hat{F}_n(y) = \sum_{n=1}^{\infty} M_2\left(\frac{\left|\hat{F}_n(x)\right|}{\rho}\right)$$

and consider

$$\lim_{\substack{n\in\mathbb{N} \\ 0 \leq \hat{F}_n(y) \leq \delta}} M_1\left(\left|\hat{F}_n(y)\right|\right) = \lim_{\substack{n\in\mathbb{N} \\ \hat{F}_n(y) \leq \delta}} M_1\left(\left|\hat{F}_n(y)\right|\right) + \lim_{\substack{n\in\mathbb{N} \\ \hat{F}_n(y) > \delta}} M_1\left(\left|\hat{F}_n(y)\right|\right)$$

By the Remark 1.1, we have

$$\lim_{\substack{n\in\mathbb{N} \\ \hat{F}_n(y) \leq \delta}} M_1\left(\left|\hat{F}_n(y)\right|\right) \leq M_1(2) \lim_{\substack{n\in\mathbb{N} \\ \hat{F}_n(y) \leq \delta}} M_1\left(\left|\hat{F}_n(y)\right|\right). \tag{2.11}$$

For $\hat{F}_n(y) > \delta$, we have

$$\hat{F}_n(y) < \frac{\hat{F}_n(y)}{\delta} < 1 + \frac{\hat{F}_n(y)}{\delta}$$

Since M_1 is non decreasing and convex, it follows that

$$M_1\left(\hat{F}_n(y)\right) < M_1\left(1 + \frac{\hat{F}_n(y)}{\delta}\right) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1\left(\frac{2\hat{F}_n(y)}{\delta}\right).$$

Since M_1 satisfies $\Delta_2 -$ condition, we have

$$M_1\left(\hat{F}_n(y)\right) \leq \frac{1}{2} K \frac{\hat{F}_n(y)}{\delta} M_1(2) + \frac{1}{2} K \frac{\hat{F}_n(y)}{\delta} M_1(2) = K \frac{\hat{F}_n(y)}{\delta} M_1(2).$$

Hence

$$\lim_{\substack{n\in\mathbb{N} \\ \hat{F}_n(y) > \delta}} M_1\left(\hat{F}_n(y)\right) \leq \max\left(1, K\delta^{-1} M_1(2)\right) \lim_{\substack{n\in\mathbb{N} \\ \hat{F}_n(y) > \delta}} \hat{F}_n(y). \tag{2.12}$$

From above equations (2.10), (2.11) and (2.12), we have $(x_n) \in c_0^I\left(M_1 \cdot M_2, \hat{F}\right)$. Thus $c_0^I\left(M_2, \hat{F}\right) \subseteq c_0^I\left(M_1 \cdot M_2, \hat{F}\right)$. The other cases can be proved similarly.

(b) Let $(x_n) \in c_0^I\left(M_1, \hat{F}\right) \cap c_0^I\left(M_2, \hat{F}\right)$. Then there exists $\rho > 0$ such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_1 \left(\frac{\left| \hat{F}_n(x) \right|}{\rho} \right) \geq \varepsilon \right\} \in I$$

and

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_2 \left(\frac{\left| \hat{F}_n(x) \right|}{\rho} \right) \geq \varepsilon \right\} \in I.$$

The rest of the proof follows from the following equality

$$\lim_{n \in \mathbb{N}} (M_1 + M_2) \left(\frac{\left| \hat{F}_n(y) \right|}{\rho} \right) = \lim_{n \in \mathbb{N}} (M_1) \left(\frac{\left| \hat{F}_n(y) \right|}{\rho} \right) + \lim_{n \in \mathbb{N}} (M_2) \left(\frac{\left| \hat{F}_n(y) \right|}{\rho} \right).$$

Let be take $M_2(x) = x$ and $M_1(x) = x$ for all $x \in [0, \infty)$ in the Theorem (2.3) (i). We have the following result.

Corollary: $W(\hat{F}) \subseteq W(M, \hat{F})$ for $W = c_0^I, c^I, m_0^I, m^I$.

Proof: The proof of the result easy, so omitted.

Theorem.2.4: The spaces $c_0^I(M, \hat{F})$ and $m_0^I(M, \hat{F})$ are solid and monotone.

Proof: We will prove the result for $c_0^I(M, \hat{F})$ and for $m_0^I(M, \hat{F})$ the result can be proved similarly. Let $x = (x_n) \in c_0^I(M, \hat{F})$, then for given $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M \left(\frac{\left| \hat{F}_n(x) \right|}{\rho} \right) \geq \varepsilon \right\} \in I. \tag{2.13}$$

Let $\alpha = (\alpha_n)$ be a sequence of scalars with $|\alpha| \leq 1$ for all $n \in$ ℕ.

Therefore,

$$M \left(\frac{\left| \hat{F}_n(\alpha x) \right|}{\rho} \right) \leq |\alpha| M \left(\frac{\left| \hat{F}_n(x) \right|}{\rho} \right) \leq M \left(\frac{\left| \hat{F}_n(x) \right|}{\rho} \right) \text{ for all } n \in \mathbb{N}.$$

Thus, from the above inequality and (2.13), we have

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M \left(\frac{\left| \hat{F}_n(\alpha x) \right|}{\rho} \right) \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M \left(\frac{\left| \hat{F}_n(x) \right|}{\rho} \right) \geq \varepsilon \right\} \in I$$

implies that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left| \hat{F}_n(\alpha x) \right|}{\rho} \right) \geq \varepsilon \right\} \in I$$

Therefore, $(\alpha x_n) \in c_0^I(M, \hat{F})$. The monotone of the spaces follows from the Lemma 1.1.

Proposition 2.5: The spaces $c^I(M, \hat{F})$ and $m^I(M, \hat{F})$ are neither monotone nor solid in general.

Proof: The proof of this result follows from the following example.

Example 2.1. Let $I = I_f$ and $M(x) = x^2$ for all $x \in [0, \infty)$. Consider the K– step space E_K of E defined as follows:

Let $(x_n) \in E$ and $(y_n) \in E_K$ be such that

$$y_n = \begin{cases} x_n, & \text{if } n \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_n) as $x_n = 1$ for all $n \in \mathbb{N}$. Then $(x_n) \in c^I(M, \hat{F})$ but its K– step space preimage does not belong to $c^I(M, \hat{F})$. Thus $c^I(M, \hat{F})$ is not monotone. Hence $c^I(M, \hat{F})$ is not solid by Lemma 1.1.

Proposition.2.6: The spaces $c^I(M, \hat{F})$ and $c_0^I(M, \hat{F})$ are not convergent free in general.

Proof: The proof of this result follows from the following example.

Example 2.2. Let $I = I_f$ and $M(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequences (x_n) and (y_n) defined by $x_n = \frac{1}{n}$ and $y_n = n$ for all $n \in \mathbb{N}$. Then (x_n) belongs to $c^I(M, \hat{F})$ and $c_0^I(M, \hat{F})$, but (y_n) does not belongs to $c^I(M, \hat{F})$ and $c_0^I(M, \hat{F})$. Hence the spaces are not convergence free.

Theorem.2.7: The spaces $c_0^I(M, \hat{F})$ and $c^I(M, \hat{F})$ are sequence algebra.

Proof: We prove that $c_0^I(M, \hat{F})$ is sequence algebra. For the space $c^I(M, \hat{F})$ the result can be proved similarly. Let $x = (x_n), y = (y_n) \in c_0^I(M, \hat{F})$. Then

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left| \hat{F}_n(x) \right|}{\rho_1} \right) \geq \varepsilon \right\} \in I \text{ for some } \rho_1 > 0$$

and

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(y)\right|}{\rho_2} \right) \geq \varepsilon \right\} \in I \text{ for some } \rho_2 > 0.$$

Let $\rho = \rho_1\rho_2 > 0$. Then we can show that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x) \cdot \hat{F}_n(y)\right|}{\rho} \right) \geq \varepsilon \right\} \in I.$$

Thus, $(x_n \cdot y_n) \in c_0^I(M, \hat{F})$. Hence, $c_0^I(M, \hat{F})$ is sequence algebra.

Theorem.2.8: Let M be an Orlicz function. Then $c_0^I(M, \hat{F}) \subset c^I(M, \hat{F}) \subset \ell_\infty^I(M, \hat{F})$ and the inclusion are proper.

Proof: Let $(x_n) \in c^I(M, \hat{F})$. Then there exists $\ell \in \mathbb{R}$ and $\rho > 0$ such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M\left(\frac{\left|\hat{F}_n(x) - \ell\right|}{\rho} \right) \geq \varepsilon \right\} \in I.$$

We have

$$M\left(\frac{\left|\hat{F}_n(x)\right|}{2\rho} \right) \leq \frac{1}{2} M\left(\frac{\left|\hat{F}_n(x) - \ell\right|}{\rho} \right) + \frac{1}{2} M\left(\frac{|\ell|}{\rho} \right).$$

Taking supremum over n on both sides we get $(x_n) \in \ell_\infty^I(M, \hat{F})$. The inclusion $c_0^I(M, \hat{F}) \subset c^I(M, \hat{F})$ is obvious. The inclusion is proper follows from the following example.

Example 2.3. Let $I = I_d$ and $M(x) = x^2$ for all $x \in [0, \infty)$.

(a). Consider the sequence (x_n) defined by $x_n = 1$ for all $n \in \mathbb{N}$. Then (x_n) belongs to $c_0^I(M, \hat{F})$, but $(x_n) \notin c_0^I(M, \hat{F})$.

(b). Consider the sequence (y_n) defined as

$$y_n = \begin{cases} 2, \text{if } n \text{ is even} \\ 0, \text{ otherwise.} \end{cases}$$

Then $(y_n) \in \ell_\infty^I(M, \hat{F})$, but $(y_n) \notin c^I(M, \hat{F})$.

3. CONCLUSIONS

In this present paper, we have introduced and studied some new Fibonacci difference sequence spaces, $c_0^I\left(M,\hat{F}\right)$, $c^I\left(M,\hat{F}\right)$, $\ell_\infty^I\left(M,\hat{F}\right)$ and $\ell_\infty\left(M,\hat{F}\right)$ generated by Fibonacci difference matrix $\hat{F}$ and Orlicz function. Some inclusion relations concerning these spaces are studied. Further, some topological and algebraic properties on these spaces are investigated. These definitions and results provide new tools to deal with the convergence problems of sequences occurring in many branches of science and engineering.

Acknowledgement

The authors would like to record their gratitude to the reviewer for her/his careful reading and making some useful corrections which improved the presentation of the paper.

REFERENCES

[1] M. Basarir, F. Basar and EE Kara. On The Spaces of Fibonacci Diffeerence Absolutely p-Summable, Null and Convergent Sequences. *Sarajevo J. Math*, 12(25): 2, 2016.

[2] M. Candan and E. E. Kara. A study on topological and geometrical characteristics of new banach sequence spaces. *Gulf J. Math*, 3(4):67-84, 2015.

[3] M. Candan. A new approach on the spaces of generalized Fibonacci difference null and convergent sequences. *Math. terna*, 5(1):191-210, 2015.

[4] M. Candan. Vector valued Orlicz sequence space generalized with an infinite matrix and some of its specific characteristics. *Gen*, 29(2):1-16, 2015.

[5] M. Candan and K. Kayaduman. Almost convergent sequence space derived by generalized Fibonacci matrix and Fibonacci core. *Brithish J. Math. Comput. Sci*, 7(2):150-167, 2015.

[6] M. Candan and G. Kilnc. A different look for paranormed riesz sequence space derived by Fibonacci matrix. *Konuralp Journal of Mathematics*, 3(2):62-76, 2015.

[7] S. Demiriz, E. E. Kara, and M. Bacarir. On the Fibonacci almost convergent sequence space and Fibonacci core. *Kyungpook Mathematical Journal*, 55(2), 2015.

[8] H. Fast. Sur la convergence statistique. *Colloquium Mathematicae*. 2(3-4): 241-244, 1951.

[9] E. E. Kara and M Basarir. An application of Fibonacci numbers into infinite toeplitz matrices. *Caspian J. Math. Sci*, 1(1):43-47, 2012.

[10] E. E. Kara. Some topological and geometrical properties of new banach sequence spaces. *Journal of Inequalities and Applications*, 2013(1):38, 2013.

[11] E. E. Kara and S. Demiriz. Some new paranormed difference sequence spaces derived by Fibonacci numbers. *arXiv preprint arXiv*:1309.0154, 2013.

[12] Vakeel A Khan. On a new sequence space defined by Orlicz functions. *Commun. Fac. Sci Univ. Ank. Series A*, 1(57):25-33, 2008.

[13] Vakeel A Khan, Hira Fatima, Sameera AA Abdullah, and Kamal MAS Alshlool. On paranorm BV_σ^i – convergent double sequence spaces defined by an Orlicz function. *Analysis*, 37(3):157-167, 2017.

[14] Vakeel A Khan and Sabiha Tabassum. On ideal convergent difference double sequence spaces in n-normed spaces defined by Orlicz function. *Theory and Applications of Mathematics & Computer Science*, 3(1):90-98, 2013.

[15] Vakeel A Khan , Rami K. A. Rababah, Kamal M. A. S. Alshlool, Sameera A. A. Abdullah, A. Ahmad. On Ideal Convergence Fibonacci Difference Sequence Spaces. *Advanced in Difference Equations*, 2018(1):199, 2018.

[16] G. Kilinc, M. Candan. Some generalized Fibonacci difference spaces defined by a sequence of modulus functions. *Facta Universitatis, Series: Mathematics and Informatics*, 095-116, 2017.

[17] Thomas Koshy. Fibonacci and Lucas numbers with applications, 1. John Wiley & Sons, 2017.

[18] P. Kostyrko, M. Macaj, and T. Salat. Statistical convergence and *I*-convergence. *Real Analysis Exchange*, 1999.

[19] P. Kostyrko, W. Wilczynski, Tibor _Salat, et al. *I*-convergence. *Real Analysis Exchange*, 26(2):669-686, 2000.

[20] Krasnoselskii, M. Aleksandrovich, and I. Bronislavovich Rutitskii. Convex functions and Orlicz spaces. *P. Noordhoff*, 1961.

[21] Joram Lindenstrauss and Lior Tzafriri. On Orlicz sequence spaces. Israel Journal of Mathematics, 10(3):379-390, 1971.

[22] H. Nakano. Concave modulars. *Journal of the Mathematical society of Japan*, 5(1):29-49, 1953.

[23] S. D. Parashar and B. Choudhary. Sequence spaces defined by Orlicz functions. *Indian Journal of Pure and Applied Mathematics*, 25:419-419, 1994.

[24] T. Salat, B. C. Tripathy, and M. Ziman. On some properties of *I*-convergence. *Tatra Mt. Math. Publ*, 28(2):274-286, 2004.

[25] T. Sal_at, B. C. Tripathy, and M. Ziman. On *I*-convergence field. *Ital. J. Pure Appl. Math*, 17(5):1-8, 2005.

[26] H. Steinhaus. Sur la convergence ordinaire et la convergence asymptotique. *In Colloq. Math*, v 2(1), 73-74, 1951.

[27] H. Toutenburg., I. J. Maddox, Elements of functional analysis. *Cambridge university press 1970. preis 50 8. net. Biometrical Journal*, 12(3):197-197, 1970.

[28] B. C. Tripathy and B. Hazarika. Some *I*-convergent sequence spaces defined by Orlicz functions. *Acta Mathematicae Applicatae Sinica (English Series)*, 27(1):149-154, 2011.