Advances in the Theory of Nonlinear Analysis and its Applications 6 (2022) No. 2, 148–156.
https://doi.org/10.31197/atnaa.1012869
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis
and its Applications

ISSN: 2587-2648 Peer-Reviewed Scientific Journal

Semilinear parabolic diffusion systems on the sphere with Caputo-Fabrizio operator

Tran Thanh Binh[a]

[a]*Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam*

Abstract

PDEs on spheres have many important applications in physical phenomena, oceanography and meteorology, geophysics. In this paper, we study the parabolic systems with Caputo-Fabrizio derivative. In order to establish the existence of the mild solution, we use the Banach fixed point theorem and some analysis of Fourier series associated with several evaluations of the spherical harmonics function. Some of the techniques on upper and lower bounds of the Mittag-Lefler functions are also applied. This is one of the first research results on the systems of parabolic diffusion on the sphere.

Keywords: Parabolic systems; Banach fixed point theory; Regularity.
2010 MSC: 35R11, 35B65, 26A33.

1. Introduction

Partial diffferential equations on the sphere and their analysis were investigated by many authors, such as many authors, for example [13, 14, 15, 16, 17, 9, 12, 33, 34, 35, 22, 28]. These equations play a role in modeling a number of physical phenomena that occur in the earth's surface or in earthquakes and seismic events. When studying natural phenomena, many external factors occur, so equations with classical derivatives cannot fully describe these models. The appearance of fractional Caculus contributed to a clarification and more complete in the simulation. Fractional analysis has many applications in mechanics, physics and engineering science, etc. We would like to share many published works on these issues such as E. Karapinar et al [1, 2, 3, 4, 5, 6, 7, 8], H. Jafari and his group [18, 19, 20, 21]. When examining mathematical models, depending on the models, there will be many corresponding derivatives. Each type of derivative

Email address: tranthanhbinh@tdmu.edu.vn (Tran Thanh Binh)

Received October 21, 2021, Accepted January 8, 2021, Online January 11, 2022

has its advantages and disadvantages. The most prominent among the fractional derivatives is the Caputo derivative. This is the derivative that contains the singularity kernels. In [26], the authors invented the CaputoâĂŞFabrizio fractional derivative with purpose of avoiding singular kernels. It is also the convolution of the exponential function and the first order derivative. The Caputo-Fabrizio derivative is an operator that has been widely applied to a number of derivative modes in many fields, such as biology, physics, control systems, materials science, and dynamics. liquid learning [30].

In this paper, we consider the systems of parabolic problem with Caputo-Fabrizio derivative on the unit sphere $S^2 \subset R^3$ as follows

$$\begin{cases} {}_{\mathrm{CF}}D_t^{\alpha} u - \Delta^* u = F(u,v) & (x,t) \quad \text{in } \mathbf{S}^2 \times (0,T), \\ {}_{\mathrm{CF}}D_t^{\alpha} v - \Delta^* v = G(u,v) & (x,t) \quad \text{in } \mathbf{S}^2 \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \quad \text{in } \mathbf{S}^2, \end{cases} \tag{1}$$

where ${}_{\mathrm{CF}}D_t^{\alpha}$ is Caputo-Fabrizio operator for fractional derivatives which is defined as (see [29])

$${}_{\mathrm{CF}}D_t^{\alpha} v(t) = \frac{H(r)}{1-\alpha} \int_0^t \mathcal{D}_\alpha(t-\nu) \frac{\partial v(\nu)}{\partial \nu} \mathrm{d}\nu, \quad \text{for } t \geq 0,$$

where we denote by the kernel $\mathcal{D}_\alpha(z) = \exp\left(-\frac{\alpha}{1-\alpha} z\right)$ and $H(r)$ satisfies $H(0) = H(1) = 1$, (see e.g. [26, 27]) and Δ^* is the Laplace-Beltrami operator. So far, only a few special cases of fractional partial differential systems have been studied [31, 32], there are still many other systems that have not been studied, especially the models on the sphere.

This article is organized as follows. Section 2 gives some preliminary and mild solution. In Section 3, we present our main results including two main theorems. Finally, the proof of some theorems is completed in section 4.

2. Preliminaries

Spherical harmonics are polynomials which satisfy $\Delta_x Y(x) = 0$ (where Δ_x is the Laplacian operator in R^3) and are restricted to the surface of the Euclidean sphere S^2. The eigenvalues for $-\Delta^*$ in R^3 are

$$\theta_n = n^2 + n,\ n = 0, 1, 2, ...$$

and the eigenfunctions corresponding to θ_n are the spherical harmonics $\mathbf{X}_n(x)$

$$\Delta^* \mathbf{X}_n(x) = -\theta_n \mathbf{X}_n(x).$$

The space of all spherical harmonics of degree n on $\mathbf{S}^2$, denoted by V_n, has an orthonormal basis $\{\mathbf{X}_{nk}(x) : n = 1, 2, 3, ...\mathcal{M}(2,n)\}$ where

$$\mathcal{M}(2,0) = 1,\ \mathcal{M}(2,n) = \frac{2n+1}{\Gamma(2)},\ n \geq 1.$$

Noting that any function $f \in L^2(S^2)$ can be expressed in the form of spherical harmonics

$$f = \sum_{n=0}^{\infty} \sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \widehat{f}_{nk} \mathbf{X}_{nk},\ \widehat{f}_{nk} = \int_{S^2} f \overline{\mathbf{X}}_{nk} dS,$$

where dS is the surface measure of the unit sphere. Let us define $H^p(\mathbf{S}^2)$ for $p > 0$ by

$$H^p(\mathbf{S}^2) = \left\{ \psi \in L^2(\mathbf{S}^2) : \sum_{n=0}^{\infty} \sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \left(n^2+n+1\right)^p |\widehat{\psi}_{nk}|^2 < \infty \right\} \tag{2}$$

with the following norm

$$\|\psi\|_{H^p(\mathbf{S}^2)} = \sqrt{\sum_{n=0}^{\infty}\sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \Big(n^2+n+1\Big)^p |\widehat{\psi}_{nk}|^2}.$$

Definition 2.1. The function (u,w) is called a mild solution of Problem (1) if it satisfies that

$$\begin{aligned} u(t) &= \mathbf{D}_r(t)u_0 + \int_0^t \mathbf{D}_r(t-s)F(u(s),w(s))ds \\ w(t) &= \mathbf{D}_r(t)w_0 + \int_0^t \mathbf{D}_r(t-s)G(u(s),w(s))ds \end{aligned} \tag{3}$$

where $\mathbf{D}_r(t)$ is defined by

$$\mathbf{D}_r(t)f = \sum_{n=0}^{\infty}\sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \frac{1}{1+(1-r)(n^2+n)} \exp\left(\frac{-\alpha(n^2+n)}{1+(1-r)(n^2+n)}t\right) \widehat{f}_{nk}\mathbf{X}_{nk}.$$

Now we have the following Lemma.

Lemma 2.2. *Let $f \in H^{q-1}(\mathbf{S}^2)$. Then we have*

$$\Big\|\mathbf{D}_r(t)f\Big\|_{H^q(\mathbf{S}^2)} \le \sqrt{\frac{1}{1-r}}\|f\|_{H^{q-1}(\mathbf{S}^2)}. \tag{4}$$

Proof. Let us assume that $f \in H^q(\mathbf{S}^2)$. Then we get the following equality

$$\Big\|\mathbf{D}_r(t)f\Big\|^2_{H^q(\mathbf{S}^2)} = \sum_{n=0}^{\infty}\sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \frac{\Big(n^2+n+1\Big)^q}{1+(1-r)(n^2+n)} \exp\left(\frac{-2\alpha(n^2+n)}{1+(1-r)(n^2+n)}t\right) |\widehat{f}_{nk}|^2. \tag{5}$$

Since $1 > 1-r$, we get that

$$\frac{\Big(n^2+n+1\Big)^q}{1+(1-r)(n^2+n)} \le \frac{1}{1-r}\Big(n^2+n+1\Big)^{q-1}. \tag{6}$$

It follows from (5) that

$$\Big\|\mathbf{D}_r(t)f\Big\|^2_{H^q(\mathbf{S}^2)} \le \frac{1}{1-r}\sum_{n=0}^{\infty}\sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \Big(n^2+n+1\Big)^{q-1}|\widehat{f}_{nk}|^2 = \frac{1}{1-r}\|f\|^2_{H^{q-1}(\mathbf{S}^2)}, \tag{7}$$

which allows us to obtain that the desired result (4). □

Theorem 2.3. *Let $(u_0,w_0) \in H^{p-1}(\mathbf{S}^2) \times H^{p-1}(\mathbf{S}^2)$ and $F(0,0) = G(0,0) = 0$. Let us assume that*

$$\Big\|F(u_1,w_1) - F(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} \le K_f\left(\|u_1-u_2\|_{H^p(\mathbf{S}^2)} + \|w_1-w_2\|_{H^p(\mathbf{S}^2)}\right) \tag{8}$$

and

$$\Big\|G(u_1,w_1) - G(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} \le K_g\left(\|u_1-u_2\|_{H^p(\mathbf{S}^2)} + \|w_1-w_2\|_{H^p(\mathbf{S}^2)}\right). \tag{9}$$

Then Problem (1) has a unique solution (u,w) on the space $\left(L^\infty_\nu(0,T;H^p(\mathbf{S}^2))\right)^2$. Then we get

$$\begin{aligned} &\Big\|u(.,t)\Big\|_{H^p(\mathbf{S}^2)} + \Big\|w(.,t)\Big\|_{H^p(\mathbf{S}^2)} \\ &\le \left(\sqrt{\frac{1}{1-r}}\Big\|u_0\Big\|_{H^{p-1}(\mathbf{S}^2)} + \sqrt{\frac{1}{1-r}}\Big\|w_0\Big\|_{H^{p-1}(\mathbf{S}^2)}\right)\exp\Big(\sqrt{\frac{1}{1-r}}(K_f+K_g)C_pt\Big). \end{aligned} \tag{10}$$

Proof. For any $\nu \geq 0$, denote by $\left(L^\infty_\nu(0,T;H^p(\mathbf{S}^2))\right)^2$ the function space $\left(L^\infty(0,T;H^p(\mathbf{S}^2))\right)^2$ associated with the norm

$$\|(u,v)\|_{\nu,p} := \max_{0\le t\le T} \Big\| \exp(-\nu t) u(\cdot,t) \Big\|_{H^p(\mathbf{S}^2)} + \max_{0\le t\le T} \Big\| \exp(-\nu t) v(\cdot,t) \Big\|_{H^p(\mathbf{S}^2)},$$

for any $(u,w) \in \left(L^\infty(0,T;H^p(\mathbf{S}^2))\right)^2$. Let us define the operator

$$\mathbf{N}(u,w)(t) = \Big(\mathcal{N}_1(u,w)(t), \mathcal{N}_2(u,w)(t)\Big) \tag{11}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are given by

$$\begin{aligned} \mathcal{N}_1(u,w)(t) &= \mathbf{D}_r(t)u_0 + \int_0^t \mathbf{D}_r(t-s) F(u(s),w(s)) ds \\ \mathcal{N}_2(u,w)(t) &= \mathbf{D}_r(t)w_0 + \int_0^t \mathbf{D}_r(t-s) G(u(s),w(s)) ds. \end{aligned} \tag{12}$$

If $(u,w)=(0,0)$ then using the condition $F(0,0)=G(0,0)=0$, we have that

$$\Big\|\mathcal{N}_1(u,w)(t)\Big\|_{H^p(\mathbf{S}^2)} = \Big\|\mathbf{D}_r(t)u_0\Big\|_{H^p(\mathbf{S}^2)} \le \Big\|u_0\Big\|_{H^{p-1}(\mathbf{S}^2)} \tag{13}$$

and

$$\Big\|\mathcal{N}_2(u,w)(t)\Big\|_{H^p(\mathbf{S}^2)} = \Big\|\mathbf{D}_r(t)w_0\Big\|_{H^p(\mathbf{S}^2)} \le \Big\|w_0\Big\|_{H^{p-1}(\mathbf{S}^2)}. \tag{14}$$

From two above observations, we deduce that $\mathbf{N}(u,w) \in \left(L^\infty_\nu(0,T;H^p(\mathbf{S}^2))\right)^2$ for any $\nu > 0$.

Take any (u_1,w_1) and (u_2,w_2) in the space $\left(L^\infty_\nu(0,T;H^p(\mathbf{S}^2))\right)^2$. We get that

$$\begin{aligned} &\Big\|\mathcal{N}_1(u_1,w_1) - \mathcal{N}_1(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} \\ &= \Big\| \int_0^t \mathbf{D}_r(t-s)F(u_1(s),w_1(s))ds - \int_0^t \mathbf{D}_r(t-s)F(u_2(s),w_2(s))ds \Big\|_{H^p(\mathbf{S}^2)} \\ &\le \int_0^t \Big\|F(u_1(s),w_1(s)) - F(u_2(s),w_2(s))\Big\|_{H^{p-1}(\mathbf{S}^2)} ds. \end{aligned} \tag{15}$$

From the condition of globally Lipschitz of the source function F, we find that

$$\begin{aligned} &\int_0^t \Big\|F(u_1(s),w_1(s)) - F(u_2(s),w_2(s))\Big\|_{H^{p-1}(\mathbf{S}^2)} ds \\ &\le C_p \int_0^t \Big\|F(u_1(s),w_1(s)) - F(u_2(s),w_2(s))\Big\|_{H^{p}(\mathbf{S}^2)} ds \\ &\le K_f C_p \int_0^t \Big\|u_1(s)-u_2(s)\Big\|_{H^p(\mathbf{S}^2)} ds + K_f C_p \int_0^t \Big\|w_1(s)-w_2(s)\Big\|_{H^p(\mathbf{S}^2)} ds. \end{aligned} \tag{16}$$

Hence, we find that

$$\begin{aligned} &e^{-\nu t}\Big\|\mathcal{N}_1(u_1,w_1) - \mathcal{N}_1(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} \\ &\le K_f C_p \int_0^t e^{-\nu(t-s)} e^{-\nu s} \Big\|u_1(s)-u_2(s)\Big\|_{H^p(\mathbf{S}^2)} ds + K_f C_p \int_0^t e^{-\nu(t-s)} e^{-\nu s} \Big\|w_1(s)-w_2(s)\Big\|_{H^p(\mathbf{S}^2)} ds \\ &\le K_f C_p \left(\int_0^t e^{-\nu(t-s)} ds\right) \max_{0\le t\le T} \Big\| \exp(-\nu t)\left(u_1(\cdot,t) - u_2(.,t)\right)\Big\|_{H^p(\mathbf{S}^2)} \\ &\qquad + K_f C_p \left(\int_0^t e^{-\nu(t-s)} ds\right) \max_{0\le t\le T} \Big\| \exp(-\nu t)\left(w_1(\cdot,t) - w_2(.,t)\right)\Big\|_{H^p(\mathbf{S}^2)}. \end{aligned} \tag{17}$$

It is obvious to see that

$$\int_0^t e^{-\nu(t-s)} ds = \frac{1-e^{-\nu t}}{\nu} \le \frac{1}{\nu}. \tag{18}$$

Combining (17) and (18), we find that

$$\begin{aligned} e^{-\nu t}\Big\|\mathcal{N}_1(u_1,w_1)-\mathcal{N}_1(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} &\le \frac{K_f C_p}{\nu} \max_{0\le t\le T} \Big\| \exp(-\nu t)\left(u_1(\cdot,t)-u_2(.,t)\right)\Big\|_{H^p(\mathbf{S}^2)} \\ &+ \frac{K_f C_p}{\nu} \max_{0\le t\le T} \Big\| \exp(-\nu t)\left(w_1(\cdot,t)-w_2(.,t)\right)\Big\|_{H^p(\mathbf{S}^2)}. \end{aligned} \tag{19}$$

By a similar argument as above, we find that

$$\begin{aligned} e^{-\nu t}\Big\|\mathcal{N}_2(u_1,w_1)-\mathcal{N}_2(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} &\le \frac{K_g C_p}{\nu} \max_{0\le t\le T} \Big\| \exp(-\nu t)\left(u_1(\cdot,t)-u_2(.,t)\right)\Big\|_{H^p(\mathbf{S}^2)} \\ &+ \frac{K_g C_p}{\nu} \max_{0\le t\le T} \Big\| \exp(-\nu t)\left(w_1(\cdot,t)-w_2(.,t)\right)\Big\|_{H^p(\mathbf{S}^2)}. \end{aligned} \tag{20}$$

From two above observation, we arrive at the following estimate

$$\begin{aligned} e^{-\nu t}\Big\|\mathcal{N}_1(u_1,w_1)-\mathcal{N}_1(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} &+ e^{-\nu t}\Big\|\mathcal{N}_2(u_1,w_1)-\mathcal{N}_2(u_2,w_2)\Big\|_{H^p(\mathbf{S}^2)} \\ &\le \frac{K_f C_p + K_g C_p}{\nu} \|(u_1,w_1)-(u_2,w_2)\|_{\nu,p}. \end{aligned} \tag{21}$$

The right hand side of (21) is independent of t, we find that

$$\Big\|\mathcal{N}(u_1,w_1)-\mathcal{N}(u_2,w_2)\Big\|_{\nu,p} \le \frac{K_f C_p + K_g C_p}{\nu} \|(u_1,w_1)-(u_2,w_2)\|_{\nu,p}. \tag{22}$$

By choose ν large enough, we can say that $\mathcal{N}$ is a contraction mapping. So, there exists a function (u,w) which is a solution of

$$\mathbf{N}(u,w) = (u,w).$$

Moreover, we get

$$\begin{aligned} \big\|u(t)\big\|_{H^p(\mathbf{S}^2)} &\le \Big\|\mathbf{D}_r(t)u_0\Big\|_{H^p(\mathbf{S}^2)} + \Big\|\int_0^t \mathbf{D}_r(t-s)F(u(s),w(s))ds\Big\|_{H^p(\mathbf{S}^2)} \\ &\le \sqrt{\frac{1}{1-r}}\big\|u_0\big\|_{H^{p-1}(\mathbf{S}^2)} + \sqrt{\frac{1}{1-r}}\Big\|\int_0^t F(u(s),w(s))ds\Big\|_{H^{p-1}(\mathbf{S}^2)}. \end{aligned} \tag{23}$$

Since the fact that $\|v\|_{H^{p-1}(\mathbf{S}^2)} \le C_p \|v\|_{H^p(\mathbf{S}^2)}$ and Lipschitz condition of F as in (8), we know that

$$\begin{aligned} \Big\|\int_0^t F(u(s),w(s))ds\Big\|_{H^{p-1}(\mathbf{S}^2)} &\le C_p \int_0^t \Big\|F(u(s),w(s))\Big\|_{H^p(\mathbf{S}^2)} ds \\ &\le K_f C_p \int_0^t \left(\Big\|u(.,s)\Big\|_{H^p(\mathbf{S}^2)} + \Big\|w(.,s)\Big\|_{H^p(\mathbf{S}^2)}\right) ds. \end{aligned} \tag{24}$$

Combining (23) and (24), we arrive at

$$\big\|u(.,t)\big\|_{H^p(\mathbf{S}^2)} \le \sqrt{\frac{1}{1-r}}\big\|u_0\big\|_{H^{p-1}(\mathbf{S}^2)} + \sqrt{\frac{1}{1-r}} K_f C_p \int_0^t \left(\Big\|u(.,s)\Big\|_{H^p(\mathbf{S}^2)} + \Big\|w(.,s)\Big\|_{H^p(\mathbf{S}^2)}\right) ds. \tag{25}$$

By a similar techniques as above, we get that

$$\left\|w(.,t)\right\|_{H^p(\mathbf{S}^2)} \leq \sqrt{\frac{1}{1-r}}\left\|w_0\right\|_{H^{p-1}(\mathbf{S}^2)} + \sqrt{\frac{1}{1-r}}K_g C_p \int_0^t \left(\left\|u(.,s)\right\|_{H^p(\mathbf{S}^2)} + \left\|w(.,s)\right\|_{H^p(\mathbf{S}^2)}\right) ds. \tag{26}$$

From two previous estimate, we infer that

$$\begin{aligned}&\left\|u(.,t)\right\|_{H^p(\mathbf{S}^2)} + \left\|w(.,t)\right\|_{H^p(\mathbf{S}^2)} \\ &\leq \sqrt{\frac{1}{1-r}}\left\|u_0\right\|_{H^{p-1}(\mathbf{S}^2)} + \sqrt{\frac{1}{1-r}}\left\|w_0\right\|_{H^{p-1}(\mathbf{S}^2)} \\ &+ \sqrt{\frac{1}{1-r}}(K_f+K_g)C_p \int_0^t \left(\left\|u(.,s)\right\|_{H^p(\mathbf{S}^2)} + \left\|u(.,s)\right\|_{H^p(\mathbf{S}^2)}\right) ds. \end{aligned} \tag{27}$$

Hence, by using Gronwall's inequality, we obtain that the desired result (10). □

Theorem 2.4. *Let F, G as in Theorem (2.3). Then there exists a positive constant $\overline{C}(p, r, K_f, K_g, T)$ such that*

$$\left\|\frac{d}{dt}u(t)\right\|_{H^{p-1}(\mathbf{S}^2)} + \left\|\frac{d}{dt}w(t)\right\|_{H^{p-1}(\mathbf{S}^2)} \leq \overline{C}(p, r, K_f, K_g, T)\left(\left\|u_0\right\|_{H^{p-1}(\mathbf{S}^2)} + \left\|w_0\right\|_{H^{p-1}(\mathbf{S}^2)}\right). \tag{28}$$

Proof. Let us continue to treat the regularity result for first derivative of (u, w). Set $\overline{D}_r(t)$ is defined by

$$\overline{D}_r(t)f = \sum_{n=0}^{\infty}\sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \frac{-r(n^2+n)}{\left(1+(1-r)(n^2+n)\right)^2} \exp\left(\frac{-\alpha(n^2+n)}{1+(1-r)(n^2+n)}t\right)\widehat{f}_{nk}\mathbf{X}_{nk}.$$

Then we get the following equality

$$\left\|\overline{D}_r(t)f\right\|^2_{H^q(\mathbf{S}^2)} = \sum_{n=0}^{\infty}\sum_{k=1}^{\frac{2n+1}{\Gamma(2)}} \frac{r^2\left(n^2+n+1\right)^q(n^2+n)^2}{\left(1+(1-r)(n^2+n)\right)^2} \exp\left(\frac{-2\alpha(n^2+n)}{1+(1-r)(n^2+n)}t\right)|\widehat{f}_{nk}|^2. \tag{29}$$

Since $1 > 1-r$ and $n^2+n < n^2+n+1$, we get that

$$\frac{\left(n^2+n+1\right)^q(n^2+n)^2}{1+(1-r)(n^2+n)} \leq \frac{1}{1-r}\left(n^2+n+1\right)^{q+1}. \tag{30}$$

It follows from (5) that

$$\left\|\overline{D}_r(t)f\right\|^2_{H^q(\mathbf{S}^2)} \leq \frac{1}{1-r}\sum_{n=0}^{\infty}\sum_{k=1}^{\frac{2n+1}{\Gamma(2)}}\left(n^2+n+1\right)^{q+1}|\widehat{f}_{nk}|^2 = \frac{1}{1-r}\left\|f\right\|^2_{H^{q+1}(\mathbf{S}^2)}. \tag{31}$$

Hence, we find that

$$\left\|\overline{D}_r(t)f\right\|_{H^q(\mathbf{S}^2)} \leq \sqrt{\frac{1}{1-r}}\left\|f\right\|_{H^{q+1}(\mathbf{S}^2)}. \tag{32}$$

Since the fomula

$$\frac{d}{dt}\int_0^t G(t,s)ds = \int_0^t G_t(t,s)ds + G(t,t).$$

We get

$$\frac{d}{dt}u(t) = \overline{D}_r(t)u_0 + \int_0^t \overline{D}_r(t-s)F(u(s),w(s))ds + F(u(t),w(t)),$$
$$\frac{d}{dt}w(t) = \overline{D}_r(t)w_0 + \int_0^t \overline{D}_r(t-s)G(u(s),w(s))ds + G(u(t),w(t)). \tag{33}$$

This implies that

$$\Big\|\frac{d}{dt}u(t)\Big\|_{H^{p-1}(\mathbf{S}^2)} \le \Big\|\overline{D}_r(t)u_0\Big\|_{H^{p-1}(\mathbf{S}^2)} + \Big\|\int_0^t \overline{D}_r(t-s)F(u(s),w(s))ds\Big\|_{H^{p-1}(\mathbf{S}^2)}$$
$$+ \Big\|F(u(t),w(t))\Big\|_{H^{p-1}(\mathbf{S}^2)} \tag{34}$$

Using (32), we get that

$$\Big\|\overline{D}_r(t)u_0\Big\|_{H^{p-1}(\mathbf{S}^2)} \le \sqrt{\frac{1}{1-r}}\Big\|u_0\Big\|_{H^p(\mathbf{S}^2)}. \tag{35}$$

Using again (32) and Lipschitz condition of F as in (8), we infer that

$$\Big\|\int_0^t \overline{D}_r(t-s)F(u(s),w(s))ds\Big\|_{H^{p-1}(\mathbf{S}^2)} \le \sqrt{\frac{1}{1-r}}\int_0^t \Big\|F(u(s),w(s))\Big\|_{H^p(\mathbf{S}^2)}ds$$
$$\le K_f\sqrt{\frac{1}{1-r}}\int_0^t \Big(\|u(.,s)\|_{H^p(\mathbf{S}^2)} + \|w(.,s)\|_{H^p(\mathbf{S}^2)}\Big)ds \tag{36}$$

and

$$\Big\|F(u(t),w(t))\Big\|_{H^{p-1}(\mathbf{S}^2)} \le K_f\Big(\|u(.,t)\|_{H^p(\mathbf{S}^2)} + \|w(.,t)\|_{H^p(\mathbf{S}^2)}\Big). \tag{37}$$

Combining (34), (35), (36), and (37), we have the following assertion right away

$$\Big\|\frac{d}{dt}u(t)\Big\|_{H^{p-1}(\mathbf{S}^2)} \le \sqrt{\frac{1}{1-r}}\Big\|u_0\Big\|_{H^p(\mathbf{S}^2)}$$
$$+ K_f\sqrt{\frac{1}{1-r}}\int_0^t \Big(\|u(.,s)\|_{H^p(\mathbf{S}^2)} + \|w(.,s)\|_{H^p(\mathbf{S}^2)}\Big)ds$$
$$+ K_f\Big(\|u(.,t)\|_{H^p(\mathbf{S}^2)} + \|w(.,t)\|_{H^p(\mathbf{S}^2)}\Big). \tag{38}$$

By a similar argument as above, we also get that

$$\Big\|\frac{d}{dt}w(t)\Big\|_{H^{p-1}(\mathbf{S}^2)} \le \sqrt{\frac{1}{1-r}}\Big\|w_0\Big\|_{H^p(\mathbf{S}^2)}$$
$$+ K_g\sqrt{\frac{1}{1-r}}\int_0^t \Big(\|u(.,s)\|_{H^p(\mathbf{S}^2)} + \|w(.,s)\|_{H^p(\mathbf{S}^2)}\Big)ds$$
$$+ K_g\Big(\|u(.,t)\|_{H^p(\mathbf{S}^2)} + \|w(.,t)\|_{H^p(\mathbf{S}^2)}\Big). \tag{39}$$

From two recent observation, we can deduce that

$$\Big\|\frac{d}{dt}u(t)\Big\|_{H^{p-1}(\mathbf{S}^2)} + \Big\|\frac{d}{dt}w(t)\Big\|_{H^{p-1}(\mathbf{S}^2)}$$
$$\le \sqrt{\frac{1}{1-r}}\Big\|u_0\Big\|_{H^p(\mathbf{S}^2)} + \sqrt{\frac{1}{1-r}}\Big\|w_0\Big\|_{H^p(\mathbf{S}^2)}$$
$$+ (K_g+K_f)\sqrt{\frac{1}{1-r}}\int_0^t \Big(\|u(.,s)\|_{H^p(\mathbf{S}^2)} + \|w(.,s)\|_{H^p(\mathbf{S}^2)}\Big)ds$$
$$+ (K_g+K_f)\Big(\|u(.,t)\|_{H^p(\mathbf{S}^2)} + \|w(.,t)\|_{H^p(\mathbf{S}^2)}\Big). \tag{40}$$

Since (10), we find that

$$\begin{aligned}&\|u(.,t)\|_{H^p(\mathbf{S}^2)} + \|w(.,t)\|_{H^p(\mathbf{S}^2)}\\ &\le \left(\sqrt{\frac{1}{1-r}}\|u_0\|_{H^{p-1}(\mathbf{S}^2)} + \sqrt{\frac{1}{1-r}}\|w_0\|_{H^{p-1}(\mathbf{S}^2)}\right)\exp\Big(\sqrt{\frac{1}{1-r}}(K_f+K_g)C_pT\Big). \end{aligned} \quad (41)$$

From two above observation, we can deduce that

$$\Big\|\frac{d}{dt}u(t)\Big\|_{H^{p-1}(\mathbf{S}^2)} + \Big\|\frac{d}{dt}w(t)\Big\|_{H^{p-1}(\mathbf{S}^2)} \le \overline{C}(p,r,K_f,K_g,T)\left(\|u_0\|_{H^{p-1}(\mathbf{S}^2)} + \|w_0\|_{H^{p-1}(\mathbf{S}^2)}\right). \quad (42)$$

□

Acknowledgements

We would like to thank the editor and referees for their valuable comments and suggestions.

References

[1] R.S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, Mathematical Methods in the Applied Sciences https://doi.org/10.1002/mma.665

[2] H. Afshari, E, Karapınar, A discussion on the existence of positive solutions of the boundary value problems via-Hilfer fractional derivative on b-metric spaces, Advances in Difference Equations volume 2020, Article number: 616 (2020)

[3] H. Afshari, S. Kalantari, E. Karapınar; Solution of fractional differential equations via coupled fixed point, Electronic Journal of Differential Equations,Vol. 2015 (2015), No. 286, pp. 1-12

[4] B. Alqahtani, H. Aydi, E. Karapınar, V. Rakocevic, A Solution for Volterra Fractional Integral Equations by Hybrid Contractions, Mathematics 2019, 7, 694.

[5] E. Karapınar, A. Fulga, M. Rashid, L. Shahid, H. Aydi, Large Contractions on Quasi-Metric Spaces with an Application to Nonlinear Fractional Differential-Equations, Mathematics 2019, 7, 444.

[6] A.Salim, B. Benchohra, E. Karapınar, J.E. Lazreg, Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations, Adv Differ Equ 2020, 601 (2020)

[7] E. Karapınar, T. Abdeljawad, F. Jarad, Applying new fixed point theorems on fractional and ordinary differential equations, Advances in Difference Equations, 2019, 2019:421

[8] A. Abdeljawad, R.P. Agarwal, E. Karapınar, P.S. Kumari, Solutions of he Nonlinear Integral Equation and Fractional Differential Equation Using the Technique of a Fixed Point with a Numerical Experiment in Extended b-Metric Space, Symmetry 2019, 11, 686.

[9] N.H. Tuan, Y. Zhou, T.N. Thach, N.H. Can, Initial inverse problem for the nonlinear fractional Rayleigh-Stokes equation with random discrete data, Commun. Nonlinear Sci. Numer. Simul. 78 (2019), 104873, 18 pp.

[10] N.H. Tuan, L.N. Huynh, T.B. Ngoc, Y. Zhou, On a backward problem for nonlinear fractional diffusion equations, Appl. Math. Lett. 92 (2019), 76–84.

[11] Q.T. Le Gia, Approximation of parabolic PDEs on spheres using collocation method, Adv. Comput. Math., 22 (2005), 377–397.

[12] Q.T.L. Gia, N.H. Tuan, T. Tran, Solving the backward heat equation on the unit sphere, ANZIAM J. (E) 56 (2016), pp. C262–C278.

[13] Q.T. Le Gia, Approximation of parabolic PDEs on spheres using collocation method, Adv. Comput. Math., 22 (2005), 377–397.

[14] Q.T. Le Gia, Galerkin approximation of elliptic PDEs on spheres, J. Approx. Theory , 130 (2004), 125–149.

[15] Q.T. Le Gia, I.H. Sloan, T. Tran, Overlapping additive Schwarz preconditioners for elliptic PDEs on the unit sphere, Math. Comp. 78 (2009), no. 265, 79–101

[16] Z. Brzeźniak, B. Goldys, Q.T. Le Gia, Random attractors for the stochastic Navier-Stokes equations on the 2D unit sphere, J. Math. Fluid Mech. 20 (2018), no. 1, 227–253.

[17] N.D. Phuong, N. H. Luc, *Note on a Nonlocal Pseudo-Parabolic Equation on the Unit Sphere*, Dynamic Systems and Applications 30 (2021) No.2, 295–304.

[18] N.H. Luc, H. Jafari, P. Kumam, N.H. Tuan, *On an initial value problem for time fractional pseudoâĂŘparabolic equation with Caputo derivarive*, Mathematical Methods in the Applied Sciences, to appear.

[19] O. Nikan, H. Jafari, A. Golbabai, Numerical analysis of the fractional evolution model for heat flow in materials with memory, Alexandria Engineering Journal 59 (4), 2627–2637

[20] R. M. Ganji, H. Jafari, S. Nemati, A new approach for solving integro-differential equations of variable order, Journal of Computational and Applied Mathematics 379, 1–13

[21] H. Jafari, H. Tajadodi, R.M. Ganji, A numerical approach for solving variable order differential equations based on Bernstein polynomials, Comput. Math. Methods 1 (2019), no. 5, e1055, 11 pp.
[22] N.H. Tuan, D. Baleanu, T.N. Thach, D. O'Regan, N.H. & Can, Approximate solution for a 2-D fractional differential equation with discrete random noise. Chaos, Solitons & Fractals, 133, 109650.
[23] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl., 1(2) (2015), pp. 1–13.
[24] M. Caputo, M. Fabrizio, Applications of new time and spatial fractional derivatives with exponential kernels, Progr. Fract. Differ. Appl., 2(2) (2016), pp. 1–11.
[25] J. Losada, J.J. Nieto, Properties of a new fractional derivative without singular kernel, Prog. Fract. Differ. Appl., 1(2) (2015), pp. 87–92.
[26] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl., 1(2) (2015), pp. 1–13.
[27] M. Caputo, M. Fabrizio, Applications of new time and spatial fractional derivatives with exponential kernels, Progr. Fract. Differ. Appl., 2(2) (2016), pp. 1–11.
[28] N.H. Luc, D. Baleanu, N.H. & Can, Reconstructing the right-hand side of a fractional subdiffusion equation from the final data, Journal of Inequalities and Applications, 2020(1), 1-15.
[29] J. Losada, J.J. Nieto, Properties of a new fractional derivative without singular kernel, Prog. Fract. Differ. Appl., 1(2) (2015), pp. 87–92.
[30] T.M. Atanacković, S. Pillipović, D. Zorica, Properties of the Caputo-Fabrizio fractional derivative and its distributional settings, Fract. Calc. Appl. Anal., 21, (2018), pp. 29–44.
[31] V. Gafiychuk and B. Datsko, Stability analysis and oscillatory structures in timefractional reaction-diffusion systems, Phys. Rev. E 75 (2007), article 055201(R).
[32] V. Gafiychuk, B. Datsko, and V. Meleshko, Mathematical modeling of time-fractional reaction-diffusion systems, J. Comput. Appl. Math. 220 (2008), 215-âĂŞ225.
[33] S. Muthaiah, M. Murugesan, N.G. & Thangaraj, Existence of solutions for nonlocal boundary value problem of Hadamard fractional differential equations. Advances in the Theory of Nonlinear Analysis and its Application, 3(3), 162-173.
[34] V. Berinde, H. Fukhar-ud-din, M. & Pacurar, On the global stability of some k-order difference equations. Results in Nonlinear Analysis, 1(1), 13-18.
[35] N.D. Phuong, L.V.C. Hoan, E. Karapınar, J. Singh, H.D. Binh, N.H. & Can, Fractional order continuity of a time semi-linear fractional diffusion-wave system. Alexandria Engineering Journal, 59(6), 4959-4968.