Turk. J. Math. Comput. Sci.
14(1)(2022) 82–90
© MatDer
DOI : 10.47000/tjmcs.1012276

Designing Generalized Cylinder with Characteristic Base Curve in Euclidean 3-space

Nabil M. Althibany

Department of Mathematics, Faculty of Applied Science, Taiz University, Taiz, Yemen.

Received: 20-10-2021 • Accepted: 11-04-2022

Abstract. Designing a surface from a given curve under some special conditions is an important problem in many practical applications. The purpose of this article is to design a generalized cylinder whose base curve is a characteristic curve in Euclidean 3- space. The main results show that the generalized cylinder with geodesic, line of curvature, or asymptotic base curve is a rectifying cylinder, a right cylinder, or a plane respectively.

2010 AMS Classification: 53A04, 53A05

Keywords: Generalized cylinder, ruled surface, geodesic, line of curvature, asymptotic.

1. Introduction

A ruled surface is constructed by the continuous motion of a straight line called the ruling through a given curve called the base curve. A generalized cylinder is a special type of ruled surface in which the ruling line moves in a constant direction along the base curve. The generalized cylinders are a class of developable ruled surfaces that can be produced from paper or sheet metal with no distortion. For this construction, the generalized cylinders have been used in many applications including geometric modeling, computer graphic, architectural designing, and manufacturing [10, 12–14].

Geodesic, asymptotic, and line of curvature are characteristic curves that lie on the surface and have been used in surface analysis. The geodesic curve gives the shortest path between two given points on curved spaces. A curve is an asymptotic or a line of curvature if its direction always points in a direction in which the surface does not bend or bend extremely respectively.

There are several articles for designing the surface or the surfaces family that possess the given curve as a characteristic curve. Wang et al. [23], Li et al. [18], and Bayram et.al [6] derived the necessary and sufficient condition for a given curve to be a geodesic, a line of curvature, and an asymptotic on a surface respectively. Later, Bilici and Bayram [5, 7] generalized this problem for the involute of a given curve. A developable surface that possesses a base curve as a line of curvature and a geodesic has been studied in [19] and [1] respectively. Recently in [2], the author studied and classified the ruled and developable surfaces whose base curve is a characteristic curve .

The purpose of this article is to design a generalized cylinder whose base curve is a characteristic curve in Euclidean 3- space. A ruled surface is parameterized by its base curve and director vector that is expressed by a linear combination of Frenet frame with angular functions as coefficients. A generalized cylinder parameterized by a ruled surface parametrization with three conditions called the cylindrical conditions. After that, additional constraints are imposed to make the base curve of the generalized cylinder is a characteristic curve. The main results show that the generalized

Email address: althibany1972@yahoo.com (N.M. Althibany)

cylinder with geodesic, line of curvature, and the asymptotic base curve is a rectifying cylinder, a right cylinder and a plane respectively. Therefore, based on the type of characteristic base curve, the generalized cylinder is classified into three types. This paper generalizes the recent work [4] dealt only with geodesic base curve and using Darboux frame.

The rest of this paper is organized as follows: In section 2, some basic notations, facts, and definitions of the space curve, ruled surface, and a generalized cylinder are reviewed. The main results are studied in section 3, where the generalized cylinder with a characteristic curve is constructed, the necessary and sufficient conditions for the base curves to be geodesic, line of curvature, or asymptotic are derived. Examples to illustrate the main results are presented in section 4. Finally, the conclusion is given in section 5.

2. Preliminaries

This section introduces some basic concepts on the classical differential geometry of space curves and ruled surfaces in three-dimensional Euclidean space. More details can be found in such standard references as [9, 20, 22].

2.1. **Curves in Euclidean 3-space.** A smooth space curve in 3-dimensional Euclidean space is parameterized by a map $\gamma : I \subseteq \mathbb{R} \to E^3$, γ is called a regular curve if $\gamma' \neq 0$ for every point of an interval $I \subseteq \mathbb{R}$, and if $|\gamma'(s)| = 1$, where $|\gamma'(s)| = \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}$, then γ is said to be of unit speed (or parameterized by arc-length s). For a unit speed regular curve $\gamma(s)$ in E^3, the unit tangent vector $t(s)$ of γ at $\gamma(s)$ is given by $t(s) = \gamma'(s)$. If $\gamma''(s) \neq 0$, the unit principal normal vector n(s) of the curve at $\gamma(s)$ is given by $n(s) = \frac{\gamma''(s)}{\|\gamma''\|}$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of γ at $\gamma(s)$. For each point of $\gamma(s)$ where $\gamma''(s) \neq 0$, we associate the Serret-Frenet frame $\{t, n, b\}$ along the curve γ. As the parameter s traces out the curve, the Serret-Frenet frame moves along γ and satisfies the following Frenet-Serret formula:

$$\begin{cases} t'(s) &= \kappa(s)n(s), \\ n'(s) &= -\kappa(s)t(s) + \tau b(s), \\ b'(s) &= -\tau(s)n(s), \end{cases} \tag{2.1}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and torsion functions. When the point moves along the unit speed curve with non-vanishing curvature and torsion, the Serret-Frenet frame $\{t, n, b\}$ is drawn to the curve at each position of the moving point, this motion consists of translation with rotation and described by the following Darboux vector:

$$\omega = \tau t + \kappa b.$$

The direction of Darboux vector is the direction of rotational axis and its magnitude gives the angular velocity of rotation. The unit Darboux vector field is defined by

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b. \tag{2.2}$$

A necessary and sufficient condition that a curve be of constant slope (or general helix) is that the ratio of torsion to curvature is constant ($\frac{\tau}{\kappa} = c$). The general helix lies on a general cylinder and also known as a cylindrical helix. The circular helix (a helix on a circular cylinder) is a special helix with both of $\kappa(s) \neq 0$ and $\tau(s)$ are constants. The Darboux vector is constant for circular helix. For the cylindrical helix, the unit Darboux vector is constant as following:

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b = \frac{c}{\sqrt{c^2 + 1}} t + \frac{1}{\sqrt{c^2 + 1}} b. \tag{2.3}$$

For a regular curve on a surface, there exists another frame $\{t(s), g(s), N(s)\}$ which is called Darboux frame. In this frame $t(s)$ is the unit tangent of the curve, $N(s)$ is the unit normal of the surface and g is a unit vector given by $g = N \times t$. The relations between Frenet frame and Darboux frame can be given by the following matrix representation:

$$\begin{pmatrix} t \\ g \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \tag{2.4}$$

Definition 2.1. A unit-speed curve on a surface is called :

(1) A geodesic if and only if its principal normal vector coincides (up to orientation) with the surface normal

$$N = \pm n.$$

(2) Asymptotic if and only if its binormal vector coincides (up to orientation) with the surface normal vector

$$N = \pm b.$$

(3) A line of curvature if and only if the following condition is satisfied

$$\tau + \frac{d\phi}{ds} = 0.$$

2.2. **Ruled Surfaces.** A ruled surface is generated by moving a straight line on a given curve and parameterized by

$$X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq \ell,\ v \in \mathbb{R}. \tag{2.5}$$

A unit regular curve $\gamma(s)$ is called a base curve, and the line passing through $\gamma(s)$ that is parallel to $D(s)$ is called the ruling. $D(s)$ is a unit director vector field that gives the direction of the ruling. The unit normal vector field (shortly surface normal) of the ruled surface is defined by

$$N(s,v) = \frac{X_s \times X_v}{|X_s \times X_v|} = \frac{(\gamma' \times D) + v(D' \times D)}{|(\gamma' \times D) + v(D' \times D)|}.$$

In particular and using (2.4), the surface normal along the base curve $\gamma(s)$ is given by

$$N(s,0) = -\sin\phi(s)n(s) + \cos\phi(s)b(s). \tag{2.6}$$

The ruled surface parameterized by (2.5) is a generalized cylinder if and only if the unit director vector $D(s)$ has constant direction, or equivalently if and only if the following condition is satisfied:

$$D'(s) = 0. \tag{2.7}$$

$D(s)$ is a unit vector field lies in the space formed by the frame $\{t, n, b\}$ and can be written using (2.4) as following:

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)g(s), \quad \text{where} \quad g(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s).$$

Therefore $D(s)$ can be decomposed as the following [21]

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)), \tag{2.8}$$

where $\theta(s)$ and $\phi(s)$ are two scalar functions called the first and second angular functions [16].

Definition 2.2. The ruled surface with base curve $\gamma(s)$ and a unit director vector D(s) (2.8) is defined by

$$\begin{cases} X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq L,\ v \in \mathbb{R}, \text{ where} \\ D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)). \end{cases} \tag{2.9}$$

In the following theorem, we give the necessary and sufficient conditions to construct a generalized cylinder as special class of ruled surface (2.9), we call them the cylindrical conditions.

Theorem 2.3. *The ruled surface parameterized by (2.9) is a generalized cylinder if and only if the following conditions are satisfied:*

$$\begin{aligned} \kappa(s)\cos\phi + \frac{d\theta}{ds} &= 0, \\ \cos\theta(s)(\kappa(s) + \cos\phi\frac{d\theta}{ds}) - \sin\theta(s)\sin\phi(s)(\frac{d\phi}{ds} + \tau) &= 0, \\ \sin\phi(s)\cos\theta(s)\frac{d\theta}{ds} + \sin\theta(s)\cos\phi(\frac{d\phi}{ds} + \tau) &= 0. \end{aligned} \tag{2.10}$$

Proof. From (2.8) we have $D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))$, by taking the derivative of $D(s)$ and using the Frenet-Serret formula of $\gamma(s)$, we get $D'(s) = -\sin\theta(s)[\kappa(s)\cos\phi + \frac{d\theta}{ds}]t(s) + [\cos\theta(s)(\kappa(s) + \cos\phi\frac{d\theta}{ds}) - \sin\theta(s)\sin\phi(s)(\frac{d\phi}{ds} + \tau)]n + [\sin\phi(s)\cos\theta(s)\frac{d\theta}{ds} + \sin\theta(s)\cos\phi(\frac{d\phi}{ds} + \tau)]b$. According to (2.7), the ruled surface parameterized by (2.9) is a cylinder if and only if $D'(s)$ vanishes, this condition is satisfied provided that (2.10) are satisfied. □

Hence, a ruled surface defined by (2.9) satisfying (2.10) is a generalized cylinder as given in the following definition.

Definition 2.4. The generalized cylinder with base curve $\gamma(s)$ is defined by

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))],\ 0 \leq s \leq L,\ v \in \mathbb{R}, \tag{2.11}$$

where the conditions (2.10) are satisfied.

The main result of this paper is the following main theorem which is proved in the next section.

Theorem 2.5. *Let $X(s, v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R}$ be a ruled surface, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature , and $D(s)$ is a unit director vector defined by (2.8), then the generalized cylinder with geodesic base curve is a rectifying cylinder, the generalized cylinder with line of curvature base curve is a right cylinder, and the generalized cylinder with asymptotic base curve is a plane.*

3. Generalized Cylinder with Characteristic Base Curve

In the preceding section, we proved that three conditions (2.10) are needed to construct a generalized cylinder (2.11) from a ruled surface parametrization (2.9). To make the base curve of the generalized cylinder (2.11) is a characteristic curve (geodesic, line of curvature, or asymptotic) we need other conditions that are investigated in this section.

3.1. **Generalized Cylinder with Geodesic Base Curve.**

Theorem 3.1. *A base curve $\gamma(s)$ of the generalized cylinder parameterized by (2.11) is a geodesic if and only if the following conditions are satisfied:*

$$\begin{aligned} \cos\phi(s) &= 0, \\ \frac{d\theta}{ds} &= 0, \\ \cos\theta(s)\kappa(s) - \sin\theta(s)\tau &= 0. \end{aligned} \tag{3.1}$$

Proof. By using definition (2.1), a base curve $\gamma(s)$ of the generalized cylinder parameterized by (2.11) is a geodesic if and only if $N = \pm n$, from (2.6), this happens if and only if $\cos\phi(s) = 0$ which is the first condition of (3.1). By substitution it in the cylindrical conditions (2.10), we get the other conditions of (3.1). □

Definition 3.2. A generalized cylinder with geodesic base curve is defined by

$$\begin{cases} X(s, v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)b(s)],\ 0 \leq s \leq L,\ v \in \mathbb{R}, \quad \text{where,} \\ \tau(s)\sin\theta(s) - \kappa(s)\cos\theta(s) = 0, \text{and} \quad \theta'(s) = 0. \end{cases}$$

Proposition 3.3. *[3] Suppose that $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ is a unit rectifying vector defined along a unit speed curve $\gamma(s)$ with non vanishing curvature and torsion, then $D(s)$ is a unit Darboux vector field if and only if $\kappa\cos\theta - \tau\sin\theta = 0$.*

Proof. Let $D(s) = \cos\theta(s)t(s) + \sin\theta(s)b(s)$ be a unit Darboux vector. From (2.2),

$$\cos\theta = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}, \quad \sin\theta(s) = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}, \quad and \quad \cot\theta = \frac{\tau}{\kappa}.$$

This implies that $\kappa\cos\theta - \tau\sin\theta = 0$, and vice versa. □

Theorem 3.4. *The ruled surfaces parameterized by (2.9) is a generalized cylinder with geodesic base curve if and only if $D(s)$ is a unit constant Darboux vector.*

Definition 3.5. A generalized cylinder with geodesic base curve is defined by

$$\begin{cases} X(s, v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R} \quad \text{,where} \\ D(s) = \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s), \quad D'(s) = 0. \end{cases}$$

As discussed in (2.3), the condition for unit Darboux vector to be constant is equivalent to the base curve is a helix. As well known, the base curve and director vector are responsible to build the generalized cylinder, so the following theorem gives the conditions that can be applied on the base curve and director vector at the same time to generate a generalized cylinder with geodesic base curve.

Theorem 3.6. *Let $X(s, v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R}$, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature and torsion, $D(s)$ is a unit direction vector defined by (2.8). Then, X(s,v) is a cylinder whose base curve is a geodesic if and only if $\gamma(s)$ is a helix and $D(s)$ is a unit Darboux vector.*

Definition 3.7. A generalized cylinder with geodesic base curve is defined by

$$\begin{cases} X(s,v) = \gamma(s) + vD(s), 0 \le s \le L, v \in \mathbb{R}, \quad \text{where,} \\ D(s) = \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}} t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}} b(s), \quad \text{and } \gamma(s) \text{ is a helix.} \end{cases} \tag{3.2}$$

The developable ruled surface whose director vector is a unit Darboux vector has been studied by many researchers and it has been called the rectifying developable surface, see for example [8, 15, 17, 22]. The generalized cylinder defined by (3.2) is a special case where the unit Darboux vector is a constant and we call it the rectifying cylinder. The rectifying cylinder inherits this property from the rectifying developable, where the classical result stated that "Every space curve is a geodesic on its rectifying developable" as given in the classical book of Struik [22, p.161].

Corollary 3.8. *A generalized cylinder with geodesic base curve parameterized by (3.2) is a rectifying cylinder.*

Theorem 3.9. *Among all generalized cylinders parameterized by (2.11), the rectifying cylinder (3.2) can be equipped with geodesic base curve.*

In fact, the rectifying cylinder not only has geodesic base curve, but it can be equipped with geodesic coordinates among all developable surfaces, this result is proved recently by the author [3].

3.2. **Generalized Cylinder with Line of Curvature Base Curve.**

Theorem 3.10. *A base curve $\gamma(s)$ of the generalized cylinder parameterized by (2.11) is a line of curvature if and only if the following conditions are satisfied:*

$$\begin{aligned} \tau + \frac{d\phi}{ds} &= 0, \\ \kappa(s)\cos\phi + \frac{d\theta}{ds} &= 0, \\ \cos\theta(s)(\kappa(s) + \cos\phi\frac{d\theta}{ds}) &= 0, \\ \sin\phi(s)\cos\theta(s)\frac{d\theta}{ds} &= 0. \end{aligned} \tag{3.3}$$

Proof. By using the definition (2.1), a base curve $\gamma(s)$ of the generalized cylinder parameterized by (2.11) is a line of curvature if and only if $\tau + \frac{d\phi}{ds} = 0$, which is the first condition of (3.3), substituting it in the cylindrical conditions (2.10), we get the other conditions of (3.3). □

Definition 3.11. A generalized cylinder with line of curvature base curve is defined by

$$\begin{cases} X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)(\cos\phi\, n(s) + \sin\phi\, b(s))],\ 0 \le s \le L,\ v \in \mathbb{R}, \quad \text{where,} \\ \tau + \frac{d\phi}{ds} = 0, \quad \kappa(s)\cos\phi + \frac{d\theta}{ds} = 0, \quad \cos\theta(s)(\kappa(s) + \cos\phi\frac{d\theta}{ds}) = 0 \quad \text{and} \quad \sin\phi(s)\cos\theta(s)\frac{d\theta}{ds} = 0. \end{cases} \tag{3.4}$$

The conditions (3.3) have been analyzed and interpreted in the following theorem.

Theorem 3.12. *For a generalized cylinder whose base curve $\gamma(s)$ is a line of curvature and parameterized by (3.4). The conditions (3.3) are satisfied if and only if $\gamma(s)$ is a plane curve and $D(s)$ is the binormal unit vector.*

Proof. Let $\gamma(s)$ be a plane curve and $D(s)$ is the binormal unit vector, then $\tau(s) = 0$ and $D(s) = b(s)$, by using (2.8), this implies that $\cos\theta = 0$ and $\cos\phi = 0$, so $\frac{d\theta}{ds} = 0$ and $\frac{d\phi}{ds} = 0$. By substituting, the conditions (3.3) are satisfied. Conversely, suppose that the conditions (3.3) are satisfied, from the third and fourth equations we get $\cos\theta = 0$ and so $\frac{d\theta}{ds} = 0$, where the second equation becomes $\kappa(s)\cos\phi = 0$, since $\kappa(s) \neq 0$ then $\cos\phi = 0$, and so $\frac{d\phi}{ds} = 0$ which implies that the first equation becomes $\tau(s) = 0$, by definition $\gamma(s)$ be a plane curve. Also by substituting $\cos\theta = 0$ and $\cos\phi = 0$ in (2.8), it follows that $D(s)$ is the binormal unit vector. □

As known, the plane curve has no binormal unit vector b(s), therefore, the binormal of plane curve coincides with the unit normal vector to the plane of the curve.

Theorem 3.13. *Let $X(s,v) = \gamma(s) + vD(s), 0 \le s \le L, v \in \mathbb{R}$, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature, $D(s)$ is a unit direction vector defined by (2.8). Then, X(s,v) is a generalized cylinder whose base curve is a line of curvature if and only if $\gamma(s)$ is a plane curve and $D(s)$ is a unit normal vector to the plane of $\gamma(s)$.*

Definition 3.14. A generalized cylinder with line of curvature base curve is defined by

$$\begin{cases} X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq L,\ v \in \mathbb{R}, \quad \text{where}, \\ D(s) = (0,0,1) \quad \text{and} \quad \gamma(s) \text{ is a plane curve.} \end{cases} \tag{3.5}$$

The generalized cylinder whose base curve is a plane curve and the director vector is a unit normal vector to the plane of the base curve is called a right generalized cylinder [11] or shortly right cylinder.

Corollary 3.15. *A generalized cylinder with line of curvature base curve parameterized by (3.5) is a right cylinder.*

Theorem 3.16. *Among all generalized cylinders parameterized by (2.11), only the right cylinder (3.5) can be equipped with line of curvature base curve.*

3.3. Generalized Cylinder with Asymptotic Base Curve.

Theorem 3.17. *A base curve $\gamma(s)$ of the generalized cylinder parameterized by (2.11) is an asymptotic if and only if the following conditions are satisfied:*

$$\begin{aligned} \sin\phi(s) = &\quad 0. \\ \frac{d\theta}{ds} + \kappa = &\quad 0. \\ \tau = &\quad 0. \end{aligned} \tag{3.6}$$

Proof. By using the definition (2.1), a base curve $\gamma(s)$ of the generalized cylinder parameterized by (2.11) is an asymptotic if and only if $N = \pm b$, from (2.6), this happens if and only if $\sin\phi(s) = 0$ which is a first condition of (3.6). By substituting it in the cylindrical conditions (2.10), we get the other conditions of (3.6). □

It is noted that in the above proof, neither $\cos\theta = 0$ nor $\sin\theta = 0$, because they are lead to $\frac{d\theta}{ds} = 0$ and hence $\kappa = 0$, which is a contradiction with $\gamma(s)$ is non vanishing curvature.

Definition 3.18. A generalized cylinder with asymptotic base curve is defined by

$$\begin{cases} X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)n(s)],\ 0 \leq s \leq L,\ v \in \mathbb{R}, \text{ where}, \\ \frac{d\theta}{ds} + \kappa = 0, \text{ and } \tau = 0. \end{cases} \tag{3.7}$$

The condition $\kappa + \frac{d\theta}{ds} = 0$ can be written as $\theta(s) = \theta_0 - \int_{s_0}^{s} \kappa ds$. If we choose $s_0 = 0$, hence $\theta_0 = \theta(0)$, then the condition becomes $\theta(s) = \theta(0) - \int_0^s \kappa ds$. For simplification we suppose that $\theta(0) = 0$ as given in the following definition.

Definition 3.19. A generalized cylinder with asymptotic base curve is defined by

$$\begin{cases} X(s,v) = \gamma(s) + v[\cos(\int_0^s \kappa ds)t(s) - \sin(\int_0^s \kappa ds)n(s)],\ 0 \leq s \leq L,\ v \in \mathbb{R}, \text{ where}, \\ \gamma(s) \text{ is a plane curve.} \end{cases} \tag{3.8}$$

Theorem 3.20. *Let $X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R}$, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature, $D(s)$ is a unit direction vector defined by (2.8). Then, X(s,v) is a generalized cylinder whose base curve is an asymptotic if and only if $\gamma(s)$ is a plane curve and $D(s) = \cos(\int_0^s \kappa ds)t(s) - \sin(\int_0^s \kappa ds)n(s)$.*

Corollary 3.21. *A generalized cylinder with asymptotic base curve parameterized by (3.8) is a plane.*

Theorem 3.22. *Among all generalized cylinders parameterized by (2.11), only the plane can be equipped with asymptotic base curve.*

Finally, using Theorems (3.9), (3.16) and (3.22), this section ended with the following theorems that classify the generalized cylinders according to the type of characteristic base curve.

Theorem 3.23. *(Classification of generalized cylinder) Let $X(s,v)$ be a generalized cylinder parametrized by (2.11), where the base curve is a characteristic curve. Then, $X(s,v)$ is either a rectifying cylinder, a right cylinder or a plane.*

The explicit classification can be obtained in the following equivalent theorem.

Theorem 3.24. *Let $X(s,v)$ be a generalized cylinder parametrized by (2.11) whose base curve is a geodesic, a line of curvature, or an asymptotic. Then, $X(s,v)$ is either a rectifying cylinder, a right cylinder or a plane, respectively.*

According to the above classification theorems, the generalized cylinder whose base curve is a characteristic curve can be classified into three different types based on the type of base curve.

Corollary 3.25. *Let $X(s, v)$ be a generalized cylinder of type a rectifying cylinder, a right cylinder or a plane. Then, the base curve is a geodesic, a line of curvature or an asymptotic respectively.*

The existence of such types can be ensured via the Theorems (3.9), (3.16), and (3.22) as the following.

Corollary 3.26. *Given a unit speed regular curve $\gamma(s)$ with non vanishing curvature. Then, there exists a rectifying cylinder, a right cylinder or a plane in which $\gamma(s)$ its a geodesic, a line of curvature or an asymptotic base curve, respectively.*

4. Examples

In this section, we give an example of a generalized cylinder whose base curve is a geodesic, a line of curvature or an asymptotic curve and draw their pictures by using Mathematica.

Example 4.1. Let $\gamma(s) = (\frac{1}{\sqrt{2}}\sin(s), \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos(s))$ be a unit speed helix curve. By direct calculation we get $t = \left(\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin(s)\right)$, $n = (-\sin(s), 0, -\cos(s))$, $b = (-\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin(s))$, $\kappa = \frac{1}{\sqrt{2}}$, and $\tau = \frac{1}{\sqrt{2}}$. According to definition (3.2), the generalized cylinder whose base curve is a geodesic can be parameterized by

$$X(s, v) = \gamma(s) + v[\frac{\tau(s)}{\sqrt{\kappa^2 + \tau^2}}t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2 + \tau^2}}b(s)],\ \ 0 \le s \le L, v \in \mathbb{R}.$$

By substituting, $\frac{\tau}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{\sqrt{2}}$ and $\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{\sqrt{2}}$, and for $0 \le s \le 2\pi$, $0 \le v \le \pi$, the constructed cylinder is a rectifying cylinder with geodesic base curve (blue) as shown in Figure 1.

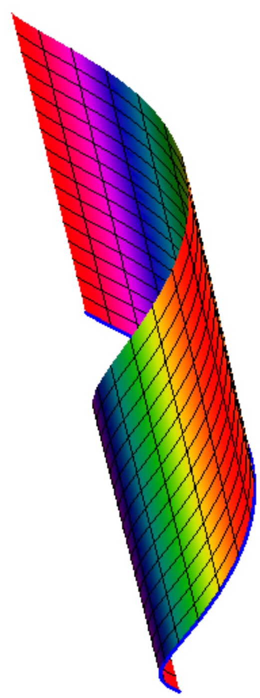

Figure 1. Rectifying cylinder with geodesic base curve.

Example 4.2. Let $\gamma(s) = (\cos(s), \sin(s), 0)$ be a unit speed plane curve. According to definition (3.5), the generalized cylinder whose base curve is a line of curvature can be parameterized by

$$X(s, v) = \gamma(s) + v\,(0, 0, 1)\,,\ \ 0 \le s \le \pi, 0 \le v \le \pi/2.$$

The constructed cylinder is a right cylinder with line of curvature base curve (blue) as shown in Figure 2.

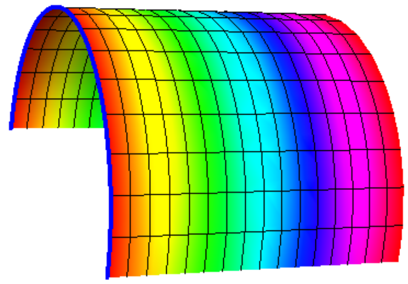

Figure 2. Right cylinder with line of curvature base curve.

Example 4.3. Let $\gamma(s) = (\cos(s), \sin(s), 0)$ be a unit speed circular plane curve, where $t = (-\sin(s), \cos(s))$, $n = (-\cos(s), -\sin(s))$ and $\kappa = 1$. From definition (3.8), the generalized cylinder with asymptotic base curve is given by

$$X(s, v) = \gamma(s) + v[\cos(s)t(s) - \sin(s)n(s)],\ \ 0 \leq s \leq \pi/2, 0 \leq v \leq \pi/2.$$

The constructed cylinder is a plane with asymptotic base curve (blue) as shown in Figure 3.

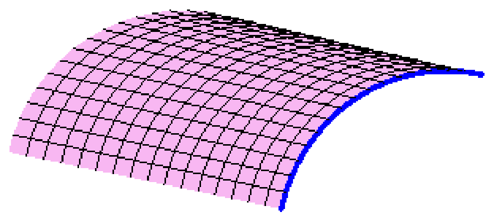

Figure 3. Generalized cylinder (plane) with asymptotic base curve.

5. Conclusion

In this paper, using a generalized cylinder parametrization (2.11), we constructed three types of a cylinder whose base curve is a characteristic. The main results asserted that the generalized cylinder with geodesic, line of curvature, or asymptotic base curve is a rectifying cylinder(3.2), a right cylinder (3.5), or a plane (3.8) respectively. Also, the base curve must be a helix as a first condition to generate a generalized cylinder with geodesic base curve, but the base curve must be a planer as a first condition to generate a generalized cylinder with line of curvature or asymptotic base curve. Among all types of generalized cylinders, only three cylinders can be equipped with characteristic base curve.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

References

[1] Al-Ghefaria, R.A., Rashad, A.B., *An approach for designing a developable surface with a common geodesic curve*, International Journal of Contemporary Mathematical Sciences, **8**(18)(2013), 875–891.

[2] Althibany, N.M., *Classification of ruled surfaces family with common characteristic curve in Euclidean 3-space*, Turkish Journal of Science, **6**(2)(2021), 61–70.

[3] Althibany, N.M., *Construction of developable surface with geodesic or line of curvature coordinates*, Journal of New Theory, **36**(2021), 75–87.

[4] Althibany, N.M., *Generating generalized cylinder with geodesic base curve according to Darboux frame*, Journal of New Theory, **37**(2021), 99–107.

[5] Bayram, E., Bilici, M., *Surface family with a common involute asymptotic curve*, International Journal of Geometric Methods in Modern Physics, **13**(05)(2016), 1650062.

[6] Bayram, E., Gŭler, F., Kasap, E., *Parametric representation of a surface pencil with a common asymptotic curve*, Computer-Aided Design, **44**(2012), 637–643.

[7] Bilici M., Bayram, E., *Surface construction with a common involute line of curvature*, Int. J. Open Problems Compt. Math, **14**(2)(2021).

[8] Bo, P., Wang, W., *Geodesic-controlled developable surfaces for modeling paper bending*, Comput. Graph. Forum, **26**(2007), 365–374.

[9] Carmo, M.D., Differential Geometry of Curves and Surfaces, Prentice-Hall, New Jersey, 1976.

[10] Chang, K.H., Product Design Modeling Using CAD/CAE, the Computer Aided Engineering Design Series, Academic Press, 2014.

[11] Georgiev, G.H., Dinkova, C.L., *Focal curves of geodesics on generalized cylinders*, ARPN J. of Engineering and Applied Sciences, **14**(11)(2019), 2058–2068.

[12] Goldman, R., An Integrated Introduction to Computer Graphics and Geometric Modeling, CRC Press, 2009.

[13] Guha, S., Computer Graphics Through OpenGL: From Theory to Experiments, Chapman and Hall/CRC, 2018.

[14] Helmut, P., Andreas, A., Michael, H., Axel, K., Architectural Geometry, Bentley Institute Press, 2007.

[15] Honda, S., *Rectifying developable surfaces of framed base curves and framed helices*, Adv. Stud. Pure Math., **78**(2018), 273–292.

[16] Honda, A., Naokawa, K., Saji, K., Umehara, M., Yamada, K., *Curved foldings with common creases and crease patterns*, Advances in Applied Mathematics, **121**(2020), 102083.

[17] Izumiya, S., Katsumi, H., Yamasaki, T., *The rectifying developable and the spherical Darboux image of a space curve*, Banach Center Publ., **50**(1999), 137–149.

[18] Li, C.Y., Wang, R.H., Zhu, C.G., *Parametric representation of a surface pencil with a common line of curvature*, Comput Aided Des., **43**(9)(2011), 1110–1117.

[19] Li, C.Y., Wang, R.H., Zhu, C.G., *An approach for designing a developable surface through a given line of curvature*, Computer-Aided Design, **45**(3)(2013), 621–627.

[20] Pressley, A.N., Elementary Differential Geometry, Springer Science & Business Media, 2010.

[21] Shtogrin, M.I., *Bending of a piecewise developable surface*, Proceedings of the Steklov Institute of Mathematics, **275**(2011), 133–154.

[22] Struik, D.J., Lectures on Classical Differential Geometry, 2nd Edition, Dover Publications Inc., New York, 1988.

[23] Wang, G.J., Tang, K., Tai, C.L., *Parametric representation of a surface pencil with a common spatial geodesic* , Computer-Aided Design, **36**(5)(2004), 447–459.