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Global Weak Solution, Uniqueness and Exponential Decay for a Class of Degenerate Hyperbolic Equation

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Abstract
This paper deals with the existence, uniqueness, and energy decay of solutions for a degenerate hyperbolic equation given by

$$K(x,t)u'' - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u - \Delta u' = 0,$$

with operator coefficient $K(x,t)$ satisfying suitable properties and $M(\cdot) \in C^1([0,\infty))$ is a function such that the greatest lower bound is zero. For global weak solutions and uniqueness, we apply the Faedo-Galerkin method. For the exponential decay, we use a theorem due to M. Nakao.

Keywords: Degenerate hyperbolic equations, Exponential decay, Global weak solution
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Dedicated in memory to Raúl Moisés Izaguirre Maguiña,
Academic Vice Rector from Universidad Nacional Mayor de San Marcos, Lima, Perú.

1. Introduction

In this work, we will be focused on the existence, uniqueness, and exponential decay of global weak solution to the problem associated with the degenerate hyperbolic equation

$$K(x,t)u'' - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u - \Delta u' = 0, \text{ in } Q = \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \text{ on } \Sigma = \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\ x \in \Omega, \tag{1.3}$$

where Ω is a bounded open set of $\mathbb{R}^n$ $(n \geq 1)$, with smooth boundary $\partial\Omega$ and $T > 0$ is a fixed but arbitrary real number. $u(x,t)$ represents the transversal displacement of a spacial variable $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ at time $t > 0$, u' denotes the derivative of u with respect to time. $M(\cdot)$ is a $C^1([0,\infty))$ function such that $M(\lambda)\geq 0$, for all $\lambda\in[0,\infty)$ and the operator coefficient $K(x,t)\in C^1([0,T],L^\infty(\Omega))$ satisfying suitable properties. By standard notation,

$$|\nabla u(x,t)|^2=\sum_{i=1}^{n}\Big|\frac{\partial u(x,t)}{\partial x_i}\Big|^2 \text{ and } \Delta u(x,t)=\sum_{i=1}^{n}\frac{\partial^2 u(x,t)}{\partial x_i^2} \text{ is the Laplace operator.}$$

Equation (1.1) with $K(x,t)=1$ has its origin in the nonlinear vibration of an a stretched string and was considered in [1]. Existence of global solution was proved for $K(x,t)\geq 0$ and $M=1$ in [2], see also [3]. For a background and physical properties of this model we refer the reader to [4]-[7].

In fact,

$$u''-M\left(\int_\Omega|\nabla u|^2\,dx\right)\Delta u+\alpha u'=0 \text{ in } Q=\Omega\times(0,T), \tag{1.4}$$

when $M(\lambda)\geq m_0>0$ is known as non-degenerate, and for $\alpha=0$, global solutions have been obtained by several authors under various assumption, see [8]-[13].

The operator coefficient $K(x,t)$ plays an important role in the asymptotic behaviour for equation (1.1). The energy of the equation (1.1) is given by

$$E(t)=\frac{1}{2}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t)))\right]$$

being

$$\widehat{M}(t)=\int_0^t M(s)\,ds \tag{1.5}$$

and

$$a(u,v)=\int_\Omega \nabla u\nabla v\,dx \text{ the Dirichlet's form, for which we write } a(u) \text{ instead of } a(u,u).$$

When $K(x,t)=1$, for non-degenerate case, with $\alpha>0$, exponential decay properties was studied in [23]-[26]. However, the decay rate of the solutions is not so fast in the degenerate case. In fact, in [1], for example was showed that the problem (1.4) was a polynomial rate of decay given by $E(t)\leq Ct^{-\left(\frac{\alpha+1}{\alpha}\right)}$.

Another example presented by J. G. Dix [27], fully transcribed here, shows that decay of solutions is not necessarily exponential. Consider for $\Omega=(0,2\pi)\in\mathbb{R}$,

$$\begin{aligned} u''-M\left(\|u_x\|^2\right)u_{xx}+u'=0,\ x\in\Omega, t\geq 1+\sqrt{2},\\ u\left(x,1+\sqrt{2}\right)=\frac{1}{\sqrt{\pi}}e^{1/(1+\sqrt{2})}\sin(x),\\ u'\left(x,1+\sqrt{2}\right)=\frac{1}{9\sqrt{\pi}}e^{1/(1+\sqrt{2})}\sin(x),\\ u(0,t)=0,\ \ u(2\pi,t)=0,\ \text{for } t\geq 1+\sqrt{2}, \end{aligned}$$

where M is the non-negative and continuous function defined as

$$M(r)=\begin{cases}\frac{1}{16}\ln^2(r)(4-4\ln(r)-\ln^2(r)), & \text{if } 1\leq r\leq e^{2/(1+\sqrt{2})},\\ 0, & \text{otherwise.}\end{cases}$$

Then $u(x,t)=\frac{1}{\sqrt{\pi}}e^{1/t}\sin(x)$ is a solution. Since

$$u'=-\frac{1}{t^2}u,\ \ u''=\left(\frac{1}{t^4}+\frac{2}{t^3}\right)u,\ \ u_x=\frac{1}{\sqrt{\pi}}e^{1/t}\cos(x),\ \ u_{xx}=-u,$$

$\|u_x\|^2 = e^{2/t}$, and $M(e^{2/t}) = \frac{1}{t^2} - \frac{2}{t^3} - \frac{1}{t^4}$ for $t \geq 1+\sqrt{2}$, it follows that u satisfies the initial-value problem. Notice that $\|u'\|$ decays polynomially rather than exponentially as $t \to \infty$. In fact, $\|u'\|^2 = \frac{1}{t^4}e^{2/t}$.

Moreover, when is considered the nonhomogeneous equation $u'' - M\left(\|u_x\|^2\right) u_{xx} + u' = f(x,t)$, and a general non-constant function M, in spite of the convergence of $\|u'\|$ to zero remains illusive, that is, was not verified it and was not presented a counter-example, was proved in [27] that if $||f(x,t)||$ is square integrable on $[0,\infty)$ then $\|u'\|$ is square integrable on $[0,\infty)$.

On the other hand, when the greatest lower bound for $M(\lambda)$ is zero, the equation (1.4) is known as degenerate, see [14]-[16]. The degenerate equation (1.1) studied in this manuscript has been considered in just a few publications, see for instance [17, 18] and references therein.

It is well known that the Cauchy problem is well-posed for strictly hyperbolic differential equations. However, in dimension one, the Cauchy problem associated with degenerate hyperbolic equations is not well-posed. See [19]. Despite this, nonlinear degenerate hyperbolic equations are one of the most important classes of partial differential equations. We present some results in the literature in several contexts. For linear and semilinear equations of Tricomi type, existence, uniqueness, and qualitative properties of weak solutions to the degenerate hyperbolic Goursat problem, which play a very important part in applied and engineering sciences, was established in [20]. In [21] was considered the generalized Riemann problem for the Suliciu relaxation system in Lagrangian coordinates. The Suliciu relaxation system can be considered as a simplified viscoelastic shallow fluid model. Recently, the mixed Cauchy problem with lateral boundary condition for noncharacteristic degenerate hyperbolic equations was analyzed in [22], where, unlike other works on mixed Cauchy that the problems under consideration are obtained in weighted spaces, authors obtained all solutions in classical Sobolev spaces. Then, in the context above, the degenerate equation gives us a feature yield several striking phenomena that require new mathematical ideas, approaches, and theories.

The outline of this manuscript is the following. In Section 2 we introduce the notation, necessary assumptions and the main results. The proof of the existence theorem is performed in section 3, in three steps: approximate problem, a priori estimates and passage to the limit in the approximated equation. The uniqueness of the solution is given in section 4. Finally in section 5 the asymptotic behaviour is studied where we prove the exponential decay by using the Nakao method.

2. Preliminaries and Main Results

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$. By $H^m(\Omega)$, m a non-negative integer, we denote the Sobolev space of order m. For $m = 0$, $H^0(\Omega) = L^2(\Omega)$. Further, we set $H_0^m(\Omega) =$ the closure of $\mathscr{D}(\Omega)$ in $H^m(\Omega)$, where $\mathscr{D}(\Omega)$ is the space of infinitely continuously differentiable functions with compact support contained in Ω. The inner product and norm in $L^2(\Omega)$ and $H_0^1(\Omega)$ are represented by $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $||\cdot||$ respectively. The space $H_0^1(\Omega)\cap H^2(\Omega)$ is equipped with the norm $|\Delta u|$.

As in [29] for $T > 0$ a real number and B a Banach space, we denote

$$L^p(0,T,B) = \left(\text{u mensurable from [0,T] into B} \;\middle|\; \begin{array}{l} \left(\int_0^T ||u(t)||_B^p\,dt\right)^{\frac{1}{p}} < \infty, \text{ if } 1 \leq p < \infty, \\ \sup\limits_{0<t<T} \text{ess}\, ||u(t)||_B < \infty, \text{ if } p = \infty. \end{array} \right).$$

From now and on, let us assume that the volume density function $K(x,t)$ satisfies:

(H.1) $K(x,t) \in C^1([0,T], L^\infty(\Omega))$, $K(x,t) \geq 0$ and $K(x,0) \geq C_0 > 0$ for some $C_0 \in \mathbb{R}$.

(H.2) $\left|\frac{\partial K(x,t)}{\partial t}\right| \leq \gamma + C(\gamma)K(x,t)$, for all $\gamma > 0$.

In this manuscript, we deal with a degenerate case, then we consider that $M(\lambda)$, $\lambda > 0$, a real function satisfying

(H.3) $M(\lambda) \in C^1([0,\infty))$ with $M(\lambda) \geq 0$, for all, $\lambda > 0$.

The well-posedness of problem (1.1) is ensured by

Theorem 2.1. *For $u_0, u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ there exists a unique function $u : [0,T] \to L^2(\Omega)$ with the following regularity*

$$u \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \tag{2.1}$$
$$u' \in L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \tag{2.2}$$
$$u'' \in L^2(0,T; H_0^1(\Omega)), \tag{2.3}$$

such that

$$K(x,t)u'' - M(a(u(t)))\Delta u - \Delta u' = 0 \text{ in } L^2(Q), \tag{2.4}$$
$$u(x,t) = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \tag{2.5}$$
$$u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\ x \in \Omega. \tag{2.6}$$

Remark 2.2. *From (2.1), (2.2), (2.3) we have that $u \in C^0([0,T], H_0^1(\Omega) \cap H^2(\Omega))$ and $u' \in C^0([0,T], H_0^1(\Omega))$ so the initial conditions (2.6) are well set.*

For asymptotic behaviour the exponential stability is given by

Theorem 2.3. *Under the hypothesis of Theorem 2.1, the energy $E(t)$ associated to equation (1.1) satisfies*

$$E(t) \le C_0 e^{-\alpha t}, \text{ for all } t \ge 0, \text{ where } C_0 \text{ and } \alpha \text{ are positive constants.}$$

3. Existence of Solution

The aim of this section is to prove the theorem (2.1). For this goal, we use the Faedo-Galerkin method, a standard technique well described in the book by Temam [30].

3.1 Step 1. Perturbed approximate problem

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^1(\Omega) \cap H^2(\Omega)$ consisting of eigenvectors of the operator $-\Delta$, that is,

$$-\Delta w_j = \lambda_j w_j,\ j = 1,2,\cdots,n,\cdots$$

where $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le \cdots$, $\lambda_n \to \infty$ as $n \to \infty$, $w_j\big|_{\partial\Omega} = 0,\ j = 1,2,\cdots$, and $V_m = [w_1,\cdots,w_m]$ is the $H_0^1(\Omega) \cap H^2(\Omega)$ subspace generated by the first m eigenfunctions.

For all $w \in V_m$, let

$$u_{\varepsilon m}(t) = \sum_{j=1}^{m} g_{j\varepsilon m}(t) w_j,\ 0 < \varepsilon < 1,$$

be a local solution of the approximated problem

$$((K+\varepsilon)u''_{\varepsilon m}, v) + M(a(u_{\varepsilon m}))\, a(u_{\varepsilon m}, v) + a(u'_{\varepsilon m}, v) = 0,\ \forall v \in V_m \tag{3.1}$$
$$u_{\varepsilon m}(0) = u_{0m} \longrightarrow u_0 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega), \tag{3.2}$$
$$u'_{\varepsilon m}(0) = u_{1m} \longrightarrow u_1 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega), \tag{3.3}$$

which exists in a interval $[0, T_{\varepsilon m})$, $0 < T_{\varepsilon m} \le T$, by virtue of Carathéodory's theorem, see [28]. The extension of the solution to the whole interval $[0,T]$ is a consequence of the following priori estimates.

3.2 Step 2. Priori estimates

(I) Replacing $w = u'_{\varepsilon m}(t)$ in perturbed approximate equation (3.1), we get

$$\frac{1}{2}\frac{d}{dt}(K, u'^2_{\varepsilon m}) + \frac{\varepsilon}{2}\frac{d}{dt}|u'_{\varepsilon m}|^2 + \frac{1}{2}M(a(u_{\varepsilon m}))\frac{d}{dt}a(u_{\varepsilon m}) + ||u'_{\varepsilon m}||^2 = \frac{1}{2}(\frac{\partial K}{\partial t}, u'^2_{\varepsilon m}). \tag{3.4}$$

From (1.5) we get

$$\frac{d}{dt}\widehat{M}(a(u_{\varepsilon m})) = M(a(u_{\varepsilon m}))\frac{d}{dt}a(u_{\varepsilon m}),$$

then, (H.2), (3.4) leads to

$$\frac{d}{dt}\left[(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))\right]+2||u_{\varepsilon m}'||^2\leq \gamma\mu||u_{\varepsilon m}'||^2+C(\gamma)(K,u_{\varepsilon m}'^2),$$

where $\mu^{1/2}$ is the Poincaré constant. Performing integration from 0 to t, $0<t\leq T_{\varepsilon m}$ we obtain

$$(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))+(2-\gamma\mu)\int_0^t||u_{\varepsilon m}'||^2\,ds\leq (K(0),u_{1m}^2)+\varepsilon|u_{1m}|^2+\widehat{M}(a(u_{0m}))+C(\gamma)\int_0^t(K,u_{\varepsilon m}'^2)\,ds. \tag{3.5}$$

Since $K(0)\in L^\infty(\Omega)$, by using (3.2), (3.3) and choosing $\gamma<2/C$ we obtain

$$(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))+(2-\gamma\mu)\int_0^t||u_{\varepsilon m}'||^2\,ds\leq C_1+C(\gamma)\int_0^t(K,u_{\varepsilon m}'^2)\,ds, \tag{3.6}$$

being $C_1>0$ a real constant independent of ε,m and t. Now, applying Gronwall's inequality in (3.6), we come to

$$(K,u_{\varepsilon m}'^2)+\varepsilon|u_{\varepsilon m}'|^2+\widehat{M}(a(u_{\varepsilon m}))+(2-\gamma\mu)\int_0^t||u_{\varepsilon m}'||^2\,ds\leq C_2,$$

with $C_2>0$ a real constant independent of ε,m and t. Therefore,

$$\begin{aligned}(K^{1/2}u_{\varepsilon m}')&\text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\ (\sqrt{\varepsilon}\,u_{\varepsilon m}')&\text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\ (u_{\varepsilon m}')&\text{ is bounded in } L^2(0,T;H_0^1(\Omega)).\end{aligned} \tag{3.7}$$

From (3.7) and of Fundamental Theorem of Calculus, that is, $u_{\varepsilon m}(t)=u_{\varepsilon m}(0)+\int_0^t u_{\varepsilon m}'(s)\,ds$, we have

$$(u_{\varepsilon m})\text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)). \tag{3.8}$$

(II) Replacing $v=u_{\varepsilon m}''(t)$ in equation (3.1), we get

$$(K,u_{\varepsilon m}''^2)+\varepsilon|u_{\varepsilon m}''|^2+M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}'')+\frac{1}{2}\frac{d}{dt}||u_{\varepsilon m}'||^2=0. \tag{3.9}$$

Note that

$$\begin{aligned}M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}'')&=M(a(u_{\varepsilon m}))\left[\frac{d}{dt}a(u_{\varepsilon m},u_{\varepsilon m}')-a(u_{\varepsilon m}')\right]\\ &=\frac{d}{dt}\left[M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}')\right]-2M'(a(u_{\varepsilon m}))a(u_{\varepsilon m},u_{\varepsilon m}')a(u_{\varepsilon m},u_{\varepsilon m}')-M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m}').\end{aligned}$$

Thereby

$$\begin{aligned}\left|\int_0^t M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}'')\,ds\right|&\leq \left|M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u_{\varepsilon m}')\right|+\left|M(a(u_{0m}))\,a(u_{0m},u_{1m}')\right|\\ &\quad+2\int_0^t\left|M'(a(u_{\varepsilon m}))a(u_{\varepsilon m},u_{\varepsilon m}')^2\right|\,ds+\int_0^t\left|M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m}')\right|\,ds.\end{aligned}$$

Since, $M(\lambda)\in C^1([0,\infty))$, then

$$M(a(u_{\varepsilon m}))\leq \sup_{m\geq 1}\{M(\lambda)\,:\,0\leq\lambda\leq \sup||u_{\varepsilon m}||_{L^\infty(0,T;H_0^1(\Omega))}\}\leq c$$

and

$$M'(a(u_{\varepsilon m}))\leq \sup_{m\geq 1}\{M(\lambda)\,:\,0\leq\lambda\leq \sup||u_{\varepsilon m}||_{L^\infty(0,T;H_0^1(\Omega))}\}\leq \overline{c},$$

with $c,\overline{c}$ positive constants independent of ε,m and t.

Then,

$$\left|\int_0^t M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u''_{\varepsilon m})\,ds\right| \le c\,\|u_{\varepsilon m}\|\,\|u'_{\varepsilon m}\| + C_3 + 2\overline{c}\int_0^t \|u_{\varepsilon m}\|^2\,\|u'_{\varepsilon m}\|^2\,ds + c\int_0^t \|u'_{\varepsilon m}\|^2\,ds.$$

From (3.7) and (3.8) we have

$$\left|\int_0^t M(a(u_{\varepsilon m}))\,a(u_{\varepsilon m},u''_{\varepsilon m})\,ds\right| \le C_4 + \alpha\,\|u'_{\varepsilon m}\|^2, \text{ with } C_4,\alpha \text{ positive constants independent of } \varepsilon,m \text{ and } t. \tag{3.10}$$

Integrating (3.9) from 0 to t, $0<t\le T$, and using the estimate (3.10) we obtain

$$\int_0^t (K,u''^2_{\varepsilon m})\,ds + \varepsilon\int_0^t |u''_{\varepsilon m}|^2\,ds + (\frac{1}{2}-\alpha)||u'_{\varepsilon m}||^2 \le C_4. \tag{3.11}$$

Choosing properly $0<\alpha<1/2$ we obtain directly from estimate (3.11)

$$\begin{aligned} &(K^{1/2}u''_{\varepsilon m}) \text{ is bounded in } L^2(Q),\\ &(\sqrt{\varepsilon}\,u''_{\varepsilon m}) \text{ is bounded in } L^2(Q),\\ &(u'_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)). \end{aligned} \tag{3.12}$$

(III) Now we will get an estimate for $u''_{\varepsilon m}(t)$. At this point we have an additional degree of difficulty. We first obtain an estimate for $u''_{\varepsilon m}(0)$. In this direction, taking $t=0$ and $v=u''_{\varepsilon m}(0)$ in equation (3.1) we obtain

$$((K(0),u''^2_{\varepsilon m}(0)) + \varepsilon|u''_{\varepsilon m}(0)|^2 + M(a(u_{0m}))\,a(u_{0m},u''_{\varepsilon m}(0)). + a(u'_{1m},u''_{\varepsilon m}(0)) = 0.$$

Since $K(0)\ge C_0>0$ we have

$$(C_0+\varepsilon)|u''_{\varepsilon m}(0)|^2 \le |M(a(u_{0m}))\Delta u_{0m} + \Delta u_{1m}|\,|u''_{\varepsilon m}(0)|,$$

therefore

$$|u''_{\varepsilon m}(0)| \le \tilde{c}, \text{ where } \tilde{c} \text{ is a positive constant independent of } \varepsilon,m \text{ and } t. \tag{3.13}$$

Deriving the approximate equation (3.1) with respect to t and making $v=u''_{\varepsilon m}(t)$ we obtain

$$(Ku'''_{\varepsilon m},u''_{\varepsilon m}) + (\frac{\partial K}{\partial t}u''_{\varepsilon m},u''_{\varepsilon m}) + \varepsilon(u'''_{\varepsilon m},u''_{\varepsilon m}) + \frac{d}{dt}\left[M(a(u_{\varepsilon m}))\right]a(u_{\varepsilon m},u''_{\varepsilon m}) + M(a(u_{\varepsilon m}))a(u'_{\varepsilon m},u''_{\varepsilon m}) + a(u''_{\varepsilon m}) = 0,$$

that is,

$$\frac{1}{2}\frac{d}{dt}(K,u''^2_{\varepsilon m}) + \frac{1}{2}(\frac{\partial K}{\partial t},u''^2_{\varepsilon m}) + \frac{\varepsilon}{2}\frac{d}{dt}|u''_{\varepsilon m}|^2 + \|u''_{\varepsilon m}\|^2 = -2M'(a(u_{\varepsilon m}))a(u_{\varepsilon m},u'_{\varepsilon m})a(u_{\varepsilon m},u''_{\varepsilon m}) - M(a(u_{\varepsilon m}))a(u'_{\varepsilon m},u''_{\varepsilon m}),$$

and then,

$$\frac{1}{2}\frac{d}{dt}\left[(K,u''^2_{\varepsilon m}) + \varepsilon|u''_{\varepsilon m}|^2\right] + \|u''_{\varepsilon m}\|^2 \le C_5 + \mu\frac{\gamma}{2}\|u''_{\varepsilon m}\|^2 + \frac{C(\gamma)}{2}(K,u''^2_{\varepsilon m}), \text{ with } C_5 \text{ independent of } \varepsilon,m \text{ and } t. \tag{3.14}$$

Integrating (3.14) from 0 to t, we obtain

$$\frac{1}{2}\left[(K,u''^2_{\varepsilon m}) + \varepsilon|u''_{\varepsilon m}|^2\right] + (1-\mu\frac{\gamma}{2})\int_0^t \|u''_{\varepsilon m}\|^2\,ds \le C_5 + C(\gamma)\int_0^t (K,u''^2_{\varepsilon m})\,ds + \frac{1}{2}\left[K(0),u''^2_{\varepsilon m}(0)) + \varepsilon|u''_{\varepsilon m}(0)|^2\right]. \tag{3.15}$$

By using (3.13) and Gronwall's inequality, (3.15) leads to

$$\frac{1}{2}\left[(K,u''^2_{\varepsilon m}) + \varepsilon|u''_{\varepsilon m}|^2\right] + (1-\mu\frac{\gamma}{2})\int_0^t \|u''_{\varepsilon m}\|^2\,ds \le C_6, \text{ with } C_6 \text{ a positive constant independent of } \varepsilon,m \text{ and } t.$$

Therefore,

$$(K^{1/2}u''_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.16}$$

$$(\sqrt{\varepsilon}\,u''_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.17}$$

$$(u''_{\varepsilon m}) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)). \tag{3.18}$$

(IV) Replacing $v = -\Delta u_{\varepsilon m}$ in the approximate equation (3.1), we obtain

$$((K+\varepsilon)u''_{\varepsilon m}, -\Delta u_{\varepsilon m}) + M(a(u_{\varepsilon m}))\, a(u_{\varepsilon m}, -\Delta u_{\varepsilon m}) + a(u'_{\varepsilon m}, -\Delta u_{\varepsilon m}) = 0,$$

that leads us to

$$\frac{1}{2}\frac{d}{dt}|-\Delta u_{\varepsilon m}|^2 \le K_0|-\Delta u_{\varepsilon m}||u''_{\varepsilon m}| + \varepsilon|-\Delta u_{\varepsilon m}||u''_{\varepsilon m}| + |M(a(u_{\varepsilon m}))||-\Delta u_{\varepsilon m}|^2, \text{ where } K_0 = \max_{0\le s\le T}\left(\operatorname*{supess}_{x\in\Omega} K(x,s)\right).$$

Performing integration from 0 to t, using Young's inequality and (3.18), we obtain

$$|-\Delta u_{\varepsilon m}|^2 \le C_7 + C_8\int_0^t |-\Delta u_{\varepsilon m}(s)|^2 ds.$$

Applying Gronwall's inequality we get

$$|-\Delta u_{\varepsilon m}|^2 \le C_9. \tag{3.19}$$

Then we obtain,

$$\|u_{\varepsilon m}\|^2_{H^2(\Omega)} \le C_9, \text{ where the constants } C_7, C_8, C_9 \text{ are positives and independent of } \varepsilon, m \text{ and } t.$$

In fact we have the following regularity

$$(u_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T;H^2(\Omega)). \tag{3.20}$$

(V) Replacing $v = -\Delta u'_{\varepsilon m}$ in approximated equation (3.1), we get

$$((K+\varepsilon)u''_{\varepsilon m}, -\Delta u'_{\varepsilon m}) + M(a(u_{\varepsilon m}))\, a(u_{\varepsilon m}, -\Delta u'_{\varepsilon m}) + a(u'_{\varepsilon m}, -\Delta u'_{\varepsilon m}) = 0,$$

then,

$$|-\Delta u'_{\varepsilon m}|^2 \le K_0|-\Delta u'_{\varepsilon m}||u''_{\varepsilon m}| + |M(a(u_{\varepsilon m}))||-\Delta u_{\varepsilon m}||-\Delta u'_{\varepsilon m}| + \varepsilon|u''_{\varepsilon m}||-\Delta u'_{\varepsilon m}|.$$

Performing integration from 0 to t, using Young's inequality, (3.18) and (3.19) we obtain

$$\int_0^t |-\Delta u'_{\varepsilon m}(s)|^2 ds \le C_{10} + \alpha\int_0^t |-\Delta u'_{\varepsilon m}(s)|^2 ds, \text{ thus } (1-\alpha)\int_0^t |-\Delta u'_{\varepsilon m}(s)|^2 ds \le C_{10}.$$

Then

$$\|u'_{\varepsilon m}\|^2_{H^2(\Omega)} \le C_{10},\ C_{10} \text{ independent of } \varepsilon, m \text{ and } t.$$

Therefore

$$(u'_{\varepsilon m}) \text{ is bounded in } L^2(0,T;H^2(\Omega)). \tag{3.21}$$

3.3 Step 3. Passage to the limit

From estimates (3.9), (3.12), (3.16), (3.17), (3.18), (3.20), and (3.21), there exists a subsequence of $(u_{\varepsilon m})$, denoted by same way, such that,

$$u_{\varepsilon m} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{3.22}$$

$$u'_{\varepsilon m} \rightharpoonup u' \text{ in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{3.23}$$

$$u''_{\varepsilon m} \rightharpoonup u'' \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$\sqrt{\varepsilon}\, u''_{\varepsilon m} \rightharpoonup 0 \text{ in } L^2(0,T;L^2(\Omega)).$$

$$Ku''_{\varepsilon m} \rightharpoonup Ku'' \text{ in } L^2(Q).$$

From compact immersion $H_0^1(\Omega)\cap H^2(\Omega)\hookrightarrow H_0^1(\Omega)$, by Aubin-Lions's lemma [29] follows that $u_{\varepsilon m}\to u$ in $L^2(0,T;H_0^1(\Omega))$, and so $a(u_{\varepsilon m})\to a(u)$ in $L^2(0,T)$, and, as $M\in C^1([0,\infty))$ we obtain

$$M(a(u_{\varepsilon m}))\to M(a(u)) \text{ in } L^2(0,T).$$

From (3.22) and (3.23) we wave that $\Delta u_{\varepsilon m}\rightharpoonup \Delta u$ in $L^2(Q)$, and $\Delta u'_{\varepsilon m}\rightharpoonup \Delta u'$ in $L^2(Q)$. Thereby,

$$M(a(u_{\varepsilon m}))\Delta u_{\varepsilon m}\rightharpoonup M(a(u))\Delta u \text{ in } L^2(Q).$$

Now consider the approximated equation

$$(K+\varepsilon)u''_{\varepsilon m}-M(a(u_{\varepsilon m}))\Delta u_{\varepsilon m}-\Delta u'_{\varepsilon m}=0.$$

Making the inner product in $L^2(\Omega)$ by $\varphi\in L^2(\Omega)$ we obtain

$$((K+\varepsilon)u''_{\varepsilon m},\varphi)-(M(a(u_{\varepsilon m}))\Delta u_{\varepsilon m},\varphi)-(\Delta u'_{\varepsilon m},\varphi)=0.$$

Taking the limit with $m\to\infty$ and $\varepsilon\to 0$, we get

$$((Ku'',\varphi)-(M(a(u))\Delta u,\varphi)-(\Delta u',\varphi)=0, \text{ for all } \varphi\in L^2(Q), \text{ and then (2.4) is proven.}$$

The verification of the initial data (2.6) is obtained in a standard way.

4. Uniqueness of Solution

Consider u and $\widehat{u}$ with the hypotheses of regularity (2.1), (2.2) of Theorem 2.1. Then, $w=u-\widehat{u}$ is solution of the equation

$$Kw''-(M(a(u))\Delta w-[M(a(u))-M(a(\widehat{u}))]\,\Delta\widehat{u}-\Delta w'=0, \tag{4.1}$$

with initial conditions

$$w(0)=0 \text{ and } w'(0)=0. \tag{4.2}$$

Taking the inner product in $L^2(\Omega)$ on both sides of (4.1) with w,w' and w'' respectively, we get

$$\begin{aligned}(Kw'',w)+(M(a(u))a(w)+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w)+a(w',w)&=0,\\ (Kw'',w')+(M(a(u))a(w,w')+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w')+a(w')&=0,\\ (K,w''^2)+(M(a(u))a(w,w'')+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w'')+a(w',w'')&=0,\end{aligned}$$

that is

$$\begin{aligned}(Kw'',w)+(M(a(u))\|w\|^2+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w)+\frac{1}{2}\frac{d}{dt}\|w\|^2&=0,\\ \frac{1}{2}\frac{d}{dt}(K,w'^2)-\frac{1}{2}(\frac{\partial K}{\partial t},w'^2)+\frac{1}{2}(M(a(u))\frac{d}{dt}\|w\|^2+\|w'\|^2+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w')&=0,\\ (K,w''^2)+(M(a(u))a(w,w'')+[M(a(u))-M(a(\widehat{u}))]\,a(\widehat{u},w'')+\frac{1}{2}\frac{d}{dt}\|w'\|^2&=0.\end{aligned}$$

Adding the last three equations above and integrating from 0 to t, we obtain

$$\begin{aligned}&\int_0^t(K,w''^2)\,ds+\frac{1}{2}(K,w'^2)+\frac{1}{2}M(a(u))\|w\|^2+\frac{1}{2}\|w\|^2+\frac{1}{2}\|w'\|^2+\int_0^t\|w'\|^2\,dx\\ &\qquad=\int_0^t\left\{\frac{1}{2}(\frac{\partial K}{\partial t},w'^2)-(Kw'',w)-M(a(u))\|w\|^2-M(a(u))a(w,w'')\right\}ds\\ &\qquad+\int_0^t\left\{[M(a(\widehat{u}))-M(a(u))]\,[a(\widehat{u},w)+a(\widehat{u},w')+a(\widehat{u},w'')]+M'(a(u))a(u,u')\|w\|^2\right\}ds.\end{aligned}$$

Note that

$$\frac{1}{2}(\frac{\partial K}{\partial t},w'^2)\le\delta C\|w'\|^2+C(\delta)(K,w'^2),$$

and

$$\int_0^t (Kw'', w)\, ds = (Kw', w) - \int_0^t (\frac{\partial K}{\partial t}, w'w)\, ds - \int_0^t (K, w'^2)\, ds$$
$$\leq C_1 \|w'\| \|w\| + C_2 \int_0^t \|w'\| \|w\|\, ds + C(\delta) C_1 \int_0^t \|w'\| \|w\|\, ds + \int_0^t (K, w'^2)\, ds.$$

Then we have,

$$\int_0^t (Kw'', w)\, ds \leq \alpha \|w'\|^2 + \frac{C_3}{\alpha} \|w\|^2 + \int_0^t (K, w'^2)\, ds$$
$$\leq \alpha \|w'\|^2 + C_4 \int_0^t \|w\|^2\, ds + C_5 \int_0^t \|w'\|^2\, ds + \int_0^t (K, w'^2)\, ds.$$

Besides that,

$$\begin{aligned}
[M(a(\widehat{u})) - M(a(u))] \left[a(\widehat{u}, w) + a(\widehat{u}, w') + a(\widehat{u}, w'') \right] &\leq |M'(\xi)| |a(\widehat{u}) - a(u)| \|\widehat{u}\| \|w\| + \|\widehat{u}\| \|w'\| + \|\widehat{u}\| w''\| \\
&= |M'(\xi)| |(\|\widehat{u}\| - \|u\|)(\|\widehat{u}\| + \|u\|)| \|\widehat{u}\| (\|w\| + \|w'\| + \|w''\|) \\
&\leq |M'(\xi)| |\widehat{u} - u\|)(\|\widehat{u}\| + \|u\|)| \|\widehat{u}\| (\|w\| + \| \|w'\| + \|w''\|) \\
&= |M'(\xi)| \|w\| (\|\widehat{u}\| + \|u\|) \|\widehat{u}\| (\|w\| + \| \|w'\| + \|w''\|) \\
&\leq C_6 \|w\|^2 + C_7 \|w'\|^2 + C_8 \|w\| \|w''\|
\end{aligned}$$

and

$$M(a(u)) a(w, w'') = M(a(u)) \left[\frac{d}{dt} a(w, w') - a(w') \right]$$
$$= \frac{d}{dt} \left[M(a(u)) a(w, w') \right] - 2M(a(u)) a(u, u') a(w, w') - M(a(u)) a(w'),$$

then,

$$\int_0^t M(a(u)) a(w, w')\, ds \leq C_9 \|w\| \|w'\| + C_{10} \int_0^t \|w\| \|w'\|\, ds + C_{11} \int_0^t \|w'\|^2\, ds$$
$$\leq \alpha \|w'\|^2 + C_{12} \int_0^t \|w\|^2\, ds + C_{13} \int_0^t \|w'\|^2\, ds.$$

Therefore,

$$\frac{1}{2}(K, w'^2) + \frac{1}{2} M(a(u)) \|w\|^2 + \frac{1}{2} \|w\|^2 + \left(\frac{1}{2} - 2\alpha \right) \|w'\|^2$$
$$\leq \int_0^t \left[(1 + C(\gamma))(K, w'^2) + M(a(u)) \|w\|^2 + C_{14} \|w\|^2 + C_5 \|w'\|^2 \right] ds + C_8 \int_0^t \|w\| \|w''\|\, ds.$$

Then,

$$(K, w'^2) + M(a(u)) \|w\|^2 + \|w\|^2 + (1 - 4\alpha) \|w'\|^2$$
$$\leq c \int_0^t \left[(K, w'^2) + M(a(u)) \|w\|^2 + \|w\|^2 + (1 - 4\alpha) \|w'\|^2 \right] ds + \overline{c} \int_0^t \|w\| \|w''\|\, ds.$$

Now, we denote

$$\varphi(t) = (K, w'^2) + M(a(u)) \|w\|^2 + \|w\|^2 + (1 - 4\alpha) \|w'\|^2$$

and we obtain

$$\varphi(t) \leq c \int_0^t \varphi(s)\, ds + \overline{c} \int_0^t g(s) \varphi^{1/2}(s)\, ds, \text{ where } g(s) = \|w''\| \in L^1(0, T).$$

Then, we have $\varphi(t) = 0,$ for all $t \in [0, T]$ and finally $w = 0$, that is, $u = \widehat{u}$ which proves the uniqueness of solution.

5. Asymptotic Behaviour

In this section we prove the exponential decay of solution to the problem (1.1)-(1.3). Let start by present the following result:

Lemma 5.1 (Nakao's Lemma, [31])**.** *Suppose that $E(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\operatorname*{supess}_{t\le s\le t+1} E(s) \le C[E(t)-E(t+1)],\ \ \textit{for } t\ge 0, \textit{ where } C \textit{ is a positive constant.}$$

Then, we have

$$E(t)\le Ce^{-\alpha t},\ \textit{with } \alpha=\frac{1}{C+1},\ \textit{for all } t\ge 0.$$

The main result of this section is given by the following theorem:

Theorem 5.2. *Under the hypotheses of Theorem 2.1, the energy associated with the system (1.1)-(1.3) satisfies*

$$E(t)\le Ce^{-\alpha t},\ \ \textit{for all } t\ge 0, \textit{ where } C \textit{ and } \alpha \textit{ are positive constants.}$$

Proof. Multiplying (1.1) by u_t and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t))\right]+\|u'(t)\|^2=\frac{1}{2}(\frac{\partial K}{\partial t},u'(t)),\ \text{where, } \widehat{M}(t)=\int_0^t M(s)\,ds.$$

By (H.2) we have

$$|(\frac{\partial K}{\partial t},u'^2(t))|\le\gamma|u'^2(t))|^2+C(\gamma)|(K,u'^2(t))|\le\mu(\delta+C(\gamma)K_0)|u'(t)|^2,$$

with

$$K_0=\max_{t\le s\le T}\left(\operatorname*{supess}_{x\in\Omega}K(x,s)\right),\ \text{and } \mu>0 \text{ is a constant such that } |\varphi|^2\le\mu\|\varphi\|^2,\ \varphi\in H_0^1(\Omega).$$

Whence follows that

$$\frac{1}{2}\frac{d}{dt}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t))\right]+[1-\mu(\gamma+C(\gamma)K_0)]\,\|u'(t)\|^2\le 0, \tag{5.1}$$

where $\gamma>0$ is sufficiently small such that $1-\mu(\gamma+C(\gamma)K_0)>0$.

Now, its important to remember that $E(t)=\frac{1}{2}\left[|K^{1/2}u'(t)|^2+\widehat{M}(a(u(t)))\right]$.

Integrating (5.1) from t to $t+1$, we obtain

$$\int_t^{t+1}|u'(s)|^2\,ds\le\mu\int_t^{t+1}\|u'(s)\|^2\,ds\le C_{15}\,[E(t)-E(t+1)]\overset{\text{def}}{=}F^2(t),\ \textit{with } C_{15}=\frac{\mu}{1-\mu(\gamma+C(\gamma)K_0)}>0. \tag{5.2}$$

Therefore, from (5.2), there exist $t_1\in\left[t,t+\frac{1}{4}\right]$ and $t_2\in\left[t+\frac{3}{4},t+1\right]$ such that $|u'(t_i)|\le 2F(t),\ \ i=1,2$.

The inner product in $L^2(\Omega)$ of (1.1) with $u(t)$ implies

$$\frac{d}{dt}(Ku'(t),u(t))-|K^{1/2}u'(t)|^2+M(a(u))a(u)+((u'(t),u(t)))=(\frac{\partial K}{\partial t}u'(t),u(t)).$$

Integrating from t_1 to t_2 and by using (H.2) we have

$$\begin{aligned}\int_{t_1}^{t_2}M(a(u))a(u)\,dt\le{}&K_0|u'(t_1)||u(t_1)|+K_0|u'(t_2)||u(t_2)|\\&+\mu K_0\int_{t_1}^{t_2}\|u'(s)\|^2\,ds+\int_{t_1}^{t_2}\|u'(s)\|\|u(s)\|\,ds\\&+\gamma\sqrt{\mu}\int_{t_1}^{t_2}|u'(s)|\|u(s)\|\,ds+C(\gamma)K_0\sqrt{\mu}\int_{t_1}^{t_2}|u'(s)|\|u(s)\|\,ds.\end{aligned} \tag{5.3}$$

Now,

$$M(a(u))a(u) \geq m_0 a(u) = m_0\|u\|^2, \text{ where } m_0 = \min_{0\leq s\leq a(u)} M(s) > 0. \tag{5.4}$$

Then, by (5.2), (5.3) and (5.4), we obtain

$$m_0 \int_{t_1}^{t_2} \|u(s)\|^2\, ds \leq 4\mu K_0 F(t) \operatorname*{supess}_{t\leq s\leq t+1} \|u(s)\| + C_{16}F^2(t) + \frac{3}{4} m_0 \int_{t_1}^{t_2} \|u(s)\|^2\, ds,$$

where $C_{16} = \mu K_0 + \dfrac{1}{m_0} + \dfrac{\mu\gamma^2}{m_0} + \dfrac{\mu C^2(\gamma)K_0^2}{m_0} > 0.$

Then we have,

$$\int_{t_1}^{t_2} \|u(s)\|^2\, ds \leq C_{17}F(t) \operatorname*{supess}_{t\leq s\leq t+1} \|u(s)\| + C_{18}F^2(t) \stackrel{\text{def}}{=} G^2(t), \text{ being } C_{17} = \frac{4\mu K_0}{m_0} \text{ and } C_{18} = \frac{4C}{m_0}. \tag{5.5}$$

From (5.2) and (5.5) we obtain

$$\int_{t_1}^{t_2} \left[|u'(s)|^2 + \|u(s)\|^2\right] ds \leq F^2(t) + G^2(t). \tag{5.6}$$

$$\text{Thus, by (5.6) there exists } t^* \in [t_1,t_2] \text{ such that } |u'(t^*)|^2 + \|u(t^*)\|^2 \leq 2[F^2(t) + G^2(t)]. \tag{5.7}$$

Now, not that,

$$\widehat{M}(a(u((t^*)))) \leq m_1\|u(t^*)\|^2 \leq 2m_1[F^2(t) + G^2(t)], \text{ with } m_1 = \max_{0\leq s\leq a(u(t^*))} M(s). \tag{5.8}$$

$$\text{From (5.7) and (5.8), we have } E(t^*) \leq C_{16}[F^2(t) + G^2(t)]. \tag{5.9}$$

$$\text{Since that } E(t) \text{ is increasing, we obtain } \operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq E(t^*) + [1 - \mu(\gamma + C(\gamma))K_0 \int_t^{t+1} \|u'(s)\|^2\, ds. \tag{5.10}$$

Now, by (5.2), (5.9) and (5.10), we get $\operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq C_{17}[F^2(t) + F(t) \operatorname*{supess}_{t\leq s\leq t+1} \|u'(s)\| \leq C_{18}F^2(t) + \dfrac{1}{2} \operatorname*{supess}_{t\leq s\leq t+1} E(s).$

Then, by (5.2) $\operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq C[E(t) - E(t+1)]$, where $C_i,\ i = 15, 16, 17, 18$ and C are positive constants.

Therefore, by Nakao's lemma, we obtain $E(t) \leq Ce^{-\alpha t}$, with $\alpha = \dfrac{1}{C+1}$, for all $t \geq 0$.

The exponential decay of the solution was been proven. □

6. Conclusion

We prove the existence, uniqueness, and exponential stability of the solution to a degenerate hyperbolic equation where the greatest lower bound for Kirchhoff function $M(\cdot)$ is zero. We consider strong damping as a stabilization mechanism. We have improved previous results in the literature, mainly because the exponential decay for this type of problem, as far as we know, has not been previously considered.

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