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Research Article

Almost complex structures on coframe bundle with Cheeger-Gromoll metric

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Abstract

In this paper we introduce several almost complex structures compatible with Cheeger-Gromoll metric on the coframe bundle and investigate their integrability conditions.

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1. Introduction

The geometric structures of the fiber bundles over Riemannian manifold (M, g) is one of the essential topics in the differential geometry. First Sasaki [13] constructed a Riemannian metric ${}^{S}g$ on the tangent bundle $T(M)$ which depend only on the base manifold. Kowalski [8] proved that if the Sasaki metric ${}^{S}g$ is locally symmetric, then the base metric g is flat and hence ${}^{S}g$ is also flat. Musso and Tricerri [10] obtained an explicit expression of the Cheeger-Gromoll metric ${}^{CG}g$ introduced by Cheeger and Gromoll in [3] (see also [6]). Sekizawa [14] defined some geometric objects related ${}^{CG}g$. Tahara, Vanhecke and Watanabe [15] constructed several almost complex structures compatible with some natural defined Riemannian metrics on the tangent bundle of an almost Hermitian manifold. Bejan and Druţă [2] defined harmonic almost complex structures with respect to general natural metrics in the tangent bundle. In [9] Munteanu introduced Cheeger-Gromooll type metrics and showed the conditions for which the tangent bundle is almost Kahlerian or Kahlerian (see also [7]).To construct an almost Hermitian structure on the cotangent bundle $T^*(M)$ of a Riemannian manifold (M, g) Oproiu and Poroşniuc used some natural lifts of geometric objects [11]. (see also [4]).

In this paper, we construct an almost Hermitian structures on the bundle of linear coframes $F^*(M)$ over a Riemannian manifold (M, g) with the Cheeger-Gromoll metric ${}^{CG}g$. In 2 we briefly describe the definitions and results that are needed later, after which the adapted frame on coframe bundle $F^*(M)$ introduced in 3. The Cheeger-Gromoll metric ${}^{CG}g$ on $F^*(M)$ and its Levi-Civita connection ${}^{CG}\nabla$ are determined in 4. In 5 we define an almost Hermitian structures $({}^{CG}g, J_\beta), \beta = 1, 2, ..., n$, on the linear coframe bundle $F^*(M)$. The integrability conditions for almost complex structures $J_\beta, \beta = 1, 2, ..., n$, are studied in 6.

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2. Preliminaries

In this section we shall summarize briefly the main definitions and results which be used later. Let (M, g) be an $n-$dimensional Riemannian manifold. Then the linear coframe bundle $F^*(M)$ over M consists of all pairs (x, u^*), where x is a point of M and u^* is a basis (coframe) for the cotangent space $T_x^* M$ of M at x [5]. We denote by π the natural projection of $F^*(M)$ to M defined by $\pi(x, u^*) = x$. If $(U; x^1, x^2, ..., x^n)$ is a system of local coordinates in M, then a coframe $u^* = (X^\alpha) = (X^1, X^2, ..., X^n)$ for $T_x^* M$ can be expressed uniquely in the form $X^\alpha = X_i^\alpha (dx^i)_x$. From mentioned above it follows that

$$\left(\pi^{-1}(U); x^1, x^2, ..., x^n, X_1^1, X_2^1, ..., X_n^n\right)$$

is a system of local coordinates in $F^*(M)$ (see, [5]), that is $F^*(M)$ is a C^∞ manifold of dimension $n + n^2$. We note that indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1, 2, ..., n\}$, while indices $A, B, C, ...$ have range in $\{1, ..., n, n+1, ..., n+n^2\}$. We put $i_\alpha = \alpha \cdot n + i$. Obviously that indices $i_\alpha, j_\beta, k_\gamma, ...$ have range in $\{n+1, n+2, ..., n+n^2\}$. Summation over repeated indices is always implied. Let ∇ be a symmetric linear connection on M with components Γ_{ij}^k. Then the tangent space $T_{(x,u^*)}(F^*(M))$ of $F^*(M)$ at $(x, u^*) \in F^*(M)$ splits into the horizontal and vertical subspaces with respect to ∇ :

$$T_{(x,u^*)}(F^*(M)) = H_{(x,u^*)}(F^*(M)) \oplus V_{(x,u^*)}(F^*(M)). \tag{2.1}$$

We denote by $\Im_s^r(M)$ the set of all differentiable tensor fields of type (r, s) on M. From (2.1) it follows that for every $X \in \Im_0^1(F^*(M))$ is obtained unique decomposing $X = hX + vX$, where $hX \in H(F^*(M))$, $vX \in V(F^*(M))$. $H(F^*(M))$ and $V(F^*(M))$ the horizontal and vertical distributions for $F^*(M)$, respectively. Now we define naturally n different vertical lifts of $1-$form $\omega \in \Im_1^0(M)$. If Y be a vector field on M, i.e. $Y \in \Im_0^1(M)$, then $i^\mu Y$ are functions on $F^*(M)$ defined by $(i^\mu Y)(x, u^*) = X^\mu(Y)$ for all $(x, u^*) = (x, X^1, X^2, ..., X^n) \in F^*(M)$, where $\mu = 1, 2, ..., n$. The vertical lifts ${}^{V_\lambda}\omega$ of ω to $F^*(M)$ are the n vector fields such that

$${}^{V_\lambda}\omega(i^\mu Y) = \omega(Y)\delta_\mu^\lambda$$

hold for all vector fields Y on M, where $\lambda, \mu = 1, 2, ..., n$ and δ_μ^λ denote the Kronecker's delta. The vertical lifts ${}^{V_\lambda}\omega$ of ω to $F^*(M)$ have the components

$${}^{V_\lambda}\omega = \begin{pmatrix} {}^{V_\lambda}\omega^k \\ {}^{V_\lambda}\omega^{k_\mu} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_k \delta_\mu^\lambda \end{pmatrix} \tag{2.2}$$

with respect to the induced coordinates (x^i, X_i^α) in $F^*(M)$ (see, [12]).

Let $V \in \Im_0^1(M)$. The complete lift ${}^C V \in \Im_0^1(F^*(M))$ of V to the linear coframe bundle $F^*(M)$ is defined by

$${}^C V(i^\mu Y) = i^\mu(L_V Y) = X_m^\mu (L_V Y)^m$$

for all vector fields $Y \in \Im_0^1(M)$, where L_V be the Lie derivation with respect to V. The complete lift ${}^C V$has the components

$${}^C V = \begin{pmatrix} {}^C V^k \\ {}^C V^{k_\mu} \end{pmatrix} = \begin{pmatrix} V^k \\ -X_m^\mu \partial_k V^m \end{pmatrix}$$

with respect to the induced coordinates (x^i, X_i^α) in $F^*(M)$.

The horizontal lift ${}^H V \in \Im_0^1(F^*(M))$ of V to the linear coframe bundle $F^*(M)$ is defined by

$${}^H V(i^\mu Y) = i^\mu(\nabla_V Y) = X_m^\mu (\nabla_V Y)^m$$

for all vector fields $Y \in \Im_0^1(M)$, where ∇_V be the covariant derivative with respect to V. The horizontal lift ${}^H V$ has the components

$$ {}^H V = \begin{pmatrix} {}^H V^k \\ {}^H V^{k_\mu} \end{pmatrix} = \begin{pmatrix} V^k \\ X_m^\mu \Gamma_{lk}^m V^l \end{pmatrix} \tag{2.3} $$

with respect to the induced coordinates (x^i, X_i^α) in $F^*(M)$, where Γ_{ij}^k are the components of Levi-Civita connection on M.

The bracket operation of vertical and horizontal vector fields is given by the formulas

$$ \begin{array}{l} [{}^{V_\beta}\omega, {}^{V_\gamma}\theta] = 0, \\ [{}^H X, {}^{V_\gamma}\theta] = {}^{V_\gamma}(\nabla_X \theta), \\ [{}^H X, {}^H Y] = {}^H[X,Y] + \sum\limits_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X,Y)) \end{array} \tag{2.4} $$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, where R is the Riemannian curvature of g. If f is a differentiable function on M, ${}^V f = f \circ \pi$ denotes its canonical vertical lift to the $F^*(M)$.

3. Adapted frames on $F^*(M)$

Suppose (U, x^i) be a local coordinate system in M. In $U \subset M$, we put

$$ X_{(i)} = \partial/(\partial x^i), \quad \theta^{(i)} = dx^i,\ i = 1, 2, ..., n. $$

Taking into account of (2.2) and (2.3), we see that

$$ {}^H X^{(i)} = D_i = \begin{pmatrix} \delta_i^j \\ X_m^\beta \Gamma_{ij}^m \end{pmatrix}, \tag{3.1} $$

$$ {}^{V_\alpha}\theta^{(i)} = D_{i_\alpha} = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \delta_j^i \end{pmatrix} \tag{3.2} $$

with respect to the natural frame $\{\partial_j, \partial_{j_\beta}\}$. It follows that this $n + n^2$ vector fields are linearly independent and generate, respectively the horizontal distribution of linear connection ∇ and the vertical distribution of linear coframe bundle $F^*(M)$. The set $\{D_I\} = \{D_i, D_{i_\alpha}\}$ is called the frame adapted to linear connection ∇ on $\pi^{-1}(U) \subset F^*(M)$. From (2.2), (2.3), (3.1) and (3.2), we deduce that the horizontal lift ${}^H V$ of $V \in \Im_0^1(M)$ and vertical lift ${}^{V_\alpha}\omega$ for each $\alpha = 1, 2, ..., n$, of $\omega \in \Im_1^0(M)$ have respectively, components:

$$ {}^H V = V^i D_i = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \tag{3.3} $$

$$ {}^{V_\beta}\omega = \sum_i \omega_i \delta_\alpha^\beta D_{i_\alpha} = \begin{pmatrix} 0 \\ \delta_\alpha^\beta \omega_i \end{pmatrix} \tag{3.4} $$

with respect to the adapted frame $\{D_I\}$. The non-holonomic objects ${\Omega_{IJ}}^K$ of the adapted frame $\{D_I\}$ are defined by

$$ [D_I, D_J] = {\Omega_{IJ}}^K D_K $$

and have the following non-zero components:

$$ \left(\begin{array}{c} {\Omega_{ij_\beta}}^{k_\gamma} = -{\Omega_{j_\beta i}}^{k_\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j, \\ {\Omega_{ij}}^{k_\gamma} = X_m^\gamma {R_{ijk}}^m, \end{array} \right. $$

where ${R_{ijk}}^m$ local components of the Riemannian curvature R.

4. The Cheeger-Gromoll metric on the linear coframe bundle

Definition 4.1. Let (M, g) be an $n-$dimensional Riemannian manifold. A Riemannian metric $\tilde{g}$ on the linear coframe bundle $F^*(M)$ is said to be natural with respect to g on M if

$$\tilde{g}({}^HX, {}^HY) = g(X, Y),$$

$$\tilde{g}({}^HX, {}^{V_\alpha}\omega) = 0$$

for all $X, Y \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$.

For any $x \in M$ the scalar product on the cotangent space T_x^*M is defined by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j$$

for all $\omega, \theta \in \Im_1^0(M)$.

The Cheeger-Gromoll metric ${}^{CG}g$ is a positive definite metric on linear coframe bundle $F^*(M)$ which is described in terms of lifted vector fields as follows.

Definition 4.2. Let g be a Riemannian metric on a manifold M. Then the Cheeger-Gromoll metric is a Riemannian metric ${}^{CG}g$ on the linear coframe bundle $F^*(M)$ such that

$$\begin{aligned} &{}^{CG}g({}^HX, {}^HY) = {}^V(g(X, Y)) = g(X, Y) \circ \pi, \\ &{}^{CG}g({}^{V_\alpha}\omega, {}^HY) = 0, \\ &{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = 0, \quad \alpha \neq \beta, \\ &{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta) = \tfrac{1}{1+r_\alpha^2}(g^{-1}(\omega, \theta) + g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\alpha)) \end{aligned} \tag{4.1}$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$,.where $r_\alpha^2 = \|X^\alpha\|^2 = g^{-1}(X^\alpha, X^\alpha)$.

We note that the Cheeger-Gromoll metric on the cotangent bundle of Riemannian manifold introduced by Salimov and Agca and studied in [1].

From (4.1) we determine that metric ${}^{CG}g$ has components

$$\begin{aligned} &{}^{CG}g_{ij} = {}^{CG}g(D_i, D_i) = {}^V(g(\partial_i, \partial_j)) = g_{ij}, \\ &{}^{CG}g_{i_\alpha j} = {}^{CG}g(D_{i_\alpha}, D_j) = 0, \\ &{}^{CG}g_{i_\alpha j_\beta} = {}^{CG}g(D_{i_\alpha}, D_{j_\beta}) = 0, \quad \alpha \neq \beta, \\ &{}^{CG}g_{i_\alpha j_\alpha} = {}^{CG}g(D_{i_\alpha}, D_{j_\alpha}) = \tfrac{1}{1+r_\alpha^2}(g^{-1}(dx^i, dx^j) \\ &+ g^{-1}(dx^i, X_r^\alpha)g^{-1}(dx^j, X_s^\alpha)) = \tfrac{1}{1+r_\alpha^2}(g^{ij} + g^{ir}g^{js}X_r^\alpha X_s^\alpha) \end{aligned}$$

with respect to the adapted frame $\{D_I\}$ of linear coframe bundle $F^*(M)$.

The Levi-Civita connection ${}^{CG}\nabla$ satisfies the following relations

i) ${}^{CG}\nabla_{{}^HX}{}^HY = {}^H(\nabla_XY) + \frac{1}{2}\sum\limits_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X, Y))$,

ii) ${}^{CG}\nabla_{{}^HX}{}^{V_\beta}\theta = {}^{V_\beta}(\nabla_X\theta) + \frac{1}{2h_\beta}{}^H(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\theta}))$,

iii) ${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^HY = \frac{1}{2h_\alpha}{}^H(X^\alpha(g^{-1} \circ R(\ \ , Y)\overset{\leftrightarrow}{\omega}))$,

iv) ${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta = 0$ for $\alpha \neq \beta$,

$$\begin{aligned} &{}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}\theta = -\tfrac{1}{h_\alpha}({}^{CG}g({}^{V_\alpha}\omega, \gamma\delta){}^{V_\alpha}\theta + {}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{V_\alpha}\omega) \\ &+ \tfrac{1+h_\alpha}{h_\alpha}{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)\gamma\delta - \tfrac{1}{h_\alpha}{}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{CG}g({}^{V_\alpha}\omega, \gamma\delta)\gamma\delta \end{aligned}$$

for all $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$, where $\tilde{\omega} = g^{-1} \circ \omega, R(\ \ , X)\tilde{\omega} \in \Im_1^1(M)$, $h_\alpha = 1 + r_\alpha^2$, R and $\gamma\delta$ denotes respectively the Riemanniian curvature of g and the canonical vertical vector field on $F^*(M)$ with local expression $\gamma\delta = X_i^\sigma D_{i_\sigma}$.

5. Almost complex structures on $(F^*(M), {}^{CG}g)$

First of all, let us introduce the almost complex structures J_β, $\beta = 1, 2, ..., n$, which are compatible with ${}^{CG}g$ on the linear coframe bundle $F^*(M)$. Suppose that for each $\beta = 1, 2, ..., n$, J_β is defined to be the following form

$$\begin{cases} J_\beta{}^H X = a_1{}^{V_\beta}\tilde{X} + b_1 X^\beta(X)^{V_\beta} X^\beta, \\ J_\beta{}^{V_\gamma}\omega = 0, \quad \beta \neq \gamma, \\ J_\beta{}^{V_\beta}\omega = a_2{}^H\tilde{\omega} + b_2 g^{-1}(X^\beta, \omega)^H \tilde{X}^\beta, \end{cases} \tag{5.1}$$

where $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$, $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$ and a_1, a_2, b_1 and b_2 are functions on colinear frame bundle $F^*(M)$ determined by conditions

$$J_\beta^2 = -I, \tag{5.2}$$

$${}^{CG}g(J_\beta{}^H X, J_\beta{}^H X) = {}^{CG}g({}^H X, {}^H X) = g(X, Y). \tag{5.3}$$

Substituting (5.1) into (5.2), we obtain:

$$J_\beta^{2H} X = J_\beta(J_\beta{}^H X) = J_\beta(a_1{}^{V_\beta}\tilde{X} + b_1 X^\beta(X)^{V_\beta} X^\beta)$$

$$= a_1(J_\beta{}^{V_\beta}\tilde{X}) + b_1 X^\beta(X)(J_\beta{}^{V_\beta} X^\beta) = a_1(a_2{}^H X + b_2 g^{-1}(X^\beta, \tilde{X})^H \tilde{X}^\beta)$$

$$+ b_1 X^\beta(X)(a_2{}^H \tilde{X}^\beta + b_2 g^{-1}(X^\beta, X^\beta)^H \tilde{X}^\beta) = a_1 a_2{}^H X$$

$$+ a_1 b_2 g^{-1}(X^\beta, \tilde{X})^H \tilde{X}^\beta + b_1 a_2 X^\beta(X)^H \tilde{X}^\beta$$

$$+ b_2 b_1 X^\beta(X)(h_\beta - 1)^H \tilde{X}^\beta = a_1 a_2{}^H X + (b_1 a_2 + b_2 b_1$$

$$+ b_2 b_1(h_\beta - 1))X^\beta(X)^H \tilde{X}^\beta = -{}^H X,$$

from which it follows that

$$a_1 a_2 = -1, \tag{5.4}$$

$$a_1 b_2 + b_1 a_2 + b_2 b_1(h_\beta - 1) = 0. \tag{5.5}$$

Direct calculations using (5.1) and (5.3) give

$${}^{CG}g(J_\beta{}^H X, J_\beta{}^H X) = {}^{CG}g(a_1{}^{V_\beta}\tilde{X} + b_1 X^\beta(X)^{V_\beta} X^\beta, a_1{}^{V_\beta}\tilde{X}$$

$$+ b_1 X^\beta(X)^{V_\beta} X^\beta) = a_1^{2CG} g({}^{V_\beta}\tilde{X}, {}^{V_\beta}\tilde{X}) + a_1 b_1 X^\beta(X)^{CG} g({}^{V_\beta}\tilde{X}, {}^{V_\beta} X^\beta)$$

$$+ b_1 a_1 X^\beta(X)^{CG} g({}^{V_\beta} X^\beta, {}^{V_\beta}\tilde{X}) + b_1^2 X^\beta(X)^{CG} g({}^{V_\beta} X^\beta, {}^{V_\beta} X^\beta)$$

$$= \frac{a_1^2}{h_\beta}(g^{-1}(\tilde{X}, \tilde{X}) + g^{-1}(\tilde{X}, X^\beta) g^{-1}(\tilde{X}, X^\beta))$$

$$+ \frac{a_1 b_1 X^\beta(X)}{h_\beta}(g^{-1}(\tilde{X}, X^\beta) + g^{-1}(\tilde{X}, X^\beta) g^{-1}(X^\beta, X^\beta))$$

$$+ \frac{b_1 a_1 X^\beta(X)}{h_\beta}(g^{-1}(X^\beta, \overset{\leftrightarrow}{X}) + g^{-1}(X^\beta, X^\beta) g^{-1}(\tilde{X}, X^\beta))$$

$$+ \frac{b_1^2 X^\beta(X) X^\beta(X)}{h_\beta}(g^{-1}(X^\beta, X^\beta) + g^{-1}(X^\beta, X^\beta) g^{-1}(X^\beta, X^\beta))$$

$$= \frac{a_1^2}{h_\beta} g(X, X) + \left(\frac{a_1^2}{h_\beta} + 2a_1 b_1 + b_1^2(h_\beta - 1)\right)(X^\beta(X))^2 = g(X, X).$$

From the last relation we obtain:

$$\frac{a_1^2}{h_\beta} = 1, \tag{5.6}$$

$$\frac{a_1^2}{h_\beta} + 2a_1b_1 + b_1^2(h_\beta - 1) = 0. \tag{5.7}$$

Using (5.6) and (5.4), we get first $a_1 = \pm\sqrt{h_\beta}$ and $a_2 = \mp\frac{1}{\sqrt{h_\beta}}$. Without lost of the generality we can take $a_1 = \sqrt{h_\beta}$ and $a_2 = -\frac{1}{\sqrt{h_\beta}}$. Then for these values from (5.7) we get

$$b_1^2(h_\beta - 1) + 2\sqrt{h_\beta}b_1 + 1 = 0,$$

from which it follows

$$b_1 = \frac{-\sqrt{h_\beta} \pm 1}{h_\beta - 1}.$$

We can take $b_1 = \frac{-\sqrt{h_\beta}+1}{h_\beta-1} = -\frac{1}{\sqrt{h_\beta}+1}$. Then by using of (5.5) we obtain:

$$\sqrt{h_\beta}b_2 + \frac{1}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)} - b_2\frac{1}{\sqrt{h_\beta}}(h_\beta - 1) = 0,$$

or

$$b_2 = \frac{-1}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)}.$$

Therefore, we have the almost complex structures J_β, $\beta = 1, 2, ..., n$, on linear coframe bundle $F^*(M)$

$$\begin{cases} J_\beta{}^H X = \sqrt{h_\beta}{}^{V_\beta}\tilde{X} - \frac{1}{\sqrt{h_\beta}+1} X^\beta(X)^{V_\beta}X^\beta, \\ J_\beta{}^{V_\gamma}\omega = 0, \quad \beta \neq \gamma, \\ J_\beta{}^{V_\beta}\omega = -\frac{1}{\sqrt{h_\beta}}\left({}^H\tilde{\omega} + \frac{1}{(\sqrt{h_\beta}+1)} g^{-1}(X^\beta, \omega)^H\tilde{X}^\beta\right), \end{cases} \tag{5.8}$$

which are satisfies the compability conditions (5.3) with the Cheeger-Gromoll metric ${}^{CG}g$.

Remark 5.1. Taking into account that equality $J_\beta{}^{V_\gamma}\omega = 0$ holds for $\gamma \neq \beta$, of interest is the case when $\gamma = \beta$.

Now it follows by a direct computations that

$${}^{CG}g(J_\beta{}^H X, J_\beta{}^{V_\beta}\omega) = {}^{CG}g({}^H X, {}^{V_\beta}\omega),$$

$${}^{CG}g(J_\beta{}^{V_\beta}\omega, J_\beta{}^{V_\beta}\theta) = {}^{CG}g({}^{V_\beta}\omega, {}^{V_\beta}\theta),$$

whenever

$${}^{CG}g(J_\beta{}^H X, J_\beta{}^H X) = {}^{CG}g({}^H X, {}^H X).$$

Indeed, using (4.1) and (5.8), we have

$$ {}^{CG}g(J_\beta{}^HX, J_\beta{}^{V_\beta}\omega) = {}^{CG}g(\sqrt{h_\beta}{}^{V_\beta}\tilde{X} $$

$$ -\tfrac{1}{\sqrt{h_\beta}+1}X^\beta(X)^{V_\beta}X^\beta, -\tfrac{1}{\sqrt{h_\beta}}({}^H\tilde{\omega} + \tfrac{1}{\sqrt{h_\beta}+1}g^{-1}(X^\beta,\omega)^H\tilde{X}^\beta)) $$

$$ = -\delta_\beta^\gamma{}^{CG}g({}^{V_\beta}\tilde{X}, {}^H\tilde{\omega}) - \tfrac{1}{\sqrt{h_\beta}+1}g^{-1}(X^\beta,\omega)^{CG}g({}^{V_\beta}\tilde{X}, {}^H\tilde{X}^\beta) $$

$$ +\tfrac{1}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)}X^\beta(X)^{CG}g({}^{V_\beta}X^\beta, {}^H\tilde{\omega}) $$

$$ +\tfrac{1}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)^2}X^\beta(X)g^{-1}(X^\beta,\omega)^{CG}g({}^{V_\beta}X^\beta, {}^H\tilde{X}^\beta) $$

$$ = 0 = {}^{CG}g({}^HX, {}^{V_\beta}\omega). $$

Similarly we get

$$ {}^{CG}g(J_\beta{}^{V_\beta}\omega, J_\beta{}^{V_\beta}\theta) = {}^{CG}g(-\tfrac{1}{\sqrt{h_\beta}}({}^H\tilde{\omega} $$

$$ +\tfrac{1}{\sqrt{h_\beta}+1}g^{-1}(X^\beta,\omega)^H\tilde{X}^\beta), -\tfrac{1}{\sqrt{h_\beta}}({}^H\tilde{\theta} + \tfrac{1}{\sqrt{h_\beta}+1}g^{-1}(X^\beta,\theta)^H\tilde{X}^\beta)) $$

$$ = \tfrac{1}{h_\beta}{}^{CG}g({}^H\tilde{\omega}, {}^H\tilde{\theta}) + \tfrac{1}{h_\beta(\sqrt{h_\beta}+1)}g^{-1}(X^\beta,\theta)^{CG}g({}^H\tilde{\omega}, {}^H\tilde{X}^\beta) $$

$$ +\tfrac{1}{h_\beta(\sqrt{h_\beta}+1)}g^{-1}(X^\beta,\omega)^{CG}g({}^H\tilde{X}^\beta, {}^H\tilde{\theta}) $$

$$ +\tfrac{1}{h_\beta(\sqrt{h_\beta}+1)^2}g^{-1}(X^\beta,\omega)g^{-1}(X^\beta,\theta)^{CG}g({}^H\tilde{X}^\beta, {}^H\tilde{X}^\beta) $$

$$ = \tfrac{1}{h_\beta}g^{-1}(\omega,\theta) + \tfrac{2}{h_\beta(\sqrt{h_\beta}+1)}g^{-1}(X^\beta,\omega)g^{-1}(X^\beta,\theta) $$

$$ +\frac{1}{h_\beta(\sqrt{h_\beta}+1)^2}g^{-1}(X^\beta,\omega)g^{-1}(X^\beta,\theta)(h_\beta - 1) $$

$$ = \frac{(\sqrt{h_\beta}+1)g^{-1}(\omega,\theta) + (\sqrt{h_\beta}+1)g^{-1}(X^\beta,\omega)g^{-1}(X^\beta,\theta)}{h_\beta(\sqrt{h_\beta}+1)} $$

$$ = \frac{1}{h_\beta}(g^{-1}(\omega,\theta) + g^{-1}(X^\beta,\omega)g^{-1}(X^\beta,\theta)) = {}^{CG}g({}^{V_\beta}\omega, {}^{V_\beta}\theta). $$

Thus the following theorem holds.

Theorem 5.2. *The triple* $(F^*(M), {}^{CG}g, J_\beta)$ *is an almost Hermitian manifold for any* $\beta = 1, 2, ..., n$.

6. The integrability of J_β, $\beta = 1, 2, ..., n$

It is known that the almost complex structure J of a Riemannian manifold (M, g) is inteqrable if and only if its Nijenhuis tensor

$$ N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0 $$

for all $X, Y \in \Im_0^1(M)$ ([16, p. 118]).

The Nijenhuis tensor of an almost complex structure J_β on $F^*(M)$ for any $\beta = 1, 2, ..., n$, is given by

$$ N_{J_\beta}(\tilde{X},\tilde{Y}) = [\tilde{X},\tilde{Y}] + J_\beta[J_\beta\tilde{X},\tilde{Y}] + J_\beta[\tilde{X},J_\beta\tilde{Y}] - [J_\beta\tilde{X},J_\beta\tilde{Y}], \tag{6.1} $$

where $\tilde{X}, \tilde{Y} \in \Im_0^1(F^*(M))$. It is easy to check that the values $N_{J_\beta}({}^H X, {}^{V_\gamma}\theta)$ and $N_{J_\beta}({}^{V_\alpha}\omega, {}^{V_\gamma}\theta)$ of the Nijenhuis tensor N_{J_β} can be expressed in terms of the values $N_{J_\beta}({}^H X, {}^H Y)$ of this tensor, where $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$. Indeed, using (5.2) and (6.1), we have

$$N_{J_\beta}({}^H X, {}^{V_\gamma}\theta) = [{}^H X, {}^{V_\gamma}\theta] + J_\beta[J_\beta{}^H X, {}^{V_\gamma}\theta] + J_\beta[{}^H X, J_\beta{}^{V_\gamma}\theta]$$

$$-[J_\beta{}^H X, J_\beta{}^{V_\gamma}\theta] = [{}^H X, \delta_\beta^\gamma J_\beta{}^H W] + J_\beta[J_\beta{}^H X, \delta_\beta^\gamma J_\beta{}^H W]$$

$$+J_\beta[{}^H X, J_\beta(\delta_\beta^\gamma J_\beta{}^H W)] - [J_\beta{}^H X, J_\beta(\delta_\beta^\gamma J_\beta{}^H W)] = \delta_\beta^\gamma[{}^H X, J_\beta{}^H W$$

$$+\delta_\beta^\gamma J_\beta[J_\beta{}^H X, J_\beta{}^H W] - \delta_\beta^\gamma J_\beta[{}^H X, {}^H W] + \delta_\beta^\gamma[J_\beta{}^H X, {}^H W]$$

$$= -\delta_\beta^\gamma J_\beta N_{J_\beta}({}^H X, {}^H W),$$

where

$$ {}^{V_\gamma}\theta = \delta_\beta^\gamma J_\beta{}^H W = \delta_\beta^\gamma(\sqrt{h_\beta}{}^{V_\beta}\tilde{W} - \tfrac{1}{\sqrt{h_\beta}+1} X^\beta(W){}^{V_\beta}X^\beta)$$

$$= \delta_\beta^\gamma{}^{V_\beta}(\sqrt{h_\beta}\tilde{W} - \tfrac{1}{\sqrt{h_\beta}+1} X^\beta(W)X^\beta), \ W \in \Im_0^1(M).$$

Similarly, we have

$$N_{J_\beta}({}^{V_\alpha}\omega, {}^{V_\gamma}\theta) = [{}^{V_\alpha}\omega, {}^{V_\gamma}\theta] + J_\beta[J_\beta{}^{V_\alpha}\omega, {}^{V_\gamma}\theta] + J_\beta[{}^{V_\alpha}\omega, J_\beta{}^{V_\gamma}\theta]$$

$$-[J_\beta{}^{V_\alpha}\omega, J_\beta{}^{V_\gamma}\theta] = [\delta_\beta^\alpha J_\beta{}^H Z, \delta_\beta^\gamma J_\beta{}^H W] + J_\beta[J_\beta(\delta_\beta^\alpha J_\beta{}^H Z, \delta_\beta^\gamma J_\beta{}^H W]$$

$$+J_\beta[\delta_\beta^\alpha J_\beta{}^H Z, J_\beta(\delta_\beta^\gamma J_\beta{}^H W)] - [J_\beta(\delta_\beta^\alpha J_\beta{}^H Z), J_\beta(\delta_\beta^\gamma J_\beta{}^H W)]$$

$$= \delta_\beta^\alpha\delta_\beta^\gamma[J_\beta{}^H Z, J_\beta{}^H W] - \delta_\beta^\alpha\delta_\beta^\gamma J_\beta[{}^H Z, J_\beta{}^H W] - \delta_\beta^\alpha\delta_\beta^\gamma J_\beta[J_\beta{}^H Z, {}^H W]$$

$$-\delta_\beta^\alpha\delta_\beta^\gamma[{}^H Z, {}^H W] = -\delta_\beta^\alpha\delta_\beta^\gamma N_{J_\beta}({}^H Z, {}^H W),$$

where ${}^{V_\alpha}\omega = \delta_\beta^\alpha J_\beta{}^H Z, \ Z \in \Im_0^1(M)$. Therefore, we have

Lemma 6.1. *An almost complex structure J_β on $(F^*(M), {}^{CG}g)$ for each $\beta = 1, 2, ..., n$, is inteqrable if and only if $N_{J_\beta}({}^H X, {}^H Y) = 0$ for any $X, Y \in \Im_0^1(M)$.*

Let us calculate

$$N_{J_\beta}({}^H X, {}^H Y) = [{}^H X, {}^H Y] + J_\beta[J_\beta{}^H X, {}^H Y] + J_\beta[{}^H X, J_\beta{}^H Y]$$

$$-[J_\beta{}^H X, J_\beta{}^H Y].$$

Before calculating $N_{J_\beta}({}^H X, {}^H Y)$ it is necessary to prove the following.

Lemma 6.2. *Let (M, g) be a Riemannian manifold and $f : R \to R$ a smooth function. Then for all $X \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, we have*

$$1. {}^{V_\beta}\omega\left(f(r_\alpha^2)\right) = 2\delta_\alpha^\beta f'(r_\alpha^2) g^{-1}(\omega, X^\alpha), \tag{6.2}$$

$$2. {}^H X(g^{-1}(X^\alpha, \theta) = g(X^\alpha, \nabla_X\theta), \tag{6.3}$$

where $r_\alpha^2 = g^{-1}(X^\alpha, X^\alpha)$.

Proof. Direct calculations using (3.3) and (3.4) give

$$1. \quad {}^{V_\beta}\omega(f(r_\alpha^2)) = \omega_i\delta_\sigma^\beta f'(r_\alpha^2)\partial_{i_\sigma}(g^{rs}X_r^\alpha X_s^\alpha)$$

$$= \omega_i \delta_\sigma^\beta f'(r_\alpha^2) g^{rs} (\delta_\alpha^\sigma \delta_r^i X_s^\alpha + \delta_\alpha^\sigma \delta_s^i X_r^\alpha) = 2\omega_i \delta_\alpha^\beta f'(r_\alpha^2) g^{is} X_s^\alpha$$

$$= 2\delta_\alpha^\beta f'(r_\alpha^2) g^{-1}(\omega, X^\alpha),$$

$$2. \quad {}^H X(g^{-1}(X^\alpha, \theta)) = (X^i D_i)(g^{-1}(X^\alpha, \theta)) = X^i(\partial_i$$

$$+ X_l^\sigma \Gamma_{ip}^l \partial_{p_\sigma})(g^{-1}(X^\alpha, \theta)) = X^i \partial_i (g^{rs} X_r^\alpha \theta_s)$$

$$+ X^i X_l^\sigma \Gamma_{ip}^l \partial_{p_\sigma} (g^{rs} X_r^\alpha \theta_s) = X^i (\partial_i g^{rs}) X_r^\alpha \theta_s$$

$$+ X^i g^{rs} X_r^\alpha \partial_i \theta_s + X^i X_l^\sigma \Gamma_{ip}^l g^{rs} \delta_\sigma^\alpha \delta_r^p \theta_s = X^i (-\Gamma_{im}^r g^{ms}$$

$$- \Gamma_{im}^s g^{rm}) X_r^\alpha \theta_s + X^i g^{rs} X_r^\alpha \partial_i \theta_s + X^i X_l^\alpha \Gamma_{ir}^l g^{rs} \theta_s$$

$$= -X^i \Gamma_{im}^r g^{ms} X_r^\alpha \theta_s - X^i \Gamma_{im}^s g^{rm} X_r^\alpha \theta_s + X^i g^{rs} X_r^\alpha \partial_i \theta_s$$

$$+ X^i X_l^\alpha \Gamma_{ir}^l g^{rs} \theta_s = X^i g^{rs} X_r^\alpha \partial_i \theta_s - X^i \Gamma_{im}^s g^{rm} X_r^\alpha \theta_s$$

$$= X_r^\alpha X^i (\partial_i \theta_s - \Gamma_{is}^m \theta_m) g^{rs} = X_r^\alpha (\nabla_X \theta)_s g^{rs} = g^{-1}(X^\alpha, \nabla_X \theta).$$

This completes the proof of the lemma.
Direct calculations using (2.4), (3.3), (3.4), (5.8), (6.2) and (6.3) give

$$[{}^H X, {}^H Y] = {}^H [X, Y] + \sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X, Y)),$$

$$J_\beta [J_\beta {}^H X, {}^H Y] = J_\beta [\sqrt{h_\beta} {}^{V_\beta} \tilde{X} - \frac{1}{\sqrt{h_\beta + 1}} X^\beta (X) {}^{V_\beta} X^\beta, {}^H Y]$$

$$= J_\beta (\sqrt{h_\beta} [{}^{V_\beta} \tilde{X}, {}^H Y] - \tfrac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}^\beta, X) [{}^{V_\beta} X^\beta, {}^H Y]$$

$$+ \tfrac{1}{\sqrt{h_\beta + 1}} {}^H Y (g(\tilde{X}^\beta, X)) {}^{V_\beta} X^\beta = J_\beta \left(-\sqrt{h_\beta}\, {}^{V_\beta} (\nabla_Y \tilde{X}) \right.$$

$$\left. + \tfrac{1}{\sqrt{h_\beta + 1}} \left(g^{-1}(X^\beta, \tilde{X}) {}^{V_\beta} (\nabla_Y X^\beta) + {}^H Y (g^{-1}(X^\beta, \tilde{X}))\, {}^{V_\beta} X^\beta \right) \right)$$

$$= J_\beta \left(-\sqrt{h_\beta}\, {}^{V_\beta} (\nabla_Y \tilde{X}) + \tfrac{1}{\sqrt{h_\beta + 1}} g^{-1}(\nabla_Y \tilde{X}, X^\beta) {}^{V_\beta} X^\beta \right)$$

$$= J_\beta (-J_\beta {}^H (\nabla_Y X)) = -J_\beta^2 {}^H (\nabla_Y X) = {}^H (\nabla_Y X),$$

$$J_\beta [{}^H X, J_\beta {}^H Y] = -J_\beta [J_\beta {}^H Y, {}^H X] = -{}^H (\nabla_X Y),$$

$$[J_\beta {}^H X, J_\beta {}^H Y] = [\sqrt{h_\beta} {}^{V_\beta} \tilde{X} - \tfrac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}^\beta, X) {}^{V_\beta} X^\beta, \sqrt{h_\beta} {}^{V_\beta} \tilde{Y}$$

$$- \tfrac{1}{\sqrt{h_\beta + 1}} g(\tilde{X}^\beta, Y) {}^{V_\beta} X^\beta] = [\sqrt{h_\beta} {}^{V_\beta} \tilde{X}, \sqrt{h_\beta} {}^{V_\beta} \tilde{Y}]$$

$$+[\sqrt{h_\beta}^{V_\beta}\tilde{X}, -\tfrac{1}{\sqrt{h_\beta}+1}g(\tilde{X}^\beta, Y)^{V_\beta}X^\beta]$$

$$+[-\tfrac{1}{\sqrt{h_\beta}+1}g(\tilde{X}^\beta, X)^{V_\beta}X^\beta, \sqrt{h_\beta}^{V_\beta}\tilde{Y}]$$

$$+[-\tfrac{1}{\sqrt{h_\beta}+1}g(\tilde{X}^\beta, X)^{V_\beta}X^\beta, -\tfrac{1}{\sqrt{h_\beta}+1}g(\tilde{X}^\beta, Y)^{V_\beta}X^\beta]$$

$$= \sqrt{h_\beta}^{V_\beta}\tilde{X}(\sqrt{h_\beta})^{V_\beta}\tilde{Y} - \sqrt{h_\beta}^{V_\beta}\tilde{Y}(\sqrt{h_\beta})^{V_\beta}\tilde{X}$$

$$+\frac{1}{\sqrt{h_\beta}+1}g\left(\tilde{X}^\beta, Y\right){}^{V_\beta}X^\beta\left(\sqrt{h_\beta}\right){}^{V_\beta}\tilde{X} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta}+1}g(\tilde{X}^\beta, Y)[{}^{V_\beta}X^\beta, {}^{V_\beta}\tilde{X}]-$$

$$-\frac{1}{\sqrt{h_\beta}+1}g(\tilde{X}^\beta, X)^{V_\beta}X^\beta(\sqrt{h_\beta})^{V_\beta}\tilde{Y} - -\frac{\sqrt{h_\beta}}{\sqrt{h_\beta}+1}g(\tilde{X}^\beta, X)[{}^{V_\beta}X^\beta, {}^{V_\beta}\tilde{Y}]$$

$$= g^{-1}(X^\beta, \tilde{X})^{V_\beta}\tilde{Y} - g^{-1}(X^\beta, \tilde{Y})^{V_\beta}\tilde{X}$$

$$+\tfrac{1}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)}g^{-1}(X^\beta, \tilde{Y})g^{-1}(X^\beta, X^\beta)^{V_\beta}\tilde{X} - \tfrac{\sqrt{h_\beta}}{\sqrt{h_\beta}+1}g^{-1}(X^\beta, \tilde{Y})^{V_\beta}\tilde{X}$$

$$-\tfrac{1}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)}g^{-1}(X^\beta, \tilde{X})g^{-1}(X^\beta, X^\beta)^{V_\beta}\tilde{Y} + \tfrac{\sqrt{h_\beta}}{\sqrt{h_\beta}+1}g^{-1}(X^\beta, \tilde{X})^{V_\beta}\tilde{Y}$$

$$= {}^{V_\beta}\left(g^{-1}(X^\beta, \tilde{X})\tilde{Y} - g^{-1}(X^\beta, \tilde{Y})\tilde{X}\right)\left(1 - \tfrac{{r_\beta}^2}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)} + \tfrac{\sqrt{h_\beta}}{\sqrt{h_\beta}+1}\right).$$

Therefore,

$$N_{J_\beta}({}^HX, {}^HY) = {}^H[X,Y] + \sum_{\sigma=1}^{n}(X^\sigma \circ R(X,Y)) + {}^H((\nabla_Y X) - (\nabla_X Y))$$

$$-{}^{V_\beta}\left(g^{-1}(X^\beta, \tilde{X})\tilde{Y} - g^{-1}(X^\beta, \tilde{Y})\tilde{X}\right)\left(1 - \frac{{r_\beta}^2}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta}+1}\right)$$

$$= \sum_{\sigma=1}^{n}(X^\sigma \circ R(X,Y)) - {}^{V_\beta}\left(g^{-1}(X^\beta, \tilde{X})\tilde{Y}\right.$$

$$\left. - g^{-1}(X^\beta, \tilde{Y})\tilde{X}\right)\left(1 - \frac{{r_\beta}^2}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)} + \frac{\sqrt{h_\beta}}{\sqrt{h_\beta}+1}\right)$$

$$= \sum_{\sigma=1}^{n}(X^\sigma \circ R(X,Y)) - \frac{1+\sqrt{h_\beta}+h_\beta}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)}{}^{V_\beta}\left(g^{-1}(X^\beta, \tilde{X})\tilde{Y} - g^{-1}(X^\beta, \tilde{Y})\tilde{X}\right).$$

Thus, the following theorem holds. □

Theorem 6.3. *An almost complex structure J_β on $(F^*(M), {}^{CG}g)$ for each $\beta = 1, 2, ..., n$, is integrable if and only if*

$$\gamma R(X,Y) = \sum_{\sigma=1}^{n}(X^\sigma \circ R(X,Y))$$

$$= \frac{1+\sqrt{h_\beta}+h_\beta}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)}{}^{V_\beta}\left(g^{-1}(X^\beta, \tilde{X})\tilde{Y} - g^{-1}(X^\beta, \tilde{Y})\tilde{X}\right).$$

References

[1] F. Agca and A. Salimov, *Some notes concerning Cheeger-Gromoll metrics*, Hacet. J. Math. Stat. **42**(5), 533-549, 2013.

[2] C.L. Bejan and S.L. Druţă-Romaniuc, *Harmonic almost complex structures with respect to general natural metrics*, Mediterr. J. Math. **11**(1), 123-136, 2013.

[3] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. Math. **96**, 413-443, 1972.

[4] S.L. Druţă-Romaniuc, *Cotangent bundles with general natural Kahler structures*, Rev. Roumaine Math. Pures Appl. **54**(1), 13-23, 2009.

[5] H. Fattayev and A. Salimov, *Diagonal lifts of metrics to coframe bundle*, Proc. IMM NAS Azerbaijan **44**(2), 328-337, 2018.

[6] S. Gudmondson and E. Kappos, *On the geometry of the tangent bundles*, Expo. Math. **20**(1), 1-41, 2002.

[7] Z. Hou and L. Sun, *Geometry of tangent bundle with Cheeger-Gromoll type metric*, J. Math. Anal. Appl. **402**, 493-504, 2013.

[8] O. Kowalski, *Curvatures of the induced Riemannian metric of the tangent bundle of Riemannian manifold*, J. Reine Angew. Math. **250**, 124-129, 1971

[9] M. Munteanu, *Cheeger-Gromoll type metrics on the tangent bundle*, Sci. Ann. Univ. Agric. Sci. Vet. Med. **49**(2), 257-268, 2006.

[10] E. Musso and F. Tricerri, *Riemannian metrics on tangent bundles*, Ann. Math. Pura. Appl. **150** (4), 1-20, 1988.

[11] V. Oproiu and D. Poroşniuc, *A Kahler Einstein structure on the cotangent bundle of a Riemannian manifold*, An. Ştiint. Univ. Al. I. Cuza, Iaşi **49**, s. I, Mathematics f.2, 399-414, 2003.

[12] A. Salimov and H. Fattayev, *Lifts of derivations in the coframe bundle*, Mediterr. J. Math. **17**(48), 1-12, 2020.

[13] S. Sasaki, *On the differential geometry of the tangent bundle of Riemannian manifolds*, Tohoku Math. J. **10**, 238-254, 1958.

[14] M. Sekizawa, *Curvatures of tangent bundles with Cheeger-Gromoll metric*, Tokyo J. Math. **14**(2), 407-417, 1991.

[15] M. Tahara, L. Vanhecke and Y. Watanabe, *New structures on tangent bundles*, Note Mat. **18**(1), 131-141, 1998.

[16] K. Yano and S. Ishihara, *Tangent and cotangent bundles*, Marsel Dekker Inc., New York, 1973.